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A group arrival retrial *G* - queue with multi optional stages of service, orbital search and server breakdown

J Radha, KIndhira and V M Chandrasekaran

Department of Mathematics, School of Advanced Sciences, VIT University, Vellore-632014,India

E-mail: kindhira@vit.ac.in

Abstract: Agroup arrival feedback retrial queue with *k* optional stages of service and orbital search policy is studied. Any arriving group of customer finds the server free, one from the group enters into the first stage of service and the rest of the group join into the orbit. After completion of the i^{th} stage of service, the customer under service may have the option to choose $(i+1)^{th}$ stage of service with θ_i probability, with p_I probabilitymay join into orbit as feedback customer or may leave the system with $q_i = \begin{Bmatrix} 1-p_i-\theta_i, i=1,2,...k-1 \\ 1-p_i, i=k \end{Bmatrix}$ probability. Busy server may get to breakdown due to the arrival of negative customers and the service channel will fail for a short interval of time.At the completion of service or repair, the server searches for the customer in the orbit (if any) with probability α or remains idle with probability$1-\alpha$. By using the supplementary variable method, steady state probability generating function for system size, some system performance measures are discussed.

1. Introduction

Nowadays we retry to get the service such as telephone calls, ATMs, good output from the candidates and searching data etc., Retrial is very common in queues. Artalejo[1], Artalejo and Ret al. [2] analyzed, the retrial policy in queueing systems. Sometimes the service station may get down suddenly due to malware attack. This malware is treated as negative customers. These customers disturb the server as well as the customer in service and it does not require any service. The server will be ready to give the service, after the repair process.Negative customers in queueing system are called G-queue. Recently, Wang J et al. [17] and Zhang M et al. [18] discussed the concept of G queue.

After service or repair completion, the idle server may search for the customer in the orbit to provide service. Gao S et al. [10] analyzed the retrial G queue with this concept. The service has many stages in nature. Here a server gives the multi optional stages of service. Autors like, Bagyam JEA et al. [3], Choudhury etal. [4],[6] and Radhaet al. [15] are surveyed the multi stage and two phase service in queueing system. This work has the application in the communication network.

2. Model description

2.1 Arrival process

Units arriving the system in batches with Poisson arrival rate λ. Let X_k, the number of units in the k^{th} batch, where k =1,2 ,3,… with common distribution $\Pr[X_k = n] = \chi_n$, n= 1,2,3… The PGF (probability generating function) of X *is* $X(z)$.The first and second moments are $E(X)$ and $E(X(X-1))$.

2.2 Retrial process

If there is no space to wait, one from the arriving unit begins service (if the server is free) and rest are waiting in the orbit. If an arriving batch finds the server either busy or on vacation or breakdown, then the batch joins into an orbit. HereInter-retrial times form an arbitrary distribution $R(x)$ with corresponding Laplace-Stieltijes transform (LST) $R^*(s)$.

2.3 Service process

Here a server gives k stages of service. The First Stage Service (FSS) is followed by $i$stages of service. The service time $S_i$for$i=1,2,...k$ has a distribution (general) function $S_i(x)$ having LST $S_i^*(s)$ and first and second moments are $E(S_i)$ and $E(S_i^2)$, (i = 1,2,…k).

2.4 Feedback rule

After completion of i^{th} stage of service the customer may go to $(i+1)^{th}$ stage with probability θ_i or may join into the orbit as feedback customer with probability p_i or leaves the system with probability $q_i = 1-\theta_i - p_i$ for i = 1,2,…k-1. If the customer in the last k^{th}stage may join to the orbit with probability p_k or leaves the system with probability $q_k = 1 - p_k$. From this model, the service time or the time required by the customer to complete the service cycle is a random variable S is given by $S = \sum_{i=1}^{k} \Theta_{i-1} S_i$ having the LST $S^*(s) = \prod_{i=1}^{k} \Theta_{i-1} S_i^*(s)$ and the expected value is $E(S) = \sum_{i=1}^{k} \Theta_{i-1} E(S_i)$, where $\Theta_i = \theta_1 \theta_2 ... \theta_i$ and $\Theta_0 = 1$.

2.5Repair process

The negative customers enters the station with Poisson arrival rate δ. The repair time G_i has the distributions function $G_i(y)$and LST $G_i^*(s)$for (i=1,2,…k).

2.6 An orbital search

An idle server enters the orbit to search the customer with probability α or remains idle with probability $1-\alpha$. Here, the search time is negligible.

Various stochastic processes involved in the system are assumed to be mutually exclusive. In the steady state, let $R(0)=0, R(\infty)=1, S_i(0)=0, S_i(\infty)=1, i=1,2,...k$ are continuous at x = 0 and $G_i(0)=0, G_i(\infty)=1$ are continuous at y = 0, $(1 \le i \le k)$. Let $R^0(t)$, $S_i^0(t)$ and $G_i^0(t)$be the elapsed times for retrial, service on i^{th} stage and repair on i^{th} stage, $(1 \le i \le k)$ respectively. Now, a random variable at time t,

$$C(t) = \begin{cases} 0, \text{ if the server is idle,} \\ 1, \text{ if the server is busy on } i^{\text{th}} \text{ stage,} \\ 2, \text{ if the server is repair on } i^{\text{th}} \text{ stage,} \end{cases}$$

The Markov process $\{C(t), N(t); t \geq 0\}$ describes the system state, where $C(t)$ - the server state and $N(t)$ - the number in orbit at time *t*, the functions $a(x)$, $\mu_i(x)$ and $\xi_i(y)$ are the conditional completion rates for retrial, service, vacation, delay in repair and repair respectively ($1 \leq i \leq k$).

$$a(x)dx = \frac{dR(x)}{1-R(x)},\ \mu_i(x)dx = \frac{dS_i(x)}{1-S_i(x)} \text{ and } \xi_i(y)dy = \frac{dG_i(y)}{1-G_i(y)}.$$

Then define $B_i^* = S_1^* S_2^* ... S_i^*$ and $B_0^* = 1$. The first moment M_{1i} and second moment M_{2i} of B_i^* are given by

$$M_{1i} = \lim_{z \to 1} dB_i^*[A_i(z)] \big/ dz = \sum_{j=1}^{i} \lambda E(X) E(S_j)$$

$$M_{2i} = \lim_{z \to 1} d^2 B_i^*[A_i(z)] \big/ dz^2 = \sum_{j=1}^{i} \Big[\left(\lambda E(X)\right)^2 E(S_j^2) - (\lambda)^2 E(X) E(X(X-1)) E(S_j) \Big]$$

where

$$A_i(z) = \delta + b(z) \text{ and } b(z) = \lambda\left(1 - X(z)\right)$$

Let $\{t_n;\ n = 1,2,\ldots\}$ be the service period ending time or repair period ending time. In this system, $Z_n = \{C(t_n+), N(t_n+)\}$ forms an embedded Markov chain which is ergodic $\Leftrightarrow \rho < 1$, where

$$\rho = E(X)\left(1 - R^*(\lambda)\right) + \lambda E(X)\left(\frac{1 - S_i^*(\delta)}{\delta}\right)(1 + \delta g^{(1)}) + \Sigma_1 + B_1,$$

$$\Sigma_1 = \sum_{i=1}^{k} \Theta_{i-1} M_{1i} + \sum_{i=1}^{k} p_i \Theta_{i-1} - \sum_{i=1}^{k-1} \Theta_{i-1} M_{1i}$$

and

$$B_1 = \sum_{i=1}^{k} \Theta_{i-1} (M_{1i-1}(1 - S_i^*(\delta)) - M_{1i} B_{i-1}^*(\delta))$$

3. Steady state distribution

For $\{N(t),\ t \geq 0\}$, define the probabilities functions at time *t*,

- $P_0(t)$ - *Pr*(the system is empty),

At time *t* and *n* customers in the orbit,

- $P_n(x,t)$ - *Pr*(an elapsed retrial time *x* of the retrial customers),
- $\Pi_{i,n}(x,t), (1 \leq i \leq k)$ - *Pr*(elapsed service time *x* on i^{th} stage of the customer under service),
- $R_{i,n}(x,y,t), (1 \leq i \leq k)$ - *Pr* (an elapsed times for service is *x* and repair is *y* on i^{th} stage),

The stability condition exists for $t \geq 0$, $x \geq 0$, $y \geq 0$, $n \geq 0$ for$i = 1,2,\ldots k$.

$$P_0 = \lim_{t\to\infty} P_0(t),\ P_n(x) = \lim_{t\to\infty} P_n(x,t),\ \Pi_{i,n}(x) = \lim_{t\to\infty} \Pi_{i,n}(x,t),\ R_{i,n}(x,y) = \lim_{t\to\infty} R_{i,n}(x,y,t),\ \text{for } t \ge 0.$$

a. *Steady state equations*

By the supplementary variable technique (Kelison and Servi. [11]) the following governing equations are obtained for (i=1,2,..k).

$$\lambda P_0 = \sum_{i=1}^{k} q_i \int_0^\infty \mu_i(x)\Pi_{i,0}(x,t)dx + \int_0^\infty \zeta_i(x)R_{i,0}(x)dx \tag{1}$$

$$\frac{dP_n(x)}{dx} + P_n(x)[\lambda + a(x)] = 0,\ n \ge 1 \tag{2}$$

$$\frac{d\Pi_{i,0}(x)}{dx} + \Pi_{i,0}(x)[\lambda + \delta + \mu_i(x)] = 0, n \ge 1, \tag{3}$$

$$\frac{d\Pi_{i,n}(x)}{dx} + \Pi_{i,n}(x)[\lambda + \delta + \mu_i(x)] - \lambda\sum_{k=1}^{n}\chi_k \Pi_{i,n-k}(x) = 0, n = 1,2,... \tag{4}$$

$$\frac{dR_{i,0}(x)}{dx} + R_{i,0}(x)[\lambda + \xi_i(x)] = 0,\ n = 0 \tag{5}$$

$$\frac{dR_{i,n}(x)}{dx} + R_{i,n}(x)[\lambda + \xi_i(x)] - \lambda\sum_{k=1}^{n}\chi_k R_{i,n-k}(x) = 0,\ n = 1,2,... \tag{6}$$

The steady state boundary conditions at x = 0 and y = 0 are

$$P_n(0) = \sum_{i=1}^{k} q_i \int_0^\infty \mu_i(x)\Pi_{i,n}(x)dx + \sum_{i=1}^{k} p_i \int_0^\infty \mu_i(x)\Pi_{i,n-1}(x)dx + \int_0^\infty \zeta_i(x)R_{i,n}(x)dx,\ n \ge 1 \tag{7}$$

$$\Pi_{\mathrm{i},0}(0) = \lambda\chi_1 P_0 + \int_0^\infty a(x)P_1(x)dx,\ n = 0 \tag{8}$$

$$\Pi_{1,n}(0) = \left\{ \begin{array}{l} \int_0^\infty a(x)P_{n+1}(x)dx + \lambda\sum_{k=1}^{n}\chi_k \int_0^\infty P_{n-k+1}(x)dx + b\lambda\chi_{n+1}P_0 \\ +\alpha\left(\begin{array}{l} \sum_{i=1}^{k-1} q_i \int_0^\infty \mu_i(x)\Pi_{i,n}(x)dx + (1-p_k)\int_0^\infty \mu_k(x)\Pi_{k,n}(x)dx \\ +\sum_{i=1}^{k} p_i \int_0^\infty \mu_i(x)\Pi_{i,n-1}(x)dx + \int_0^\infty \zeta_i(x)R_{i,n}(x)dx \end{array} \right) \end{array} \right\},\ n \ge 1 \tag{9}$$

$$\Pi_{i,n}(0) = \theta_{i-1}\int_0^\infty \mu_{i-1}(x)\Pi_{i-1,n}(x)dx,\ n = 1,2,...\ (2 \le i \le k) \tag{10}$$

$$R_{i,n}(x,0) = \delta\int_0^\infty \Pi_{i,n}(x),\ n \ge 0,\ \text{for}(1 \le i \le k) \tag{11}$$

The normalizing condition is

$$P_0 + \sum_{n=1}^{\infty}\int_0^{\infty} P_n(x)dx + \sum_{n=0}^{\infty}\left(\sum_{i=1}^{k}\left(\int_0^{\infty}\Pi_{i,n}(x)dx + \int_0^{\infty} R_{i,n}(x)dx\right)\right) = 1 \tag{12}$$

b. *Steady state solutions*

To solve the above equations, generating functions defined for $|z|\le 1$,

$$P(x,z) = \sum_{n=1}^{\infty} P_n(x)z^n;\ P(0,z) = \sum_{n=1}^{\infty} P_n(0)z^n;\ \Pi_i(x,z) = \sum_{n=0}^{\infty}\Pi_{i,n}(x)z^n; \Pi_i(0,z) = \sum_{n=0}^{\infty}\Pi_{i,n}(0)z^n;$$

$$R_i(x,z) = \sum_{n=0}^{\infty} R_{i,n}(x)z^n;\ R_i(0,z) = \sum_{n=0}^{\infty} R_{i,n}(0)z^n$$

Now multiplying Eqns. (2) - (11) by z^n and summing over n, (n = 0,1,2...and $1\le i \le k$)

$$\frac{dP(x,z)}{dx} = -P(x,z)[\lambda + a(x)] \tag{13}$$

$$\frac{d\Pi_i(x,z)}{dx} = -\Pi_i(x,z)[\lambda(1 - X(z)) + \delta + \mu_i(x)] \tag{14}$$

$$\frac{dR_i(x,z)}{dx} = -R_i(x,z)[\lambda(1 - X(z)) + \xi_i(y)] \tag{15}$$

Atx = 0 and y = 0,

$$P(0,z) = (1-\alpha)\left(\sum_{i=1}^{k}\left\{(p_i z + q_i)\int_0^{\infty}\mu_i(x)\Pi_i(x,z)dx\right\} - \lambda P_0 + \int_0^{\infty}\zeta_i(x)R_i(x,z)dx\right) \tag{16}$$

$$\Pi_1(0,z) = \left\{\begin{array}{l} \frac{1}{z}\int_0^{\infty} a(x)P(x,z)dx + \lambda\frac{P_0 X(z)}{z} - \lambda P_0 + \lambda\frac{X(z)}{z}\int_0^{\infty}P(x,z)dx \\ +\frac{\alpha}{z}\left(\sum_{i=1}^{k}\left\{(p_i z + q_i)\int_0^{\infty}\Pi_i(x,z)\mu_i(x)dx\right\} + \int_0^{\infty}R_i(x,z)\zeta_i(x)dx\right)\end{array}\right\},\ n\ge 1. \tag{17}$$

$$\Pi_i(0,z) = \theta_{i-1}\int_0^{\infty}\mu_{i-1}(x)\Pi_{i-1}(0,z)dx, (i = 2,3,...k). \tag{18}$$

$$R_i(0,z) = \delta\int_0^{\infty}\Pi_i(x,z)dx \tag{19}$$

Solving Eqns. (13)-(19),

$$P(x,z) = P(0,z)e^{-\lambda x}[1 - R(x)] \tag{20}$$

$$\Pi_i(x,z) = \Pi_i(0,z)e^{-A_i(z)x}[1 - S_i(x)] \tag{21}$$

$$R_i(x,z) = R_i(0,z)e^{-b(z)x}[1 - G_i(x)] \tag{22}$$

where

$A_i(z) = \delta + b(z)$ and $b(z) = \lambda(1 - X(z))$

Solving the above system of equations, we obtain the following,

$$\Pi_1(0,z)=\left(\begin{array}{l}\dfrac{P(0,z)}{z}\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]+\dfrac{\lambda X(z)}{z}P_0\\+\dfrac{\alpha}{z}\left(\sum_{i=1}^{k}\left(p_iz+q_i\right)\Pi_i(0,z)S_i^*(A_i(z))+R_i(0,z)G_i^*(b(z))-\lambda P_0\right)\end{array}\right) \quad (23)$$

$$\Pi_i(0,z)=\Theta_{i-1}\Pi_1(0,z)\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right),\ (i=2,3,...k) \quad (24)$$

$$R_i(0,z)=\delta\,\Theta_{i-1}\Pi_1(0,z)\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\frac{[1-S_i^*(A_i(z))]}{A_i(z)} \quad (25)$$

$$P(0,z)=\left\{\frac{\lambda P_0(1-\alpha)\left(X(z)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)-zA_i(z)\right)}{zA_i(z)-\left(\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}\right\} \quad (26)$$

Using (26) in (23), we get,

$$\Pi_1(0,z)=\left\{\frac{\lambda P_0A_i(z)\left(\left(X(z)-\alpha\right)-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)}{zA_i(z)-\left(\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}\right\} \quad (27)$$

$$\Pi_i(0,z)=\left\{\frac{\lambda P_0A_i(z)\Theta_{i-1}(\mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z)))\left(\left(X(z)-\alpha\right)-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)}{zA_i(z)-\left(\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}\right\} \quad (28)$$

Similarly,

$$R_i(x,0,z)=\left\{\frac{\delta\lambda P_0A_i(z)\Theta_{i-1}(\mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z)))\left(1-S_i^*(A_i(z))\right)\left(\left(X(z)-\alpha\right)-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)}{zA_i(z)-\left(\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}\right\} \quad (29)$$

Using Eqns. (20) to (22) and Eqns. (26) to (29),the obtained $P(x,z),\Pi_i(x,z)$ and $R_i(x,y,z)$ are given, under $\rho < 1$,

$$P(x,z)=\left\{\frac{\lambda P_0(1-\alpha)\left(X(z)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)-zA_i(z)\right)\left(1-R(x)\right)e^{-\lambda x}}{zA_i(z)-\left(\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}\right\} \quad (30)$$

$$\Pi_i(x,z)=\left\{\frac{\lambda P_0A_i(z)\Theta_{i-1}(\mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z)))\left(\left(X(z)-\alpha\right)-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\left(1-S_i(x)\right)e^{-A_i(z)x}}{zA_i(z)-\left(\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}\right\} \quad (31)$$

$$R_i(x,y,z)=\left(\begin{array}{l}[1-S_i(x)]e^{-A_i(z)x}[1-G_i(y)]e^{-b(z)y}\\ \left\{\dfrac{\delta\lambda P_0 A_i(z)\Theta_{i-1}(B^*_{i-1}(A_{i-1}(z)))\left(1-S_i^*(A_i(z))\right)\left(\left(X(z)-\alpha\right)-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)}{zA_i(z)-\left(\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B^*_{i-1}\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}\right\}\end{array}\right) \tag{32}$$

Theorem 3.1. Under $\rho < 1$, the stationary distributions of the numbers in the system when server being idle, busy during i[th] stage and repair on i[th] stage (for $1\leq i\leq k$) are given by

$$P(z)=\left\{\frac{P_0(1-\alpha)\left(X(z)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B^*_{i-1}\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)-zA_i(z)\right)\left(1-R^*(\lambda)\right)}{zA_i(z)-\left(\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B^*_{i-1}\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}\right\} \tag{33}$$

$$\Pi_i(z)=\left\{\frac{\lambda P_0\Theta_{i-1}(B^*_{i-1}(A_{i-1}(z)))\left(\left(X(z)-\alpha\right)-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\left(1-S_i^*(A_i(z))\right)}{zA_i(z)-\left(\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B^*_{i-1}\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}\right\} \tag{34}$$

$$R_i(z)=\left(\begin{array}{l}\dfrac{\left(1-S_i^*(A_i(z))\right)\left(1-G_i^*(b(z))\right)}{A_i(z)b(z)}\\ \left\{\dfrac{\delta\lambda P_0\Theta_{i-1}(B^*_{i-1}(A_{i-1}(z)))\left(1-S_i^*(A_i(z))\right)\left(\left(X(z)-\alpha\right)-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)}{zA_i(z)-\left(\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right)\delta G_i^*\left[b(z)\right]\Theta_{i-1}\left(B^*_{i-1}\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}\right\}\end{array}\right) \tag{35}$$

where, $P_0=\dfrac{1}{\beta}\left\{1-(1-\alpha)E(X)\left(1-R^*(\lambda)\right)-\lambda E(X)\left(\dfrac{1-S_i^*(\delta)}{\delta}\right)(1+\delta g^{(1)})+\Sigma_1+B_1\right\}$ (36)

$$\beta=\left\{\begin{array}{l}\left(1-(1-\alpha)E(X)\left(1-R^*(\lambda)\right)-E(X)R^*(\lambda)\right)-(1-\alpha)\left(1-R^*(\lambda)\right)(E(X)R^*(\lambda)+E(X))\\ +\left(E(X)-(1-\alpha)E(X)\left(1-R^*(\lambda)\right)\right)\lambda\left(\dfrac{1-S_i^*(\delta)}{\delta}\right)(1+\delta g^{(1)})\end{array}\right.$$

Proof.Integrating(49) to (53) with respect to x and y (whenever needed), defined the following for ($1\leq i\leq k$) $P(z)=\int_0^\infty P(x,z)dx$, $\Pi_i(z)=\int_0^\infty \Pi_i(x,z)dx$,. $R_i(x,z)=\int_0^\infty R_i(x,y,z)dy$, $R_i(z)=\int_0^\infty R_i(x,z)dx$, Since, P_0 can be determined using (12). $P_0+P(1)+\sum_{i=1}^{k}\left(\Pi_i(1)+R_i(1)\right)=1$is obtained by setting $z=1$ in (33) to (35).

Theorem 3.2 Under $\rho < 1$, probability generating function of the system size and orbit size distribution at stationary point of time is

$$K(z)=\frac{Nr(z)}{Dr(z)} \tag{37}$$

$$Nr(z)=P_0\left\{\begin{array}{l}\left(\left(zA_i(z)-\left[1-(1-\alpha)\left(1-R^*(\lambda)\right)\right]+c\left(\begin{array}{l}(1-\alpha)\left(1-R^*(\lambda)\right)X(z)\\-\alpha-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\end{array}\right)\right)\right)[1-X(z)]\\+\left[X(z)-\alpha-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right]\left(\begin{array}{l}\sum_{i=1}^{k}zb(z)\Theta_{i-1}(B_{i-1}^*(A_{i-1}(z)))\left(1-S_i^*(A_i(z))\right)\\+\delta\left(1-G_i^*(b(z))\right)\end{array}\right)\end{array}\right\}$$

$$Dr(z)=[1-X(z)]\left(zA_i(z)-\left[\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right]\delta G_i^*[b(z)]\Theta_{i-1}\left(B_{i-1}^*[A_{i-1}(z)]\right)\left(1-S_i^*(A_i(z))\right)\right)$$

$$H(z)=\frac{Nr(z)}{Dr(z)} \tag{38}$$

$$Nr(z)=P_0\left\{\begin{array}{l}\left(\left(zA_i(z)-\left[1-(1-\alpha)\left(1-R^*(\lambda)\right)\right]+c\left(\begin{array}{l}(1-\alpha)\left(1-R^*(\lambda)\right)X(z)\\-\alpha-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\end{array}\right)\right)\right)[1-X(z)]\\+\left[X(z)-\alpha-(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right]\left(\begin{array}{l}\sum_{i=1}^{k}b(z)\Theta_{i-1}(B_{i-1}^*(A_{i-1}(z)))\left(1-S_i^*(A_i(z))\right)\\+\delta\left(1-G_i^*(b(z))\right)\end{array}\right)\end{array}\right\}$$

$$Dr(z)=[1-X(z)]\left(zA_i(z)-\left[\alpha+(1-\alpha)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\right]\delta G_i^*[b(z)]\Theta_{i-1}\left(B_{i-1}^*[A_{i-1}(z)]\right)\left(1-S_i^*(A_i(z))\right)\right)$$

where$P_0$is given in Eq. (36).

Proof. The statement is obtained by using

$$K(z)=P_0+P(z)+z\sum_{i=1}^{k}\left(\Pi_i(z)+R_i(z)\right)$$

and

$$H(z)=P_0+P(z)+\sum_{i=1}^{k}\left(\Pi_i(z)+R_i(z)\right).$$

4. Performance measures

Theorem 4.1. If $\rho < 1$is satisfied, then the following probabilities of the server state, that is the server is idle during the retrial, busy during i^{th}stage, frequency of customer loss and under repair on i^{th}stage respectively are obtained.

$$P=P_0(1-\alpha)\left(1-R^*(\lambda)\right)\left(\frac{E(X)\left[\lambda\left(\frac{1-S_i^*(\delta)}{\delta}\right)(1+\delta g^{(1)})+1\right]+\Sigma_1+B_1-1}{1-\rho}\right)$$

$$\Pi_i=\sum_{i=1}^{k}\frac{\lambda P_0\Theta_{i-1}\left(B_{i-1}^*[A_{i-1}(z)]\right)\left(1-S_i^*(A_i(z))\right)E(X)\left(1-(1-\alpha)\left(1-R^*(\lambda)\right)\right)}{\delta(1-\rho)}$$

$$F_{loss}=\delta\Pi_i=\sum_{i=1}^{k}\frac{\lambda P_0\Theta_{i-1}\left(B_{i-1}^{*}\left[A_{i-1}(z)\right]\right)\left(1-S_i^{*}(A_i(z))\right)E(X)\left(1-(1-\alpha)\left(1-R^{*}(\lambda)\right)\right)}{(1-\rho)}$$

$$R_i=\sum_{i=1}^{k}\frac{\lambda P_0 g^{(1)}\Theta_{i-1}\left(B_{i-1}^{*}\left[A_{i-1}(z)\right]\right)\left(1-S_i^{*}(A_i(z))\right)E(X)\left(1-(1-\alpha)\left(1-R^{*}(\lambda)\right)\right)}{(1-\rho)}$$

Proof.The stated formula follows by using

$$P=\lim_{z\to 1}P(z),\quad \sum_{i=1}^{k}\Pi_i=\lim_{z\to 1}\sum_{i=1}^{k}\Pi_i(z)\ \text{ and } \sum_{i=1}^{k}R_i=\lim_{z\to 1}\sum_{i=1}^{k}R_i(z).$$

Theorem 4.2. Let L_s, L_q, W_s and W_q be the average system size, average orbit size, average waiting time in the system and in the orbit respectively, then under $\rho < 1$,

$$L_q=\frac{P_0}{V^{*}(\lambda)}\left[\frac{Nr_q'''(1)Dr_q''(1)-Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

where

$$Nr_q''(1)=-2\delta\left\{\frac{E(X)}{\delta}\left(1-(1-\alpha)\left(1-R^{*}(\lambda)\right)\right)\left(\delta-\lambda-c^{1}\right)+\lambda\left(E(X)\right)^2\left(1-(1-\alpha)\left(1-R^{*}(\lambda)\right)\right)\sum_{i=1}^{k}\left(\frac{1-S_i^{*}(\delta)}{\delta}\right)(1+\delta g^{(1)})\right\}$$

$$Nr_q'''(1)=-3\delta\left(\begin{array}{l}\left(1-(1-\alpha)\left(1-R^{*}(\lambda)\right)\right)\left[\begin{array}{l}E(X(X-1))\left((1-\Sigma_1-B_1)+\dfrac{\lambda S_i^{*}(\delta)}{\delta}(1+E(X))-\lambda E(X)g^{(1)}(1-S_i^{*}(\delta))\right)\\+\lambda(E(X))^2\left(-\lambda E(X)g^{(2)}(1-S_i^{*}(\delta))+\left(\dfrac{-2+\omega-E(X(X-1))}{\delta}-2g^{(1)}L\right)-E(X)(\Sigma_2+B_2)\right)\end{array}\right]\\-\lambda(E(X))^2\left[1-(1-\alpha)\left(1-R^{*}(\lambda)\right)\right]\left(\begin{array}{l}\displaystyle\sum_{i=1}^{k}(1-S_i^{*}(\delta))\left[\begin{array}{l}\lambda\delta(E(X)g^{(2)}\\+E(X(X-1))g^{(1)})-\dfrac{E(X(X-1))}{E(X)}\end{array}\right]\\+\displaystyle\sum_{i=1}^{k}B_1(1+\delta g^{(1)})\end{array}\right)+E(X)E(X(X-1))\frac{\lambda}{\delta}\sum_{i=1}^{k}B_1(1+\delta g^{(1)})\end{array}\right)$$

$$Dr_q''(1)=-2\delta E(X)\left(1-\rho\right)$$

$$Dr_q'''(1)=3\delta\left(\begin{array}{l}(1-\alpha)E(X)\left(1-R^{*}(\lambda)\right)\left[\Sigma_1+B_1+\lambda(E(X))^2k_1+E(X(X-1))(1+E(X))\right]+\lambda E(X)E(X(X-1))\left(\dfrac{2-S_i^{*}(\delta)}{\delta}+k_1\right)\\+\lambda^2(E(X))^3g^{(2)}(1-S_i^{*}(\delta))+\lambda(E(X))^2\left(\dfrac{1-S_i^{*}(\delta)}{\delta}+g^{(1)}B_1\right)+E(X)(\Sigma_2+B_2)+E(X(X-1))(\Sigma_1+B_1-1)\end{array}\right)$$

$$k_1=\left(-\lambda E(X)g^{(2)}(1-S_i^{*}(\delta))+\left(\frac{-2+\Sigma_1-E(X(X-1))}{\delta}-2g^{(1)}B_1\right)-E(X)(\Sigma_2+B_2)\right)$$

$$c'=\delta\left\{\lambda E(X)(1-S_i^{*}(\delta))g^{(1)}+\Sigma_1+B_1\right\}$$

$$B_2 = \begin{pmatrix} \sum_{i=1}^{k}\Theta_{i-1}(-M_{2i-1}\lambda E(X)(1-S_i^*(\delta)) - M_{1i-1}\lambda E(X(X-1))(1-S_i^*(\delta)) \\ +2(\lambda E(X))^2 E(S_i)M_{1i} - (\lambda E(X))^2 E(S_i^2)B_{i-1}^*(\delta) - \lambda E(X(X-1))E(S_i) \end{pmatrix}$$

and

$$\Sigma_2 = \sum_{i=1}^{k}\Theta_{i-1}M_{2i} + 2\sum_{i=1}^{k} p_i\Theta_{i-1}M_{1i} - \sum_{i=1}^{k-1}\Theta_i M_{2i}$$

$$L_s = \frac{P_0}{V^*(\lambda)}\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

where

$$Nr_s'''(1) = Nr_q'''(1) - 6\sum_{i=1}^{k}\Theta_{i-1}\lambda E(X)^2\left(1-(1-\alpha)\left(1-R^*(\lambda)\right)\sum_{i=1}^{k}(1-S_i^*(\delta))(1+\delta g^{(1)})\right)$$

$$W_s = \frac{L_s}{\lambda E(X)} \text{ and } W_q = \frac{L_q}{\lambda E(X)}$$

Proof.The statement is obtained by using

$$L_q = \frac{Nr(z)}{Dr(z)} = \lim_{z\to 1}\frac{d}{dz}H(z) = H'(1) = \frac{P_0}{V^*(\lambda)}\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

and

$$L_s = \frac{Nr(z)}{Dr(z)} = \lim_{z\to 1}\frac{d}{dz}K(z) = K'(1) = \frac{P_0}{V^*(\lambda)}\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

$W_s$andW_q under steady- state condition due to Little's formula is, $L_s = \lambda W_s$ and $L_q = \lambda W_q$.

4.1 Special cases

Case (i):*Single phase, No retrial, No balking and reneging and No breakdown*

Let $Pr\ [X = 1] = 1$, $R^*(\lambda) \to 1$,$Pr[V = 0] = 1$, $b = 1$,$r = 1$and $\alpha_1 = \alpha_2 = 0$. Our model can be reduced to Multi stage M/G/1 feedback queueing system. The following results agree with Salehiradet al. [16].

$$K(z) = P_0\left\{\frac{\left(1-S_1^*(A_1(z))\right) + \sum_{i=2}^{k}\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}{z - \sum_{i=1}^{k}\left\{\left(p_i z + q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}}\right\}$$

Case (ii):*Single phase, No feedback, No balking & reneging and No breakdown*

Let $Pr\ [X = 1] = 1$, $k=1$, $Pr[S_k = 0] = 1$, $\theta_1 = 0$, $b = 1$, $r = 1$ and $\alpha_1 = \alpha_2 = 0$, the model reduced to M/G/1 retrial queue.

$$K(z)=\left\{\frac{\left[R^*(\lambda)-\lambda E(S_0)\right]S_0^*\left[\lambda-\lambda z\right]\left[z-1\right]}{z-\left[R^*(\lambda)+z(1-R^*(\lambda))\right]\left\{S_0^*\left[\lambda-\lambda z\right]\right\}}\right\};\ L_q=\frac{\left\{\lambda^2E(S_0{}^2)+2\lambda E(S_0)\left(1-R^*(\lambda)\right)\right\}}{2\left\{R^*(\lambda)-\lambda E(S_0)\right\}}$$

The above results agree with Gomez-Corral [11].

Case (iii):*Single phase, No balkig& reneging and No breakdown*
Let $Pr\ [X = 1] = 1,\ b = 1,\ r = 1$ and $\alpha_1 = \alpha_2 = 0$. The model reduced to Multi stage retrial queueing system with Bernoulli feedback. The following results agree with Bagyam et al.[3].

$$K(z)=P_0R^*(\lambda)\left\{\frac{\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}+z\left\{\sum_{i=1}^{k}\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)-1\right\}}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}}\right\}$$

where

$$P_0=\left\{\frac{R^*(\lambda)-\sum_{i=1}^{k}\Theta_{i-1}M_{1i}-\sum_{i=1}^{k}p_i\Theta_{i-1}+\sum_{i=1}^{k-1}\Theta_{i-1}M_{1i}}{R^*(\lambda)\left\{1-\sum_{i=1}^{k}\Theta_{i-1}\lambda E(X)E(S_i)\left(1-\alpha_iE(G_i)\right)-\sum_{i=1}^{k}\Theta_{i-1}(p_i+M_{1i})+\sum_{i=1}^{k-1}\Theta_{i-1}M_{1i}\right\}}\right\}$$

5. Numerical illustration

Here, some numerical examples are given using MATLAB. The retrial times, service times and repair times are exponentially $f(x)=ve^{-vx}, x>0$ for Erlang-2stage $f(x)=v^2xe^{-vx}, x>0$ and hyper-exponentially $f(x)=cve^{-vx}+(1-c)v^2e^{-v^2x}, x>0$ distributed. And assume the arbitrary values to the parameters satisfies$\rho<$ 1. The following tables indicate the computed values of P_0 ,P, Π_i, F_{loss} and R_i for (i=1,2,…k) respectively. For the effect of a, p, γ and ξ_i are retrial rate, feedback probability, customer loss and repair rate on FSS respectively graphs are given in Figure 1 to 6.
Table 1 indicates when a increases, then P_0 increases, L_q and P decreases for λ= 0.2; p_1= 0.2; μ_1 = 5;α_1 = 0.2; ξ_1 = 3; c =0.8; k =1. Table 2 indicates when (p_1) increases, P_0 decreases, L_q and P increases for λ= δ=2 ; a=5; μ_1 = 10; μ_2 = 8; θ_1=0.2; θ_2=0.4; α_1 = 0.2; ξ_1 = 5; ξ_2 = 3; k=2; c =0.8. Table 3 indicates that when δ increases, then the probability that server is idle P_0, L_q and F_f increases for λ= 0.5; a=5; μ_1 = 10; p_1= 0.4; α_1 = 0.1; ξ_1 =5; c= 0.8; k=1.

Figure 1-4 are given for a, p_1, γ and ξ_1.Figure 1 indicates P_0 increases for increasing a.
Figure 2 indicates L_q increasing for increasing p_1. Figure 3 indicates L_q decreases for increasing γ. Figure 4 indicates P_0 increases for increasing ξ_1. Three dimensional Figure 5 indicatesp_1 and $\xi_1$increases against $L_q$increases.P_0increasing against increasing a and γ is shown in Figure 6.

Table 1. The effect of (a) on P_0, Lq and P

Orbital search probability	Exponential			Erlang – 2 stage			Hyper – Exponential		
α_1	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
0.20	0.9109	0.0064	0.0011	0.8272	0.0148	0.0055	0.9053	0.0132	0.0033
0.30	0.9200	0.0063	0.0010	0.8446	0.0145	0.0048	0.9132	0.0129	0.0029
0.40	0.9291	0.0062	0.0009	0.8620	0.0141	0.0042	0.9211	0.0127	0.0025
0.50	0.9381	0.0062	0.0007	0.8793	0.0137	0.0035	0.9291	0.0124	0.0021
0.60	0.9472	0.0061	0.0006	0.8967	0.0133	0.0028	0.9370	0.0121	0.0017

Table 2. The effect of p_1 on P_0, Lq and P

Feedback probability	Exponential			Erlang – 2 stage			Hyper – Exponential		
p_1	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
0.10	0.9007	0.0059	0.0003	0.8076	0.0116	0.0034	0.8988	0.0103	0.0018
0.20	0.9004	0.0067	0.0010	0.8062	0.0145	0.0059	0.8982	0.0113	0.0026
0.30	0.9001	0.0076	0.0017	0.8048	0.0174	0.0084	0.8975	0.0123	0.0035
0.40	0.8997	0.0084	0.0024	0.8033	0.0206	0.0111	0.8968	0.0134	0.0045
0.50	0.8994	0.0093	0.0030	0.8017	0.0238	0.0138	0.8962	0.0145	0.0054

Table 3. The effect of δ on P_0, L_qandF_f

Negative arrival rate	Exponential			Erlang – 2 stage			Hyper – Exponential		
δ	P_0	L_q	F_f	P_0	L_q	F_f	P_0	L_q	F_f
5.00	0.8909	0.0177	0.0984	0.7876	0.0481	0.1776	0.8927	0.0240	0.1434
6.00	0.8914	0.0193	0.1156	0.7891	0.0533	0.2081	0.8941	0.0253	0.1649
7.00	0.8920	0.0206	0.1320	0.7905	0.0578	0.2368	0.8955	0.0263	0.1846
8.00	0.8925	0.0217	0.1477	0.7918	0.0618	0.2637	0.8969	0.0271	0.2027
9.00	0.8931	0.0227	0.1626	0.7930	0.0655	0.2890	0.8981	0.0278	0.2195

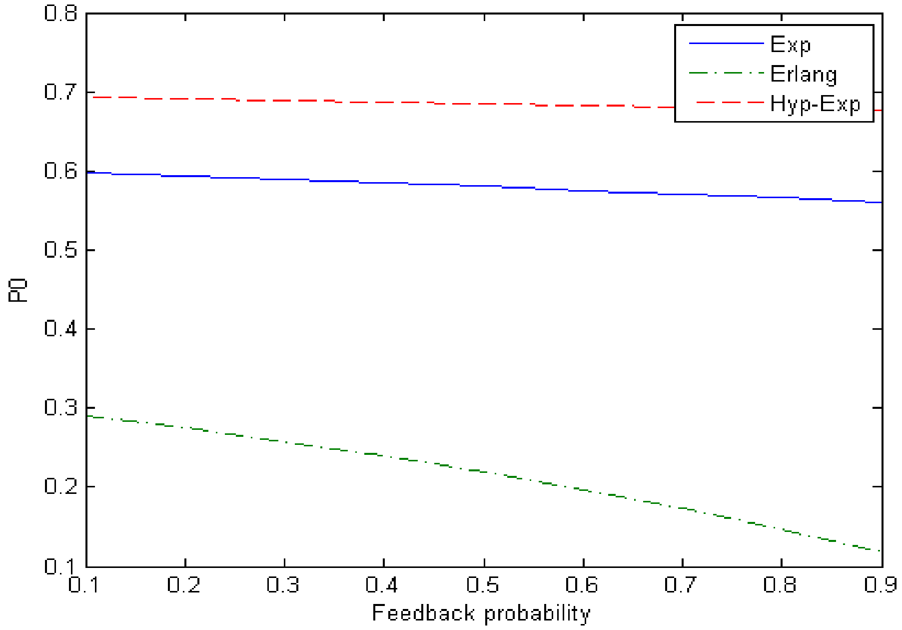

Figure 1. P_0 versus p_1

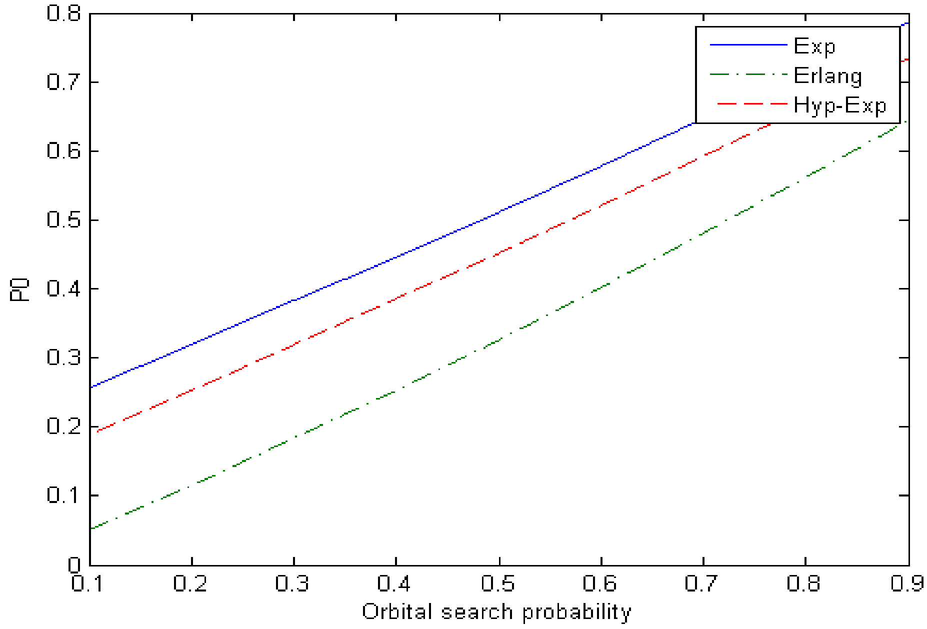

Figure 2. P_0 versus α

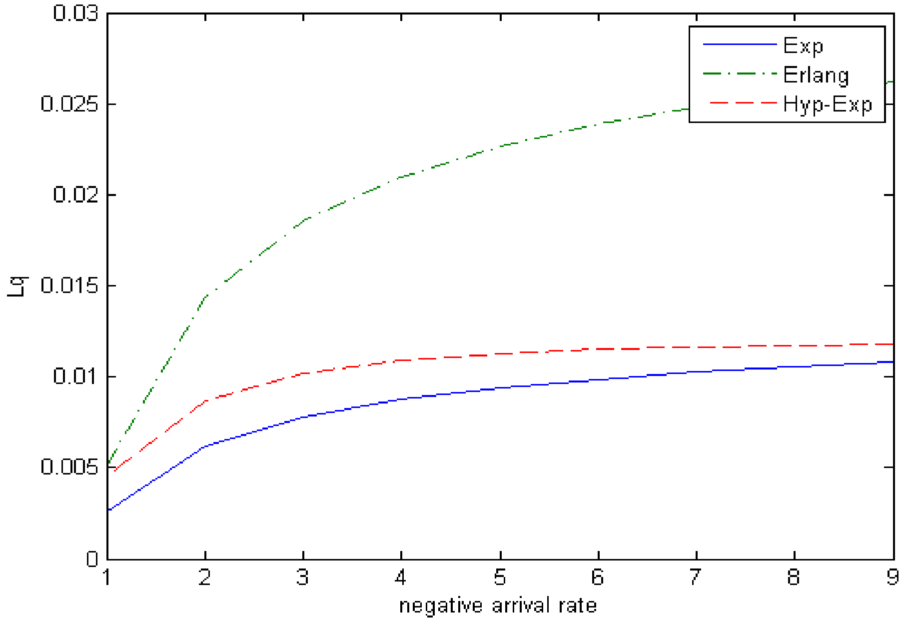

Figure 3. $L_q$versus δ

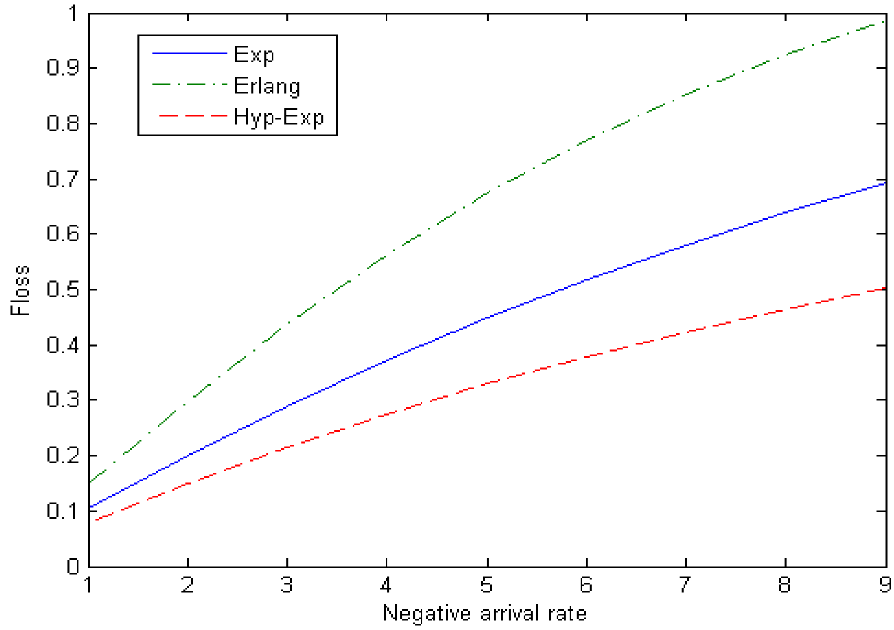

Figure 4. F_{loss}versus δ

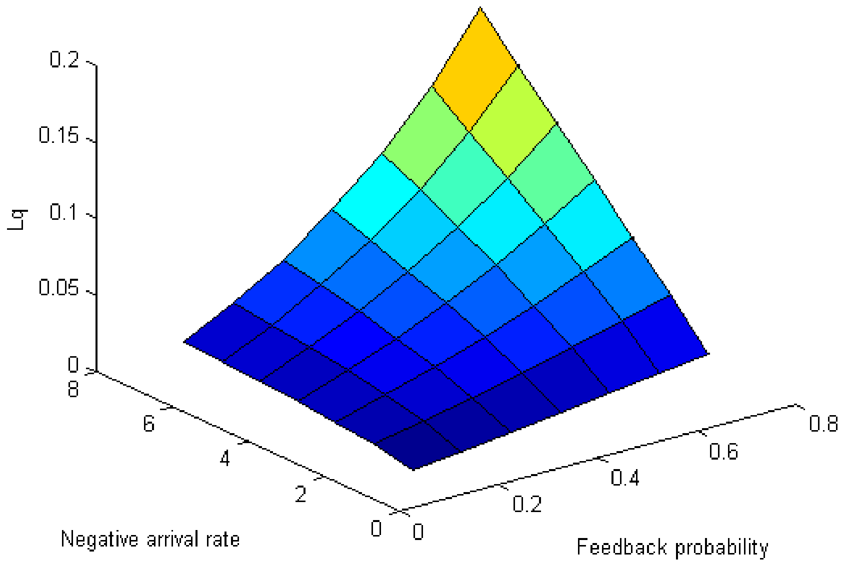

Figure 5. L_q versus p_1 and δ

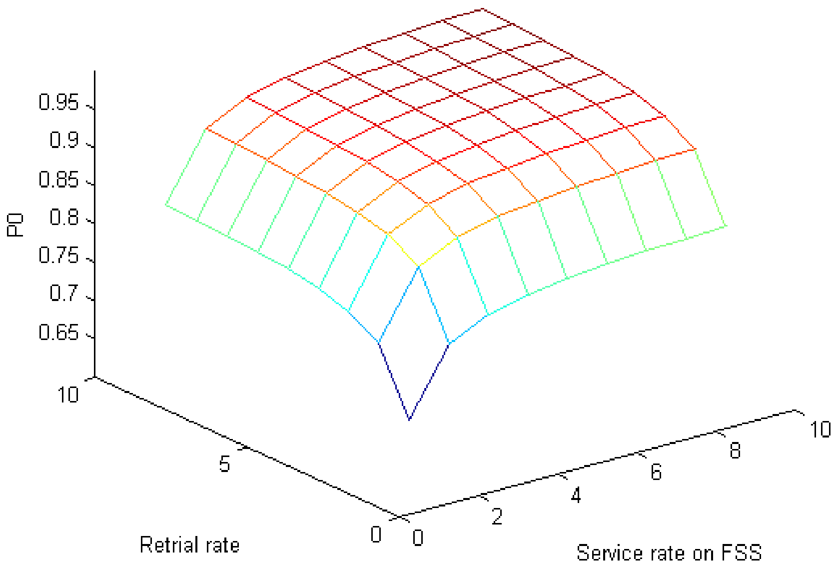

Figure 6. P_0 versus a and μ_1

6. Conclusion

A group arrival feedback retrial queue with k optional stages of service and orbital search policy are meticulously studied. The PGF of the numbers in the system and orbit are found.L_s, L_q, W_s and $W_q$are obtained. The mathematical results are validated by simulation results.

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