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A Hesitant Fuzzy based Security Approach for Fog and Mobile-Edge Computing

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***Abstract*— Fog and Mobile-Edge Computing (FMEC) is a sustainable and innovative mobile networking framework that enables the offloading of cloud services and resources at the edge of mobile cellular networks to provide high bandwidth and ultra-low latency. Nonetheless, how to handle several dynamically varying security services with the mobile user's requirements efficiently is a critical problem that hinders the development of FMEC. To address this problem, we sought to introduce an approach to selecting an appropriate security service as per the mobile user requirements in FMEC. The problem of appropriate security service selection with hesitant fuzzy information is a multi-criteria decision making problem. In this paper, we introduce a Soft Hesitant Fuzzy Rough Set (SHFRS) to solve multi-criteria decision making problems. SHFRS is introduced as an innovative extension of the hesitant fuzzy rough set theory by fusing it with the hesitant fuzzy soft set. We describe the inverse hesitant fuzzy soft set that defines the inverse hesitant fuzzy relation to determine the SHFRS upper and lower approximation operators of any hesitant fuzzy subset in the given set of parameters. We also present different special cases of SHFRS upper and lower approximation operators and discuss some fundamental theorems based on approximation operators. In addition, we propose a novel solution to multi-criteria decision making problems based on SHFRS. Finally, we assess the proposed solution by applying it to a real-time multi-criteria decision making problem of appropriate security service selection for FMEC in the existence of multi-observer hesitant fuzzy information.**

***Index Terms*— Fog and mobile-edge computing, hesitant fuzzy set, hesitant fuzzy soft set, rough set, decision making.**

I. INTRODUCTION

With the increasing number of mobile terminals, an explosive growth in global mobile traffic has been observed. According to a report from Cisco, a 74% growth in mobile traffic was recorded in 2015; this is expected to increase further by approximately eight times from 2016 to 2020 [1]. Nevertheless, the greater demands of mobile network services due to the increasing amount of mobile traffic cannot be accommodated by the conventional infrastructure of mobile networking due to lack of energy efficiency. To overcome this problem, the concept of Fog and Mobile-Edge Computing (FMEC) has been introduced as a sustainable and innovative mobile networking framework [2]. FMEC puts cloud computing capabilities such as resources and services within the access network, i.e., near the users of mobile. Thanks to its proximity to mobile users, it provides direct access to the available resources and services with high bandwidth and ultra-low latency. Moreover, FMEC increases the response of services, content, and application from the network edge and enriches the experience of the mobile user.

With the increasing expansion of mobile network such as FMEC, the need for security of mobile traffic is becoming critical. Note, however, that few research works focusing on real-time and dynamic varying security services to identify the different security needs in FMEC have been done. Kanghyo Lee et al. [3] discussed various privacy and security issues in FMEC in the context of a cloud-based Internet of Things (IoT) environment. The discussion provides the categorization of various security technologies to secure various network components such as IoT node, Fog node (i.e., FMEC server), and communication between fog nodes. Ivan Stojmenovic et al. [4] studied the various security challenges of FMEC in the context of other technologies such as wireless sensor network, smart traffic, smart grid, and so on. Similarly, other researches such as [5] discussed the security needs and challenges in mobile-edge computing. These researches mainly provide the privacy and security implications of FMEC and do not offer adequate solutions to alleviate all the security issues and challenges specifically while considering the collaboration of mobile-edge computing with other technologies, such as software-defined networking.

Therefore, our research sought to introduce an innovative approach to security service selection to select optimal security services as per the mobile user requirements in the FMEC environment. In the security service selection approach, the selection of optimal security service among several available security services with overlapping functionalities is carried out based on various Quality of Service (QoS) parameters such as processing delay and CPU usage. This is because optimal security service selection is a decision making problem that depends on multiple QoS criteria to satisfy the mobile user requirements. Therefore, to choose the optimal security service, it comes up with multi-criteria decision making based on Fuzzy logic that ranks the security services according to their functionalities based on multiple QoS criteria. Recently, a Fuzzy based security service chaining approach to find the optimal order of the necessary security services in FMEC. The approach established a Fuzzy Inference System (FIS) based scheme to obtain the goal of multi-criteria decision making [5]. However, in fuzzy-based multi-criteria decision making problems, it is challenging for decision making experts to come up with a final decision because there are always uncertainties in their choice of objects. We can consider a case wherein the degree of membership for a given object in a set is defined by two experts. The degree of membership defined by the first expert is 0.5, but the other expert defines it as 0.7. Here, the trouble in defining a collective degree of membership for the object does not arise due to some possibility distribution values [6] or a margin of error [7], [8]. Rather, it occurs due to the hesitation of experts among a set of possible values. Recently, Vicenc Torra et al. [9], [10] proposed the idea of a hesitant fuzzy set in order to handle the issue of hesitation. A Hesitant Fuzzy Set (HFS) is an extension of a fuzzy set that supports assigning the degree of membership to an object in a set as multiple values between 0 and 1. Hesitant information can be expressed more comprehensively by using HFS instead of other forms of fuzzy sets. To tackle the problem of hesitant information, many researchers [11], [12], [13] have introduced the idea and its application in decision making (for example, in multi-criteria decision making problems [13], where an optimal alternative is evaluated from several available alternatives according to multiple criteria).

In a different vein, D. Molodtsov et al. [14] introduced a new mathematical model called a soft set, which handles uncertainties that are open to the inadequacies of parameterization tools. It has applications in several different areas, including the probabilistic model, Perron integration, Riemann integration, operations research, game theory, smoothness of functions, and forecasting method [15], [16]. Pabitra Kumar Maji et al. [16] first used the concept of a soft set to solve the problem of decision making. In the past few years, many researchers have established a new extension of a soft set called fuzzy soft set by applying the theory of soft set in the fuzzy environment. Pabitra Kumar Maji et al. [17] first introduced the fuzzy soft set theory by merging the fuzzy concept with the soft set theory. Naim Cagman et al. [18] described the aggregation operator for a fuzzy soft set. Irfan Deli. et al. [19] defined a intuitionistic fuzzy parameterized soft sets and provided the application of this set to the decision making problem. Akhil Ranjan Roy et al. [20] and Feng Feng et al. [21] also applied a fuzzy soft set to the problems of decision making. José Carlos R. Alcantud et al. [22] solved the decision-making problem in the existence of sets of input parameters from multi-observers using a novel method of fuzzy soft set. The research of Fuqiang Wang et al. [23] presented a hesitant fuzzy soft set theory by applying a soft set concept in the hesitant fuzzy environment. They also defined intersection, union, "OR," "AND," and complement operations in this set. Hai-dong Zhang et al. [24] combined the soft set theory and dual hesitant fuzzy set theory and defined the dual hesitant fuzzy soft set.

To handle uncertainty, vagueness, and imprecision in data analysis, Zdzisław Pawlak et al. [25], [26] first introduced a new mathematical model that works on the key notion of an equivalence relation and is called a rough set. Nevertheless, in numerous real-world problems, the frequent utilization of the equivalence relation is too restrictive. Hence, many researchers have used non-equivalence relations in rough sets and expanded rough set models, which have been used in numerous areas (for example, in expert systems, intelligent decision-making systems, and machine learning). Many researchers [27], [28], [29] applied the notion of a rough set in the fuzzy environment and developed several extended rough set models. Moreover, the theory of rough set has also been combined with the theory of interval-valued fuzzy set, intuitionistic fuzzy set, and hesitant fuzzy set and several new rough set models have been developed.

Recently, Xibei Yang et al. [30] combined the rough set theory with the hesitant fuzzy set and proposed a hesitant fuzzy rough set. They also defined various monotonic properties for their proposed set. Haidong Zhang et al. [31] and Chao Zhang et al. [32] recommended different types of extensions for the hesitant fuzzy rough set and applied them to various problems of decision making, such as medical diagnosis and the fault diagnosis of steam turbines.

In this paper, we propose a novel extension of the hesitant fuzzy rough set theory [30] by fusing it with the hesitant fuzzy soft set -- called the Soft Hesitant Fuzzy Rough Set (SHFRS) -- to support multi-criteria decision making for security service selection in the FMEC environment. To define SHFRS, we first describe the inverse hesitant fuzzy soft set by inversing the mapping (from the set of parameters Q to the universe of objects X) defined in the hesitant fuzzy soft set. Actually, a hesitant fuzzy soft set over universe set X is described as a mapping from parameters (Q) to the set of all hesitant fuzzy sets in X. On the other hand, an inverse hesitant fuzzy soft set is defined as a mapping from the universe of objects X to the set of all hesitant fuzzy sets in Q. According to the definitions of both sets (hesitant fuzzy soft set and inverse hesitant fuzzy soft set), we can easily determine that they define a hesitant fuzzy relation between the set of parameters (Q) and set of objects (X). Subsequently, it is already known that the traditional hesitant fuzzy rough set describes the hesitant fuzzy relation of universe X [30]. Therefore, we can use the hesitant fuzzy relation described by the inverse hesitant fuzzy soft set in the traditional hesitant fuzzy rough set and define a new SHFRS with upper and lower approximations of any hesitant fuzzy set $\mathbb{K} \in HFS(Q)$ with respect to triple $(X, Q, \widetilde{\mathcal{D}}^{-1})$, where $HFS(Q)$ represents a set of all hesitant fuzzy sets in Q, and $\widetilde{\mathcal{D}}^{-1}$ is a hesitant fuzzy relation defined by an inverse hesitant fuzzy soft set. Similarly, the traditional hesitant fuzzy rough set, SHFRS, can be used for solving decision making problems. As such, we also introduce a novel decision making method based on the SHFRS theory.

The rest of this paper is organized as follows: Section II describes the hesitant fuzzy set theory and its fusion with the soft set and rough set theories; Section III introduces SHFRS and discusses some of its fundamental properties & theorems in detail; Section IV presents a novel solution for the multi-criteria decision making problem based on our proposed SHFRS theory, gives the stepwise procedure of the provided solution, and presents a security service selection approach for FMEC; Finally, Section V presents our conclusion.

II. PRELIMINARIES

In this Section, we briefly discuss the hesitant fuzzy set theory and its properties. The fundamental notion of soft and rough sets is also described, and some existing concepts related to the fusion of the hesitant fuzzy set theory with the soft set and rough set theories are defined.

A. Hesitant fuzzy set

The research of Vicenc Torra et al. [9], [10] first presented the idea of the hesitant fuzzy set, which is defined below.

Definition 1 (See [9], [10]): For a given universe of discourse X, hesitant fuzzy set $\mathcal{F}$ on X is defined in the form of function $h_{\mathcal{F}}(v)$ that takes input from X and returns a set of values between 0 and 1. The mathematical definition of hesitant fuzzy set is provided below.

$$\mathcal{F} = \{\langle v,\ h_{\mathcal{F}}(v)\rangle | v \in X\}$$

Where $h_{\mathcal{F}}(v)$ is a hesitant fuzzy element (HFE) containing a set of values between 0 and 1, which indicates the probable degrees of membership for element $v \in X$ to $\mathcal{F}$. We can describe all possible hesitant fuzzy sets in X by using a set represented by $HFS(X)$. Moreover, Vicenc Torra et al. [10] described the null and full hesitant fuzzy sets as explained below.

Definition 2 (See [6]): Hesitant fuzzy set $\mathcal{F}$ is defined as a null hesitant fuzzy set if $\mathcal{F}(v) = \{0\}$ for every v in X. This type of set is denoted by $\emptyset$. In contrast, $\mathcal{F}$ is defined as a full hesitant fuzzy set, if $h_{\mathcal{F}}(v) = \{1\}$ for every v in X. This type of set is denoted by L.

Vicenc Torra et al. [9], [10] introduced the basic operations below to deal with HFEs. Let three HFEs be $\hbar$, $\hbar_1$, and $\hbar_2$. Then:

(1) $\sim\hbar = \cup_{\beta\in\hbar} \{1-\beta\}$,
(2) $\hbar_1 \underline{\vee} \hbar_2 = \cup_{\beta_1\in\hbar_1,\, \beta_2\in\hbar_2} \max\{\beta_1,\ \beta_2\}$,
(3) $\hbar_1 \overline{\wedge} \hbar_2 = \cup_{\beta_1\in\hbar_1,\, \beta_2\in\hbar_2} \min\{\beta_1,\ \beta_2\}$.

Here, operations $\underline{\vee}$, $\underline{\wedge}$, and $\sim$ are referred to as the Supremum, Infimum, and complement operations on HFEs, respectively.

Meimei Xia et al. [33] introduced a score function in order to carry out the comparison of HFEs under the following assumption:
(a) The values in all the HFEs selected for comparison should be in increasing order.
(b) The length of all HFEs for comparison should be the same. Thus, if the length of any two HFEs is different, the HFE with shorter length is expanded with the addition of maximum values until the lengths of two HFEs are equal.

Definition 3 (See [33]): If $\mathcal{E}$ is a given HFE, then the score of $\mathcal{E}$ is computed using the following function:

$$scr(\mathcal{E}) = (1/l(\mathcal{E}))\sum_{\beta\in\mathcal{E}}\beta$$

Where $l(\mathcal{E})$ is the number of values in $\mathcal{E}$
For two HFEs $\mathcal{E}_1$ and $\mathcal{E}_2$

$$if\ scr(\mathcal{E}_1) > scr(\mathcal{E}_2), then\ \mathcal{E}_1 > \mathcal{E}_2,$$
$$if\ scr(\mathcal{E}_1) = scr(\mathcal{E}_2), then\ \mathcal{E}_1 = \mathcal{E}_2.$$

B. Soft set and its fusion with a hesitant fuzzy set

In this subsection, we discuss the fundamental theory of a soft set and its fusion with a hesitant fuzzy set. Let us assume two sets X and Q, where X describes the set of objects in the universe and Q denotes the set of parameters. The power set of X is denoted as $\mathcal{P}(X)$. According to D. Molodtsov et al. [14], a soft set over X is defined as follows:

Definition 4 (See [14]): A soft set over X is defined as pair $(\mathcal{D}, \mathcal{K})$, where $\mathcal{K} \subseteq Q$ and $\mathcal{D}$ describes the mapping from $\mathcal{K}$ to X, which is given by $\mathcal{D} : \mathcal{K} \to \mathcal{P}(X)$.

In other form, a soft set $(\mathcal{D}, \mathcal{K})$ over universe X is defined as a parametric group of subsets of universe X. For each $q \in \mathcal{K}$, $\mathcal{D}(q)$ is examined as a subset of X approximated by q or the collection of q-approximated elements of the soft set $(\mathcal{D}, \mathcal{K})$. We present a real-time example of soft set as follows:

Example 1: Let $X = \{v_1, v_2, v_3, v_4\}$ be a universe of computers, and $\mathcal{K}$ holds the parameters $\{q_1, q_2, q_3\}$ that define the characteristics of computers. Parameter q_1 stands for "high speed," q_2 describes "high storage," and q_3 stands for "low power consumption." Therefore, $\mathcal{D}(q_1) = \{v_1, v_2, v_3\}$ means that computers with high speed are v_1, v_2, v_3. Similarly, $\mathcal{D}(q_2) = \{v_2, v_4\}$ and $\mathcal{D}(q_3) = \{v_1, v_3, v_4\}$ define the computers with high storage and low power consumption as v_2, v_4 and v_1, v_3, v_4, respectively.

Many researchers have studied the soft set and its related concept. Pabitra Kumar Maji et al. [34] defined various operations on soft set (for example, union, intersection) and the concept of soft supersets and subsets. Sani Danjuma et al. [35] recommended normal parameter reduction algorithm for soft set. Moreover, Feng Feng et al. [36] investigated several forms of soft subsets and explored the relationship among them. Qinrong Feng et al. [37], Naim Çağman et al. [38], and Pabitra Kumar Maji et al. [16] described various methods of solving decision making problems using a soft set. Recently, Fuqiang Wang et al. [23] described the combination of the traditional soft set with a hesitant fuzzy set. The resultant set is called a hesitant fuzzy soft set and is defined using the definition below.

Definition 5 (See [23]): A hesitant fuzzy soft set over X is defined as pair $(\widetilde{\mathcal{D}}, \mathcal{K})$, where $\mathcal{K} \subseteq Q$, and $\widetilde{\mathcal{D}}$ describes the

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mapping from $\mathcal{K}$ to X, which is given by $\widetilde{\mathcal{D}}:\mathcal{K}\rightarrow HFS(X)$. Here, HFS(X) represents the set of all hesitant fuzzy sets in X.

Ideally, a hesitant fuzzy soft set on X is defined as a mapping from parameters to the set of all hesitant sets in X. In other words, it is a parametric group of hesitant fuzzy subsets of X. For each $q\in\mathcal{K}$, $\widetilde{\mathcal{D}}(q)$ is examined as a hesitant fuzzy subset of X, which is approximated by q. In general, $\widetilde{\mathcal{D}}(q)=\{(v,\widetilde{\mathcal{D}}(q)(v))|v\in X\}$.

As already defined in [14], [23], when both universal set X and parameter set $\mathcal{K}\subseteq Q$ are finite, then soft sets and hesitant fuzzy soft sets can be represented in table format, where each row describes an object in X and each column describes a parameter in $\mathcal{K}$. Each cell of the table that represents a soft set contains either 0 or 1, whereas each cell of the table that represents a hesitant fuzzy soft set contains a set of values between 0 and 1, indicating the probable degrees of membership for an object.

Fuqiang Wang et al. [23] described the AND ($\wedge$) operation on a hesitant fuzzy soft set as follows:

Definition 6 (See [23]): Consider $(\tilde{\mathcal{A}},\mathcal{K})$ and $(\widetilde{\mathcal{B}},\mathcal{L})$ as two hesitant fuzzy soft set; then the AND operation of these two sets is represented as $(\tilde{\mathcal{A}},\mathcal{K})\wedge(\widetilde{\mathcal{B}},\mathcal{L})$ and can be defined using the following mathematical expression:

$$(\tilde{\mathcal{A}},\mathcal{K})\wedge(\widetilde{\mathcal{B}},\mathcal{L})=(\tilde{\mathcal{R}},\mathcal{K}\times\mathcal{L})$$

Where $\tilde{\mathcal{R}}(a,b)=\tilde{\mathcal{A}}(a)\cap\widetilde{\mathcal{B}}(b)$, for all $(a,b)\in\mathcal{K}\times\mathcal{L}$.

C. Rough set and its fusion with hesitant fuzzy set

In this subsection, we discuss the fundamental theory of a rough set and its fusion with a hesitant fuzzy set. Let us assume two sets X and Q, where X describes the set of objects in the universe and Q denotes the set of parameters. The power set of X is denoted as $\mathcal{P}(X)$. According to Y.Y. Yao et al. [39] and Wei-Zhi Wu et al. [40], a rough set over X is defined as stated below.

Definition 7 (See [39], [40]): If $\mathcal{G}$ is an arbitrary crisp relation from X to Q, then, $\mathcal{G}_s:X\rightarrow\mathcal{P}(Q)$ is a set-valued function and is described using the following mathematical expression:

$$\mathcal{G}_s(v)=\{w\in Q|(v,w)\in\mathcal{G}\},\quad v\in X$$

$\mathcal{G}_s(v)$ is identified as a successor neighborhood of v for crisp relation $\mathcal{G}$. The triple $(X,Q,\mathcal{G})$ is called a generalized crisp approximation space. Upper approximation $\overline{\mathcal{G}}(\mathcal{K})$ and lower approximation $\underline{\mathcal{G}}(\mathcal{K})$ of any set $\mathcal{K}\subseteq Q$ with respect to triple $(X,Q,\mathcal{G})$ are computed using the following expression:

$$\overline{\mathcal{G}}(\mathcal{K})=\{v\in X|\mathcal{G}_s(v)\cap\mathcal{K}\neq\emptyset\},$$

$$\underline{\mathcal{G}}(\mathcal{K})=\{v\in X|\mathcal{G}_s(v)\subseteq\mathcal{K}\}.$$

Finally, a generalized crisp rough set of any set $\mathcal{K}\subseteq Q$ with respect to triple $(X,Q,\mathcal{G})$ is defined as pair $(\overline{\mathcal{G}}(\mathcal{K}),\ \underline{\mathcal{G}}(\mathcal{K}))$, where $\overline{\mathcal{G}}$ and $\underline{\mathcal{G}}$ are called upper and lower generalized crisp approximation operators, respectively.

Haidong Zhang et al. [31] described the fusion of a traditional rough set with a hesitant fuzzy set. The resultant set is called a hesitant fuzzy rough set and is defined using the following notion:

Definition 8 (See [31]): Let $\mathbb{G}$ be a hesitant fuzzy relation from X to Q; then, a hesitant fuzzy approximation space over X and Q is defined as triple $(X,Q,\mathbb{G})$. Upper approximation $\overline{\mathbb{G}}(\mathbb{K})$ and lower approximation $\underline{\mathbb{G}}(\mathbb{K})$ of any set $\mathbb{K}\in HFS(Q)$ with respect to triple $(X,Q,\mathbb{G})$ are computed using the following expression:

$$\overline{\mathbb{G}}(\mathbb{K})=\{\langle v,h_{\overline{\mathbb{G}}(\mathbb{K})}(v)\rangle|v\in X\},$$
$$\underline{\mathbb{G}}(\mathbb{K})=\{\langle v,h_{\underline{\mathbb{G}}(\mathbb{K})}(v)\rangle|v\in X\},$$

Where

$$h_{\overline{\mathbb{G}}(\mathbb{K})}(v)=\underline{\vee}_{w\in Q}\{h_{\mathbb{G}}(v,w)\,\overline{\wedge}\,h_{\mathbb{K}}(w)\},\ \ v\in X$$
$$h_{\underline{\mathbb{G}}(\mathbb{K})}(v)=\overline{\wedge}_{w\in Q}\{(1-h_{\mathbb{G}}(v,w))\,\underline{\vee}\,h_{\mathbb{K}}(w)\},\quad v\in X$$

Finally, a hesitant fuzzy rough set of any set $\mathbb{K}\in HFS(Q)$ with respect to triple $(X,Q,\mathbb{G})$ is defined as pair $(\overline{\mathbb{G}}(\mathbb{K}),\underline{\mathbb{G}}(\mathbb{K}))$, where $\overline{\mathbb{G}}$ and $\underline{\mathbb{G}}$ are called upper and lower hesitant fuzzy rough approximation operators, respectively. If we assume that $X=Q$, then the resultant set is called a hesitant fuzzy rough set over the same universe as defined by Xibei Yang et al. [30].

III. SOFT HESITANT FUZZY ROUGH SET: A NOVEL PROPOSED SET

After describing the fundamental definition and concept of hesitant fuzzy soft set and hesitant fuzzy rough set, in this section, we combine the concepts of these two sets and introduce an innovative notion of hesitant fuzzy rough set based on a hesitant fuzzy soft set called SHFRS.

As we have already discussed, the soft set describes each object v_i in universe X by defining it in object set $\mathcal{D}(q)$ corresponding to any parameter $q\subseteq\mathcal{K}$. This is illustrated in example 1, where we define object set $\mathcal{D}(q_2)=\{v_2,v_4\}$ that describes computers v_2 and v_4, which hold parameter q_2 ("high storage"). Contrary to the soft set, we can consider a general problem wherein one wants to know that what are the characteristics or parameters that computer $v_i\in X$ has. To describe this problem, we first present the following notion of an inverse soft set:

Definition 9: An inverse soft set over X is defined as pair $(\mathcal{D}^{-1},\mathcal{K})$, where $\mathcal{K}\subseteq Q$ and $\mathcal{D}^{-1}$ describes the mapping from X to $\mathcal{K}$ given by $\mathcal{D}^{-1}:X\rightarrow\mathcal{P}(\mathcal{K})$. Here, $\mathcal{P}(\mathcal{K})$ denotes the power set of parameter set $\mathcal{K}$.

From the definition above, an inverse soft set over X maps each object $v\in X$ with the group of parameters $p\in\mathcal{P}(\mathcal{K})$ held by object v. To illustrate this, we present a real-time decision making problem as follows:

Example 2: Assume that a person wants to purchase a computer from a shop. Let $X=\{\mathcal{pC}_1,\mathcal{pC}_2,\mathcal{pC}_3,\mathcal{pC}_4,\mathcal{pC}_5\}$,

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which is a set of five computers with various characteristics. Suppose that the characteristics of all computers in set X is defined by a set of $\mathcal{K} = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7\}$ parameters, where $q_1, q_2, q_3, q_4, q_5, q_6, and\ q_7$ stand for high capacity, high storage, high versatility, high diligence, high accuracy, high speed, and low power consumption, respectively. For the evaluation of an optional computer under the consideration of the given parameter, we define soft set $(\mathcal{D}, \mathcal{K})$ that depicts the "efficiency of the computer" that the person wants to purchase. The defined soft set $(\mathcal{D}, \mathcal{K})$ for this example is shown in Table 1.

Contrary to the soft set $(\mathcal{D}, \mathcal{K})$, if the person wants to know what are the characteristics or parameters of an optional computer that he/she is going to purchase, then we define an inverse soft set $(\mathcal{D}^{-1}, \mathcal{K})$ that depicts the following results:

$\mathcal{D}^{-1}(pC_1) = \{q_1, q_4, q_7\}$, $\mathcal{D}^{-1}(pC_2) = \{q_1, q_4, q_5\}$, $\mathcal{D}^{-1}(pC_3) = \{q_2\}$, $\mathcal{D}^{-1}(pC_4) = \{q_3, q_4, q_6\}$, $\mathcal{D}^{-1}(pC_5) = \{q_1, q_5, q_6\}$

The expression $\mathcal{D}^{-1}(pC_1) = \{q_1, q_4, q_7\}$ denotes that computer pC_1 define with three characteristics: high capacity, high storage, and low power consumption. Similarly, other expression can be described.

As clearly shown in Example 2 and Definition 9, an inverse soft set describes another way of representing the relationship between parameter set $\mathcal{K}$ and universe X of objects. With the notion of an inverse soft set, the basic characteristics of a given object $v_i \in X$ can be described by determining each parameter $q_i \in \mathcal{K}$ that belongs to the given object.

Table 1. Representation of soft set $(\mathcal{D}, \mathcal{K})$ in table format (in Example 2)

$\mathcal{K}$ / X	q_1	q_2	q_3	q_4	q_5	q_6	q_7
pC_1	1	0	0	1	0	0	1
pC_2	1	0	0	1	1	0	0
pC_3	0	1	0	0	0	0	0
pC_4	0	0	1	1	0	1	0
pC_5	1	0	0	0	1	1	0

Similar to the inverse soft set, the inverse hesitant fuzzy soft set can be defined by applying the concept of an inverse soft set with the hesitant fuzzy set. We define an inverse hesitant fuzzy soft set as described below.

Definition 10: An inverse hesitant fuzzy soft set over X *is defined as pair* $(\widetilde{\mathcal{D}}^{-1}, \mathcal{K})$*, where* $\mathcal{K} \subseteq Q$ *and* $\widetilde{\mathcal{D}}^{-1}: X \to HFS(\mathcal{K})$*, with* $HFS(\mathcal{K})$ *as the set of all hesitant fuzzy sets in parameter set* $\mathcal{K}$*. In general, we can denote* $\widetilde{\mathcal{D}}^{-1}(v)(q) \in [0,1]$, $\forall v \in X, q \in \mathcal{K}$.

As per this definition, it is clear that the mapping $\widetilde{\mathcal{D}}^{-1}: X \to HFS(\mathcal{K})$ describes a hesitant fuzzy relation from the universe of objects X to a set of parameters $\mathcal{K}$. In other words, we can define this hesitant fuzzy relation as follows:

For any $v_i \in X$, $q_j \in \mathcal{K}$, $\widetilde{\mathcal{D}}^{-1}(v_i)(q_j) \in HFS(X \times \mathcal{K})$.

In general, $\widetilde{\mathcal{D}}^{-1}(v_i)(q_j)$ is referred to as an arbitrary hesitant fuzzy binary relation because it does not hold the condition of an equivalence relation (reflexive, symmetric, and transitive).

Now, we describe the following definition of SHFRS by using the notion of inverse hesitant fuzzy soft set:

Definition 11: Suppose $(\widetilde{\mathcal{D}}^{-1}, Q)$ *is an inverse hesitant fuzzy soft set on* X*. Then, a soft hesitant fuzzy approximation space over* X *and* Q *is defined as triple* $(X, Q, \widetilde{\mathcal{D}}^{-1})$*. The upper approximation* $\overline{\mathcal{D}}(\mathbb{K})$ *and lower approximation* $\underline{\mathcal{D}}(\mathbb{K})$ *of any set* $\mathbb{K} \in HFS(Q)$ *with respect to triple* $(X, Q, \widetilde{\mathcal{D}}^{-1})$ *are computed using the following mathematical expression:*

$$\overline{\mathcal{D}}(\mathbb{K}) = \{\langle v, h_{\overline{\mathcal{D}}(\mathbb{K})}(v)\rangle | v \in X\},$$
$$\underline{\mathcal{D}}(\mathbb{K}) = \{\langle v, h_{\underline{\mathcal{D}}(\mathbb{K})}(v)\rangle | v \in X\},$$

Where

$$h_{\overline{\mathcal{D}}(\mathbb{K})}(v) = \underline{\vee}_{w \in Q} \{h_{\widetilde{\mathcal{D}}^{-1}}(v, w) \,\overline{\wedge}\, h_{\mathbb{K}}(w)\}, \ v \in X$$
$$h_{\underline{\mathcal{D}}(\mathbb{K})}(v) = \overline{\wedge}_{w \in Q} \{(1 - h_{\widetilde{\mathcal{D}}^{-1}}(v, w)) \,\underline{\vee}\, h_{\mathbb{K}}(w)\}, \quad v \in X.$$

Finally, the SHFRS of any set $\mathbb{K} \in HFS(Q)$ with respect to triple $(X, Q, \widetilde{\mathcal{D}}^{-1})$ is defined as pair $(\overline{\mathcal{D}}(\mathbb{K}), \underline{\mathcal{D}}(\mathbb{K}))$, where $\overline{\mathcal{D}}$ and $\underline{\mathcal{D}}$ are called upper and lower soft hesitant fuzzy rough approximation operators, respectively.

Since $\widetilde{\mathcal{D}}^{-1}$ is an arbitrary hesitant fuzzy binary relation, it is clear that, for any set $\mathbb{K} \in HFS(Q)$, $\underline{\mathcal{D}}(\mathbb{K}) \sqsubseteq \overline{\mathcal{D}}(\mathbb{K})$ does not hold. Here, $\sqsubseteq$ denotes the proper subset operation on a hesitant fuzzy set.

We can describe the following four different cases of Definition 11:

Remark 1: Let $(\widetilde{\mathcal{D}}^{-1}, Q)$ be an inverse soft set on X. In this case, upper approximation $\overline{\mathcal{D}}(\mathbb{K})$ and lower approximation $\underline{\mathcal{D}}(\mathbb{K})$ of any set $\mathbb{K} \in HFS(Q)$ with respect to triple $(X, Q, \widetilde{\mathcal{D}}^{-1})$ are computed using the following expression:

$$\overline{\mathcal{D}}(\mathbb{K}) = \{(v, h_{\overline{\mathcal{D}}(\mathbb{K})}(v)) | v \in X\},$$
$$\underline{\mathcal{D}}(\mathbb{K}) = \{(v, h_{\underline{\mathcal{D}}(\mathbb{K})}(v)) | v \in X\},$$

where

$$h_{\overline{\mathcal{D}}(\mathbb{K})}(v) = \underline{\vee}_{w \in \widetilde{\mathcal{D}}^{-1}(v)} \{h_{\mathbb{K}}(w)\}, \ v \in X$$
$$h_{\underline{\mathcal{D}}(\mathbb{K})}(v) = \overline{\wedge}_{w \in \widetilde{\mathcal{D}}^{-1}(v)} \{h_{\mathbb{K}}(w)\}, \quad v \in X.$$

Here, triple $(X, Q, \widetilde{\mathcal{D}}^{-1})$ is called a soft approximation space, and the resultant set is called a soft rough hesitant fuzzy set defined by pair $(\overline{\mathcal{D}}(\mathbb{K}), \underline{\mathcal{D}}(\mathbb{K}))$.

Remark 2: Let the given set $\mathbb{K} \in \mathcal{P}(Q)$ be the crisp set of Q. In this case, upper approximation $\overline{\mathcal{D}}(\mathbb{K})$ and lower approximation $\underline{\mathcal{D}}(\mathbb{K})$ of set $\mathbb{K} \in \mathcal{P}(Q)$ with respect to triple $(X, Q, \widetilde{\mathcal{D}}^{-1})$ are computed using the following expression:

$$\overline{\mathcal{D}}(\mathbb{K}) = \{(v, h_{\overline{\mathcal{D}}(\mathbb{K})}(v)) | v \in X\},$$
$$\underline{\mathcal{D}}(\mathbb{K}) = \{(v, h_{\underline{\mathcal{D}}(\mathbb{K})}(v)) | v \in X\},$$

Where

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$$h_{\overline{\mathcal{D}}(\mathbb{K})}(v) = \underline{\vee}_{w\in\mathbb{K}} \{h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\}, \quad v \in X$$
$$h_{\underline{\mathcal{D}}(\mathbb{K})}(v) = \overline{\wedge}_{w\notin\mathbb{K}} \{1 - h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\}, \quad v \in X.$$

Here, triple $(X, \mathcal{Q}, \widetilde{\mathcal{D}}^{-1})$ is called a soft hesitant fuzzy approximation space, and a soft hesitant fuzzy rough set of any crisp set $\mathbb{K} \in \mathcal{P}(\mathcal{Q})$ with respect to triple $(X, \mathcal{Q}, \widetilde{\mathcal{D}}^{-1})$ is defined as pair $(\overline{\mathcal{D}}(\mathbb{K}), \underline{\mathcal{D}}(\mathbb{K}))$, where $\overline{\mathcal{D}}$ and $\underline{\mathcal{D}}$ are called upper and lower soft hesitant fuzzy approximation operators, respectively.

Remark 3: Let $(\widetilde{\mathcal{D}}^{-1}, \mathcal{Q})$ be an inverse soft set on X and the given set $\mathbb{K} \in \mathcal{P}(\mathcal{Q})$ be the crisp set of $\mathcal{Q}$. In this case, upper approximation $\overline{\mathcal{D}}(\mathbb{K})$ and lower approximation $\underline{\mathcal{D}}(\mathbb{K})$ of set $\mathbb{K} \in \mathcal{P}(\mathcal{Q})$ with respect to triple $(X, \mathcal{Q}, \widetilde{\mathcal{D}}^{-1})$ are computed using the following expression:

$$\overline{\mathcal{D}}(\mathbb{K})(v) = \{w \in \mathcal{Q} | \exists v \in X, \exists w \in \widetilde{\mathcal{D}}^{-1}(v) \cap \mathbb{K} \neq \phi\}$$
$$\underline{\mathcal{D}}(\mathbb{K})(v) = \{w \in \mathcal{Q} | \exists v \in X, \exists w \in \widetilde{\mathcal{D}}^{-1}(v) \subseteq \mathbb{K}\}$$

Here, triple $(X, \mathcal{Q}, \widetilde{\mathcal{D}}^{-1})$ is called a soft approximation space, and the resultant set is called a soft rough set defined by pair $(\overline{\mathcal{D}}(\mathbb{K}), \underline{\mathcal{D}}(\mathbb{K}))$.

Remark 4: With regard to Definition 11, suppose the hesitant fuzzy elements $h_{\widetilde{\mathcal{D}}^{-1}}(v,w)$ and $h_{\mathbb{K}}(w)$ have only one element each; then hesitant fuzzy relation $\widetilde{\mathcal{D}}^{-1}$ is reduced to a fuzzy relation from X to $\mathcal{Q}$ and hesitant fuzzy set $\mathbb{K}$ is reduced to a fuzzy set, with SHFRS over X and $\mathcal{Q}$ reduced to a soft fuzzy rough set defined by Dan Meng, et al [41].

Let us now consider the example below to clarify the results presented above:

Example 3: With regard to Example 2, assume that a person is evaluating an optional computer considering various characteristics with hesitant fuzzy element. Then, all computers in X with their characteristics under the hesitant fuzzy information are illuminated by using the hesitant fuzzy soft set $(\widetilde{\mathcal{D}}, \mathcal{K})$ as presented in Table 2.

If a hesitant fuzzy set

$$\mathbb{K} = \frac{\{0.3, 0.4, 0.6\}}{q_1} + \frac{\{0.6, 0.8\}}{q_2} + \frac{\{0.3, 0.5, 0.9\}}{q_3} + \frac{\{0.5, 0.7\}}{q_4} + \frac{\{0.6, 0.7\}}{q_5} + \frac{\{0.4, 0.6\}}{q_6} + \frac{\{0.5, 0.7, 0.9\}}{q_7}$$

Then, by Definition 11, we get the hesitant fuzzy upper and lower approximation of $\mathbb{K}$ as follows:

$$h_{\overline{\mathcal{D}}(\mathbb{K})}(\wp\mathcal{C}_1) = \underline{\vee}_{q\in\mathcal{K}} \{h_{\widetilde{\mathcal{D}}^{-1}}(\wp\mathcal{C}_1, q) \,\overline{\wedge}\, h_{\mathbb{K}}(q)\}$$
$$= (\{0.3,0.4,0.5\} \,\overline{\wedge}\, \{0.3,0.4,0.6\}) \,\underline{\vee}\, (\{0.3,0.5,0.6\} \,\overline{\wedge}\, \{0.6,0.8\}) \,\underline{\vee}\, (\{0.2,0.4\} \,\overline{\wedge}\, \{0.3,0.5,0.9\}) \,\underline{\vee}\, (\{0.2,0.4,0.7\} \,\overline{\wedge}\, \{0.5,0.7\}) \,\underline{\vee}\, (\{0.3,0.5,0.7\} \,\overline{\wedge}\, \{0.6,0.7\}) \,\underline{\vee}\, (\{0.3,0.4\} \,\overline{\wedge}\, \{0.4,0.6\}) \,\underline{\vee}\, (\{0.2,0.3,0.4\} \,\overline{\wedge}\, \{0.5,0.7,0.9\})$$
$$= \{0.3,0.4,0.5\} \,\underline{\vee}\, \{0.3,0.5,0.6\} \,\underline{\vee}\, \{0.2,0.3,0.4\} \,\underline{\vee}\, \{0.2,0.4,0.5,0.7\} \,\underline{\vee}\, \{0.3,0.5,0.6,0.7\} \,\underline{\vee}\, \{0.3,0.4\} \,\underline{\vee}\, \{0.2,0.3,0.4\}$$
$$= \{0.3,0.4,0.5,0.6,0.7\}.$$

$$h_{\underline{\mathcal{D}}(\mathbb{K})}(\wp\mathcal{C}_1) = \overline{\wedge}_{q\in\mathcal{K}} \{(1 - h_{\widetilde{\mathcal{D}}^{-1}}(\wp\mathcal{C}_1, q)) \,\underline{\vee}\, h_{\mathbb{K}}(q)\}$$
$$= (\{0.5,0.6,0.7\} \,\underline{\vee}\, \{0.3,0.4,0.6\}) \,\overline{\wedge}\, (\{0.4,0.5,0.7\} \,\underline{\vee}\, \{0.6,0.7,0.8\}) \,\overline{\wedge}\, (\{0.6,0.8\} \,\underline{\vee}\, \{0.3,0.5,0.9\}) \,\overline{\wedge}\, (\{0.3,0.6,0.8\} \,\underline{\vee}\, \{0.5,0.7\})$$

$$\overline{\wedge}\, (\{0.3,0.5,0.7\} \,\underline{\vee}\, \{0.6,0.7\}) \,\overline{\wedge}\, (\{0.6,0.7\} \,\underline{\vee}\, \{0.4,0.6\}) \,\overline{\wedge}\, (\{0.6,0.7,0.8\} \,\underline{\vee}\, \{0.5,0.7,0.9\})$$
$$= \{0.5,0.6,0.7\} \,\overline{\wedge}\, \{0.6,0.8\} \,\overline{\wedge}\, \{0.6,0.8,0.9\} \,\overline{\wedge}\, \{0.5,0.6,0.7,0.8\} \,\overline{\wedge}\, \{0.6,0.7\} \,\overline{\wedge}\, \{0.6,0.7\} \,\overline{\wedge}\, \{0.6,0.7,0.8,0.9\}$$
$$= \{0.5,0.6,0.7\}.$$

Similarly, we have

$h_{\overline{\mathcal{D}}(\mathbb{K})}(\wp\mathcal{C}_2) = \{0.3, 0.4, 0.5, 0.6\}$,
$h_{\underline{\mathcal{D}}(\mathbb{K})}(\wp\mathcal{C}_2) = \{0.4, 0.5, 0.6, 0.7\}$,
$h_{\overline{\mathcal{D}}(\mathbb{K})}(\wp\mathcal{C}_3) = \{0.5, 0.6, 0.7\}$,
$h_{\underline{\mathcal{D}}(\mathbb{K})}(\wp\mathcal{C}_3) = \{0.5, 0.6\}$,
$h_{\overline{\mathcal{D}}(\mathbb{K})}(\wp\mathcal{C}_4) = \{0.5, 0.6, 0.7\}$,
$h_{\underline{\mathcal{D}}(\mathbb{K})}(\wp\mathcal{C}_4) = \{0.4, 0.5, 0.6\}$,
$h_{\overline{\mathcal{D}}(\mathbb{K})}(\wp\mathcal{C}_5) = \{0.4, 0.5\}$,
$h_{\underline{\mathcal{D}}(\mathbb{K})}(\wp\mathcal{C}_5) = \{0.5, 0.6, 0.7\}$.

We can represent upper approximation $\overline{\mathcal{D}}(\mathbb{K})$ and lower approximation $\underline{\mathcal{D}}(\mathbb{K})$ of $\mathbb{K}$ with respect to $(X, \mathcal{Q}, \widetilde{\mathcal{D}}^{-1})$ as follows:

$$\overline{\mathcal{D}}(\mathbb{K}) = \{\langle \wp\mathcal{C}_1, \{0.3, 0.5, 0.6, 0.7\}\rangle, \langle \wp\mathcal{C}_2, \{0.3, 04, 0.5, 0.6\}\rangle, \langle \wp\mathcal{C}_3, \{0.5, 0.6, 0.7\}\rangle, \langle \wp\mathcal{C}_4, \{0.5, 0.6, 0.7\}\rangle, \langle \wp\mathcal{C}_5, \{0.4, 0.5\}\rangle\}$$

$$\underline{\mathcal{D}}(\mathbb{K}) = \{\langle \wp\mathcal{C}_1, \{0.5, 0.6, 0.7\}\rangle, \langle \wp\mathcal{C}_2, \{0.4, 0.5, 0.6, 0.7\}\rangle, \langle \wp\mathcal{C}_3, \{0.5, 0.6\}\rangle, \langle \wp\mathcal{C}_4, \{0.4, 0.5, 0.6\}\rangle, \langle \wp\mathcal{C}_5, \{0.5, 0.6, 0.7\}\rangle\}$$

It can be easily verified that $\underline{\mathcal{D}}(\mathbb{K}) \nsubseteq \overline{\mathcal{D}}(\mathbb{K})$. Similarly, the four cases described in Remarks 1, 2, 3, and 4 can also be proven.

Table 2. Illustration of hesitant fuzzy soft set $(\widetilde{\mathcal{D}}, \mathcal{K})$ in table format

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$\mathcal{K}$ / X	q_1	q_2	q_3	q_4	q_5	q_6	q_7
pC_1	{0.3, 0.4, 0.5}	{0.3, 0.5, 0.6}	{0.2, 0.4}	{0.2, 0.4, 0.7}	{0.3, 0.5, 0.7}	{0.3, 0.4}	{0.2, 0.3, 0.4}
pC_2	{0.2, 0.4}	{0.2, 0.4}	{0.5, 0.6}	{0.3, 0.5}	{0.2, 0.4}	{0.3, 0.5, 0.6}	{0.3, 0.5, 0.6}
pC_3	{0.4}	{0.5}	{0.2, 0.4}	{0.4, 0.8}	{0.4, 0.6}	{0.2, 0.4}	{0.2, 0.4}
pC_4	{0.2, 0.3}	{0.4, 0.6}	{0.3, 0.5, 0.6}	{0.3, 0.5, 0.6}	{0.5, 0.7}	{0.5, 0.6, 0.7}	{0.3, 0.5, 0.7}
pC_5	{0.3, 0.4}	{0.2, 0.3}	{0.4, 0.5}	{0.2, 0.3}	{0.3}	{0.3}	{0.4, 0.5}

Similar to the hesitant fuzzy rough approximation operators, soft hesitant fuzzy rough approximation operators have various properties. We describe some properties as follows:

Proposition 1: Suppose that a soft hesitant fuzzy approximation space is defined as triple $(X, \mathcal{Q}, \widetilde{\mathcal{D}}^{-1})$; then for a given set $\mathbb{H} \in HFS(\mathcal{Q})$, we can describe the following:

(1) $h^{+}_{\overline{\mathcal{D}}(\mathbb{H})}(v) = max\{min\{h^{+}_{\widetilde{\mathcal{D}}^{-1}}(v,w), h^{+}_{\mathbb{H}}(w)\} : w \in \mathcal{Q}\}, \quad v \in X$
(2) $h^{-}_{\overline{\mathcal{D}}(\mathbb{H})}(v) = max\{min\{h^{-}_{\widetilde{\mathcal{D}}^{-1}}(v,w), h^{-}_{\mathbb{H}}(w)\} : w \in \mathcal{Q}\}, \quad v \in X$
(3) $h^{+}_{\underline{\mathcal{D}}(\mathbb{H})}(v) = min\{\max\{(1 - h^{+}_{\widetilde{\mathcal{D}}^{-1}}(v,w)), h^{+}_{\mathbb{H}}(w)\} : w \in \mathcal{Q}\}, \quad v \in X$
(4) $h^{-}_{\underline{\mathcal{D}}(\mathbb{H})}(v) = min\{\max\{(1 - h^{-}_{\widetilde{\mathcal{D}}^{-1}}(v,w)), h^{-}_{\mathbb{H}}(w)\} : w \in \mathcal{Q}\}, \quad v \in X$.

Where $h^{+}_{\widetilde{\mathcal{D}}^{-1}}(v,w)$ and $h^{+}_{\mathbb{H}}(w)$ are the upper bound of HFEs $h_{\widetilde{\mathcal{D}}^{-1}}(v,w)$ and $h_{\mathbb{H}}(w)$, respectively. $h^{-}_{\widetilde{\mathcal{D}}^{-1}}(v,w)$ and $h^{-}_{\mathbb{H}}(w)$ are the lower bound of HFEs $h_{\widetilde{\mathcal{D}}^{-1}}(v,w)$ and $h_{\mathbb{H}}(w)$, respectively.

Proof: The above results can be directly derived from the definitions of $\underline{\vee}$ and $\overline{\wedge}$ operations and Definition 11.

Proposition 2: Suppose that a soft hesitant fuzzy approximation space is defined as triple $(X, \mathcal{Q}, \widetilde{\mathcal{D}}^{-1})$; then for a given set $\mathbb{H} \in HFS(\mathcal{Q})$, we can describe the following:

(1) $\sim\overline{\mathcal{D}}(\sim\mathbb{H}) = \underline{\mathcal{D}}(\mathbb{H})$,
(2) $\sim\underline{\mathcal{D}}(\sim\mathbb{H}) = \overline{\mathcal{D}}(\mathbb{H})$.

Proof (1): $\forall v \in X$, from Definition 11, we have

$$h_{\sim\overline{\mathcal{D}}(\sim\mathbb{H})}(v) = \sim\left\{\underline{\vee}_{w\in\mathcal{Q}}\left\{h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\,\overline{\wedge}\,h_{(\sim\mathbb{H})}(w)\right\}\right\}, \quad v \in X$$
$$= \sim\left\{\underline{\vee}_{w\in\mathcal{Q}}\left\{h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\,\overline{\wedge} \sim h_{\mathbb{H}}(w)\right\}\right\}$$
$$= \overline{\wedge}_{w\in\mathcal{Q}}\left\{\sim h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\,\underline{\vee}\,h_{\mathbb{H}}(w)\right\}$$
$$= \overline{\wedge}_{w\in\mathcal{Q}}\left\{(1 - h_{\widetilde{\mathcal{D}}^{-1}}(v,w))\,\underline{\vee}\,h_{\mathbb{H}}(w)\right\}$$
$$= h_{\underline{\mathcal{D}}(\mathbb{H})}(v)$$

This implies that $\sim\overline{\mathcal{D}}(\sim\mathbb{H}) = \underline{\mathcal{D}}(\mathbb{H})$.
Proof (2): It can be proven similar to proof of (1).

Theorem 1: Suppose that a soft hesitant fuzzy approximation space is defined as triple $(X, \mathcal{Q}, \widetilde{\mathcal{D}}^{-1})$; then for two given sets $\mathbb{H}, \mathbb{I} \in HFS(\mathcal{Q})$, we can describe the following properties:

(1) $\overline{\mathcal{D}}(\mathbb{H} \Cup \mathbb{I}) = \overline{\mathcal{D}}(\mathbb{H}) \Cup \overline{\mathcal{D}}(\mathbb{I})$,
(2) $\overline{\mathcal{D}}(\mathbb{H} \Cap \mathbb{I}) \sqsubseteq \overline{\mathcal{D}}(\mathbb{H}) \Cap \overline{\mathcal{D}}(\mathbb{I})$,
(3) $\underline{\mathcal{D}}(\mathbb{H} \Cap \mathbb{I}) = \underline{\mathcal{D}}(\mathbb{H}) \Cap \underline{\mathcal{D}}(\mathbb{I})$,
(4) $\underline{\mathcal{D}}(\mathbb{H} \Cup \mathbb{I}) \sqsupseteq \underline{\mathcal{D}}(\mathbb{H}) \Cup \underline{\mathcal{D}}(\mathbb{I})$,
(5) $\mathbb{H} \sqsubseteq \mathbb{I} \Rightarrow \overline{\mathcal{D}}(\mathbb{H}) \sqsubseteq \overline{\mathcal{D}}(\mathbb{I})$,
(6) $\mathbb{H} \sqsubseteq \mathbb{I} \Rightarrow \underline{\mathcal{D}}(\mathbb{H}) \sqsubseteq \underline{\mathcal{D}}(\mathbb{I})$,
(7) $\underline{\mathcal{D}}(\mathcal{Q}) = X$,
(8) $\overline{\mathcal{D}}(\emptyset) = \emptyset$

Here, it should be noted that operations $\Cup$, $\Cap$, $\sqsubseteq$, and $\sqsupseteq$ denote union, intersection, proper subset, and proper superset operations on hesitant fuzzy sets, respectively, whereas operations $\cup$, $\cap$, $\subseteq$, and $\supseteq$ are the ordinary union, intersection, proper subset, and proper superset operations, respectively.

Proof (1): $\forall v \in X$, from Definition 11 , we have

$$h_{\overline{\mathcal{D}}(\mathbb{H}\Cup\mathbb{I})}(v) = \underline{\vee}_{w\in\mathcal{Q}}\left\{h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\,\overline{\wedge}\,h_{\mathbb{H}\Cup\mathbb{I}}(w)\right\}, \quad v \in X$$
$$= \underline{\vee}_{w\in\mathcal{Q}}\left\{h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\,\overline{\wedge}\,(h_{\mathbb{H}}(w)\,\underline{\vee}\,h_{\mathbb{I}}(w))\right\}$$
$$= \underline{\vee}_{w\in\mathcal{Q}}\left\{(h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\,\overline{\wedge}\,(h_{\mathbb{H}}(w))\,\underline{\vee}\,(h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\,\overline{\wedge}\,(h_{\mathbb{I}}(w))\right\}$$
$$= \{\underline{\vee}_{w\in\mathcal{Q}}\,(h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\,\overline{\wedge}\,h_{\mathbb{H}}(w))\}\ \underline{\vee}\ \{\underline{\vee}_{w\in\mathcal{Q}}\,(h_{\widetilde{\mathcal{D}}^{-1}}(v,w)\,\overline{\wedge}\,h_{\mathbb{I}}(w))\}$$
$$= h_{\overline{\mathcal{D}}(\mathbb{H})}(v)\,\underline{\vee}\,h_{\overline{\mathcal{D}}(\mathbb{I})}(v)$$
$$= h_{\overline{\mathcal{D}}(\mathbb{H})\Cup\overline{\mathcal{D}}(\mathbb{I})}(v).$$

This implies that $\overline{\mathcal{D}}(\mathbb{H} \Cup \mathbb{I}) = \overline{\mathcal{D}}(\mathbb{H}) \Cup \overline{\mathcal{D}}(\mathbb{I})$,
Proof (2), (3), (4): These three properties can be proven to be similar to the proof of (1).
Proof (5): $\forall v \in X$, from proposition 1, we get $h^{+}_{\overline{\mathcal{D}}(\mathbb{H})}(v) = max\{min\{h^{+}_{\widetilde{\mathcal{D}}^{-1}}(v,w), h^{+}_{\mathbb{H}}(w)\}\}$, and $h^{-}_{\overline{\mathcal{D}}(\mathbb{H})}(v) = max\{min\{h^{-}_{\widetilde{\mathcal{D}}^{-1}}(v,w), h^{-}_{\mathbb{H}}(w)\}\}$, similarly for $h^{+}_{\overline{\mathcal{D}}(\mathbb{I})}(v) = max\{min\{h^{+}_{\widetilde{\mathcal{D}}^{-1}}(v,w), h^{+}_{\mathbb{I}}(w)\}\}$ and $h^{-}_{\overline{\mathcal{D}}(\mathbb{I})}(v) = max\{min\{h^{-}_{\widetilde{\mathcal{D}}^{-1}}(v,w), h^{-}_{\mathbb{I}}(w)\}\}$. Since $\mathbb{H} \sqsubseteq \mathbb{I}$, then we have $h^{+}_{\mathbb{H}}(v) \le h^{+}_{\mathbb{I}}(v)$ and $h^{-}_{\mathbb{H}}(v) \le h^{-}_{\mathbb{I}}(v)$ for each $v \in X$, which implies that $min\{h^{-}_{\widetilde{\mathcal{D}}^{-1}}(v,w), h^{-}_{\mathbb{H}}(w)\} \le min\{h^{-}_{\widetilde{\mathcal{D}}^{-1}}(v,w), h^{-}_{\mathbb{I}}(w)\}$ for each $w \in \mathcal{Q}$, it can be concluded that $h^{-}_{\overline{\mathcal{D}}(\mathbb{H})}(v) \le h^{-}_{\overline{\mathcal{D}}(\mathbb{I})}(v)$. In the same way, we can conclude that $h^{+}_{\overline{\mathcal{D}}(\mathbb{H})}(v) \le h^{+}_{\overline{\mathcal{D}}(\mathbb{I})}(v)$. From the two conclusions above, we hold $h_{\overline{\mathcal{D}}(\mathbb{H})}(v) \le h_{\overline{\mathcal{D}}(\mathbb{I})}(v)$, which implies that $\overline{\mathcal{D}}(\mathbb{H}) \sqsubseteq \overline{\mathcal{D}}(\mathbb{I})$.
Proof (6): It can be proven to be similar to the proof of (5).

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Proof (7): $\forall w \in Q$, we have $h_Q(w) = \{1\}$. Then from Definition 11,

$$h_{\underline{\mathcal{D}}(Q)}(v) = \overline{\wedge}_{w\in Q} \{(1 - h_{\widetilde{\mathcal{D}}^{-1}}(v,w)) \underline{\vee}\, h_Q(w)\}, \quad v \in X$$
$$= \overline{\wedge}_{w\in Q} \{(1 - h_{\widetilde{\mathcal{D}}^{-1}}(v,w)) \underline{\vee}\, \{1\}\} = \{1\}$$

This implies that $\underline{\mathcal{D}}(Q) = X$.

Proof (8): $\forall w \in Q$, we have $h_\emptyset(w) = \{0\}$. Then from Definition 11,

$$h_{\overline{\mathcal{D}}(\emptyset)}(v) = \underline{\vee}_{w\in Q} \{h_{\widetilde{\mathcal{D}}^{-1}}(v,w) \,\overline{\wedge}\, h_\emptyset(w)\}, \quad v \in X$$
$$= \underline{\vee}_{w\in Q} \{h_{\widetilde{\mathcal{D}}^{-1}}(v,w) \,\overline{\wedge}\, \{0\}\} = \{0\}.$$

This implies that $\overline{\mathcal{D}}(\emptyset) = \emptyset$.

In general, the results described in theorem 1 are similar to the results presented by Xibei Yang et al. [30] for traditional hesitant fuzzy rough set.

Theorem 2: Suppose $(\widetilde{\mathcal{D}}_1^{-1}, Q)$ and $(\widetilde{\mathcal{D}}_2^{-1}, Q)$ are two inverse hesitant fuzzy soft sets on X. Then, two soft hesitant fuzzy approximation spaces over X and Q can be defined as triples $(X, Q, \widetilde{\mathcal{D}}_1^{-1})$ and $(X, Q, \widetilde{\mathcal{D}}_2^{-1})$. If $\widetilde{\mathcal{D}}_1^{-1} \sqsubseteq \widetilde{\mathcal{D}}_2^{-1}$, then we can describe the following properties:

(1) $\overline{\mathcal{D}_1}(\mathbb{H}) \sqsubseteq \overline{\mathcal{D}_2}(\mathbb{H}), \ \forall \mathbb{H} \in HFS(Q)$.
(2) $\underline{\mathcal{D}_1}(\mathbb{H}) \sqsupseteq \underline{\mathcal{D}_2}(\mathbb{H}), \ \forall \mathbb{H} \in HFS(Q)$,

Where $\overline{\mathcal{D}_1}$ and $\underline{\mathcal{D}_1}$ denote upper and lower soft hesitant fuzzy rough approximation operators, respectively with respect to triple $(X, Q, \widetilde{\mathcal{D}}_1^{-1})$ and $\overline{\mathcal{D}_2}$ and $\underline{\mathcal{D}_2}$ are upper and lower soft hesitant fuzzy rough approximation operators, respectively with respect to triple $(X, Q, \widetilde{\mathcal{D}}_2^{-1})$.

Proof (1): $\forall v \in X$, from proposition 1, we get $h^+_{\overline{\mathcal{D}_1}(\mathbb{H})}(v) = max\{min\{h^+_{\widetilde{\mathcal{D}}_1^{-1}}(v,w), h^+_{\mathbb{H}}(w)\} : w \in Q\}$ and $h^-_{\overline{\mathcal{D}_1}(\mathbb{H})}(v) = max\{min\{h^-_{\widetilde{\mathcal{D}}_1^{-1}}(v,w), h^-_{\mathbb{H}}(w)\}\}$, similarly for $h^+_{\overline{\mathcal{D}_2}(\mathbb{H})}(v) = max\{min\{h^+_{\widetilde{\mathcal{D}}_2^{-1}}(v,w), h^+_{\mathbb{H}}(w)\} : w \in Q\}$ and $h^-_{\overline{\mathcal{D}_2}(\mathbb{H})}(v) = max\{min\{h^-_{\widetilde{\mathcal{D}}_2^{-1}}(v,w), h^-_{\mathbb{H}}(w)\}\}$.
Since $\widetilde{\mathcal{D}}_1^{-1} \sqsubseteq \widetilde{\mathcal{D}}_2^{-1}$, then we have $h^+_{\widetilde{\mathcal{D}}_1^{-1}}(v,w) \le h^+_{\widetilde{\mathcal{D}}_2^{-1}}(v,w)$ and $h^-_{\widetilde{\mathcal{D}}_1^{-1}}(v,w) \le h^-_{\widetilde{\mathcal{D}}_2^{-1}}(v,w)$ for all $(v,w) \in X \times Q$, which implies that $min\{h^-_{\widetilde{\mathcal{D}}_1^{-1}}(v,w), h^-_{\mathbb{H}}(w)\} \le min\{h^-_{\widetilde{\mathcal{D}}_2^{-1}}(v,w), h^-_{\mathbb{H}}(w)\}$ for each $w \in Q$, it can be concluded that $h^-_{\overline{\mathcal{D}_1}(\mathbb{H})}(v) \le h^-_{\overline{\mathcal{D}_2}(\mathbb{H})}(v)$. In the same way, we can conclude that $h^+_{\overline{\mathcal{D}_1}(\mathbb{H})}(v) \le h^+_{\overline{\mathcal{D}_2}(\mathbb{H})}(v)$. From the two conclusions above, we hold $h_{\overline{\mathcal{D}_1}(\mathbb{H})}(v) \le h_{\overline{\mathcal{D}_2}(\mathbb{H})}(v)$, which implies that $\overline{\mathcal{D}}_1(\mathbb{H}) \sqsubseteq \overline{\mathcal{D}_2}(\mathbb{H})$.

Proof (2): It can be proven to be similar to the proof of (1).

Theorem 3: Suppose $\widetilde{\mathcal{D}}^{-1}$ is a hesitant fuzzy relation defined by an inverse hesitant fuzzy soft set on X and Q. The upper approximation and lower approximation of any set $\mathbb{H} \in HFS(Q)$ with respect to triple $(X, Q, \widetilde{\mathcal{D}}^{-1})$ are $\overline{\mathcal{D}}(\mathbb{H})$ and $\underline{\mathcal{D}}(\mathbb{H})$, respectively; then $\widetilde{\mathcal{D}}^{-1}$ is serial if one of the following conditions holds:

(1) $\underline{\mathcal{D}}(\emptyset) = \emptyset$,
(2) $\overline{\mathcal{D}}(Q) = X$,
(3) $\underline{\mathcal{D}}(\mathbb{H}) \sqsubseteq \overline{\mathcal{D}}(\mathbb{H}), \ \forall \mathbb{H} \in HFS(Q)$.

Proof: These three conditions can be validated similar to theorem 1 and by using Definition 11.

Definition 12: Let two soft hesitant fuzzy approximation spaces over universal set X and Q be $(X, Q, \widetilde{\mathcal{D}}_1^{-1})$ and $(X, Q, \widetilde{\mathcal{D}}_2^{-1})$; then:

(1) The intersection of $(X, Q, \widetilde{\mathcal{D}}_1^{-1})$ and $(X, Q, \widetilde{\mathcal{D}}_2^{-1})$ can be defined by using soft hesitant fuzzy approximation space $(X, Q, \ \widetilde{\mathcal{D}}_1^{-1} \Cap \widetilde{\mathcal{D}}_2^{-1})$.
(2) The union of $(X, Q, \widetilde{\mathcal{D}}_1^{-1})$ and $(X, Q, \widetilde{\mathcal{D}}_2^{-1})$ can be defined by using soft hesitant fuzzy approximation space $(X, Q, \ \widetilde{\mathcal{D}}_1^{-1} \Cup \widetilde{\mathcal{D}}_2^{-1})$.

Theorem 4: Let two soft hesitant fuzzy approximation spaces over universal set X and Q are $(X, Q, \widetilde{\mathcal{D}}_1^{-1})$ and $(X, Q, \widetilde{\mathcal{D}}_2^{-1})$. If $\widetilde{\mathcal{D}}^{-1} = \widetilde{\mathcal{D}}_1^{-1} \Cup \widetilde{\mathcal{D}}_2^{-1}$, then for any set $\mathbb{H} \in HFS(Q)$, the following conditions hold:

(1) $\overline{\mathcal{D}}(\mathbb{H}) = \overline{\mathcal{D}_1}(\mathbb{H}) \Cup \overline{\mathcal{D}_2}(\mathbb{H})$,
(2) $\underline{\mathcal{D}}(\mathbb{H}) = \underline{\mathcal{D}_1}(\mathbb{H}) \Cap \underline{\mathcal{D}_2}(\mathbb{H})$.

Proof (1): $\forall v \in X$, from Definition 11 , we have

$$h_{\overline{\mathcal{D}}(\mathbb{H})}(v) = \underline{\vee}_{w\in Q} \{h_{\widetilde{\mathcal{D}}^{-1}}(v,w) \,\overline{\wedge}\, h_{\mathbb{H}}(w)\}, \quad v \in X$$
$$= \underline{\vee}_{w\in Q} \left\{h_{\widetilde{\mathcal{D}}_1^{-1} \Cup \widetilde{\mathcal{D}}_2^{-1}}(v,w) \,\overline{\wedge}\, h_{\mathbb{H}}(w)\right\}$$
$$= \underline{\vee}_{w\in Q} \left\{(h_{\widetilde{\mathcal{D}}_1^{-1}}(v,w) \,\underline{\vee}\, h_{\widetilde{\mathcal{D}}_2^{-1}}(v,w)) \,\overline{\wedge}\, h_{\mathbb{H}}(w)\right\}$$
$$= \{ \underline{\vee}_{w\in Q} (h_{\widetilde{\mathcal{D}}_1^{-1}}(v,w) \,\overline{\wedge}\, h_{\mathbb{H}}(w))\} \ \underline{\vee} \{ \underline{\vee}_{w\in Q} (h_{\widetilde{\mathcal{D}}_2^{-1}}(v,w) \,\overline{\wedge}\, h_{\mathbb{H}}(w))\}$$
$$= h_{\overline{\mathcal{D}_1}(\mathbb{H})}(v) \,\underline{\vee}\, h_{\overline{\mathcal{D}_2}(\mathbb{H})}(v)$$
$$= h_{\overline{\mathcal{D}_1}(\mathbb{H}) \Cup \overline{\mathcal{D}_2}(\mathbb{H})}(v).$$

This implies that $\overline{\mathcal{D}}(\mathbb{H}) = \overline{\mathcal{D}_1}(\mathbb{H}) \Cup \overline{\mathcal{D}_2}(\mathbb{H})$.

Proof (2): It is dual of condition (1); therefore, it follows the proof and conclusion of (1).

Theorem 5: Let two soft hesitant fuzzy approximation spaces over universal set X and Q be $(X, Q, \widetilde{\mathcal{D}}_1^{-1})$ and $(X, Q, \widetilde{\mathcal{D}}_2^{-1})$. If $\widetilde{\mathcal{D}}^{-1} = \widetilde{\mathcal{D}}_1^{-1} \Cap \widetilde{\mathcal{D}}_2^{-1}$, then for any set $\mathbb{H} \in HFS(Q)$, the following conditions hold:

(1) $\overline{\mathcal{D}}(\mathbb{H}) \sqsubseteq \overline{\mathcal{D}_1}(\mathbb{H}) \Cap \overline{\mathcal{D}_2}(\mathbb{H})$,
(2) $\underline{\mathcal{D}}(\mathbb{H}) \sqsupseteq \underline{\mathcal{D}_1}(\mathbb{H}) \Cup \underline{\mathcal{D}_2}(\mathbb{H})$.

Proof (1): $\forall v \in X$, from Definition 11, we have

$$h_{\overline{\mathcal{D}}(\mathbb{H})}(v) = \underline{\vee}_{w\in Q} \{h_{\widetilde{\mathcal{D}}^{-1}}(v,w) \,\overline{\wedge}\, h_{\mathbb{H}}(w)\}, \quad v \in X$$
$$= \underline{\vee}_{w\in Q} \left\{h_{\widetilde{\mathcal{D}}_1^{-1} \Cap \widetilde{\mathcal{D}}_2^{-1}}(v,w) \,\overline{\wedge}\, h_{\mathbb{H}}(w)\right\}$$

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$$=\underline{\vee}_{w\in Q}\left\{(h_{\widetilde{\mathcal{D}}_1^{-1}}(v,w)\,\overline{\wedge}\,h_{\widetilde{\mathcal{D}}_2^{-1}}(v,w))\,\overline{\wedge}\,h_{\mathbb{H}}(w)\right\}$$
$$=\{\underline{\vee}_{w\in Q}\,(h_{\widetilde{\mathcal{D}}_1^{-1}}(v,w)\,\overline{\wedge}\,h_{\mathbb{H}}(w))\}\,\overline{\wedge}\,\{\underline{\vee}_{w\in Q}\,(h_{\widetilde{\mathcal{D}}_2^{-1}}(v,w)\,\overline{\wedge}\,h_{\mathbb{H}}(w))\} =$$
$$h_{\overline{\mathcal{D}_1}(\mathbb{H})}(v)\,\overline{\wedge}\,h_{\overline{\mathcal{D}_2}(\mathbb{H})}(v)=h_{\overline{\mathcal{D}_1}(\mathbb{H})\Cap\overline{\mathcal{D}_2}(\mathbb{H})}(v)$$

This implies that $\overline{\mathcal{D}}(\mathbb{H})\sqsubseteq\overline{\mathcal{D}_1}(\mathbb{H})\Cap\overline{\mathcal{D}_2}(\mathbb{H})$.

Proof (2): It is dual of condition (1); therefore, it follows the proof and conclusion of (1).

Remark 5: In this paper, we have described SHFRS, which is a combination of two concepts. One is hesitant fuzzy rough set over two universes (X and Q), and the other is an inverse hesitant fuzzy binary relation $\widetilde{\mathcal{D}}^{-1}$ or an inverse hesitant fuzzy mapping between X and Q. Since these two universes X and Q are completely dissimilar with different sense, the reflexive, symmetric, and transitive properties cannot be defined for the inverse hesitant fuzzy binary relation $\widetilde{\mathcal{D}}^{-1}$ because we can describe all these properties on the identical universe. As such, all of the results that have been described for a traditional hesitant fuzzy rough set [30] cannot be defined for SHFRS; thus showing that our proposed set differs from a traditional hesitant fuzzy rough set.

IV. Multi-criteria decision making based on SHFRS

In this Section, we present a novel solution for multi-criteria decision making problems based on our proposed SHFRS.

The research of A.R. Roy et al. [20] first presented a method of solving the decision making problem based on the fuzzy soft theory. Feng Feng et al. [21] cited the shortcomings of this method [20] and introduced a new level soft set technique for providing a solution to the decision making problem based on a fuzzy soft set. Fuqiang Wang et al. [23] applied this level soft set technique to hesitant fuzzy soft set-based decision making. Nonetheless, we found that, in the level soft set technique, a threshold fuzzy set must be chosen in advance by the decision maker. The final result of the decision is dependent on the threshold fuzzy set to a certain extent. Therefore, the concept of choosing a threshold fuzzy set is not appropriate for solving the decision making problem based on a fuzzy soft set. In the next subsection, a novel method that uses the concept of SHFRS for solving the decision making problem is proposed. Our proposed method does not require supplementary information (for example, threshold fuzzy set) to be delivered by decision makers or in another manner. It only uses the data information delivered by the given problem. Thus, the final decision results obtained by our proposed method are free from the influence of subjective information. Moreover, our method avoids ambiguity in the decision results for the same decision problem because it is not influenced by any information delivered by different decision experts.

A. Procedural steps for multi-criteria decision making based on SHFRS

Here, we will describe the procedural steps for our proposed method in detail.

Let $X=\{v_1,v_2,\ldots\ldots v_n\}$ and $(\widetilde{\mathcal{D}}^{-1},Q)$ be an inverse hesitant fuzzy soft set over X, where $Q=\{q_1,q_2\ldots\ldots q_m\}$. According to Definition 5, for each $q_i\in Q$, $\widetilde{\mathcal{D}}(q_i)=\left\{\left(v_1,\,\widetilde{\mathcal{D}}(q_i)(v_1)\right),\left(v_2,\,\widetilde{\mathcal{D}}(q_i)(v_2)\right),\ldots\ldots,\left(v_n,\,\widetilde{\mathcal{D}}(q_i)(v_n)\right)\right\}$.
We can compute the score of each hesitant fuzzy element by Definition 3 That is,

$$scr(\widetilde{\mathcal{D}}(q_i))=\left\{\left(v_1,\,scr\left(\widetilde{\mathcal{D}}(q_i)(v_1)\right)\right),(v_2,\,scr(\widetilde{\mathcal{D}}(q_i)(v_2))),\ldots\ldots,(v_n,\,scr(\widetilde{\mathcal{D}}(q_i)(v_n)))\right\}.$$

Step 1: Since, for a certain decision making problem, a decision maker is willing to choose an optional object in universe X with the parameter value $q_i\in Q$ as high as at each parameter index i, an optimal normal decision object on parameter set Q is first computed using the following mathematical expression:

$$\mathbb{K}=\sum_{i=1}^{|Q|}\frac{\max\{(\widetilde{\mathcal{D}}(q_i)\}}{q_i},\ q_i\in Q\ ,$$

$$\text{i.e.,}\quad \mathbb{K}(q_i)=\max\{\widetilde{\mathcal{D}}(q_i)(v_j)|v_j\in X\}$$

Where $|Q|$ is the number of parameters in set Q.
We can compute $\max\{(\widetilde{\mathcal{D}}(q_i)\}$ by calculating the score function of each hesitant fuzzy element by using Definition 3.

$$\max\{(\widetilde{\mathcal{D}}(q_i))\}=\{h_{ij}\mid max_{h_{ij}\in\widetilde{\mathcal{D}}(q_i)(v_j)}\{scr(h_{ij})\}\},\ v_j\in X$$

Step 2: According to Definition 11, soft hesitant fuzzy upper approximation $\overline{\mathcal{D}}\,(\mathbb{K})$ and soft hesitant fuzzy lower approximation $\underline{\mathcal{D}}\,(\mathbb{K})$ of optimal normal decision object $\mathbb{K}$ with respect to $\left(X,Q,\,\widetilde{\mathcal{D}}^{-1}\right)$ are calculated. Thus, for each object $v_i\in X$ in X, we get the two closest values $\overline{\mathcal{D}}\,(\mathbb{K})(v_i)$ and $\underline{\mathcal{D}}\,(\mathbb{K})(v_i)$ by using the soft hesitant fuzzy upper and lower approximations of hesitant fuzzy subset $\mathbb{K}$.

Step 3: To calculate choice value σ_i for each object $v_i\in X$, the score functions $scr(\overline{\mathcal{D}}(\mathbb{K})(v_i))$ and $scr(\underline{\mathcal{D}}(\mathbb{K})(v_i))$ of values $\overline{\mathcal{D}}(\mathbb{K})(v_i)$ and $\underline{\mathcal{D}}\,(\mathbb{K})(v_i)$, respectively, are calculated using Definition 3. Finally, we define choice value σ_i as follows:

$$\sigma_i=scr\left(\overline{\mathcal{D}}(\mathbb{K})(v_i)\right)+scr\left(\underline{\mathcal{D}}(\mathbb{K})(v_i)\right),\quad v_i\in X$$

Choice value σ_i for each object $v_i\in X$ is then calculated. For the given decision problem, the object with the highest choice value σ_i is selected as the optimal decision object. In the case of similar highest choice value σ_i for two or more objects, any one of them can be randomly selected as the optimal decision object.

B. SHFRS based security service selection approach for FMEC

In FMEC, several security services with overlapping functions are presented for the composite security service. Therefore, the composite service maker requires some key criteria to differentiate the efficiency and user satisfaction level offered by every presented security service on a certain QoS parameter. A number of non-functional, context-dependent, and domain-specific properties of services and several influential factors such as CPU usage, processing delay, overhead, and many more are included in the QoS. For the security service composition, a multi-criteria decision making process is required to select optimal services from several available security services based on QoS. Therefore, in this subsection, we present a novel method for selecting the optimal services based on SHFRS in the existence of multi-observer hesitant fuzzy information. Here, we describe the multi-observer hesitant fuzzy information as multiple hesitant fuzzy soft sets containing multi-observer hesitant fuzzy information in terms of different sets of QoS parameters defined by the FMEC security service. The method mainly consists of two phases for solving the given problem of service composition. In the first phase, the method performs an aggregation procedure with respect to input parameter set R to compute the resultant hesitant fuzzy soft set from the multiple hesitant fuzzy soft sets. Here, we use the AND operation as an aggregation procedure given in Definition 6. In the second phase, the algorithm finds the optimal decision from the resultant hesitant fuzzy soft set by using the concept of SHFRS, which is described in the earlier subsection.

To understand better how the presented method works, we take the following example of optimal security service selection problem in the FMEC environment and solve it by using the concept of SHFRS-based decision making:

Example 4: Considering the most common security services in FMEC, we choose Firewall (v_1), Network address translator (v_2), Deep packet inspection (DPI) (v_3), Load balancer (v_4), and Virtual private network (v_5) as an example represented by security service set $X = \{v_1,\ v_2,\ v_3,\ v_4,\ v_5\}$. Each service in this set is differentiated by three types of QoS parameters: processing delay, CPU usage, and memory overhead. The processing delay parameter is represented by set $\mathcal{K} = \{low(k_1),\ high(k_2),\ very\ low(k_3),\ very\ high(k_4)\}$. The CPU usage parameter is represented by set $\mathcal{L} = \{low(l_1),\ high(l_2),\ very\ low(l_3),\ very\ high(l_4), idle(l_5)\}$. The memory overhead is defined by set $\mathcal{M} = \{low(m_1),\ high(m_2),\ very\ low(m_3),\ very\ high(m_4)\}$. The universal set of parameters is defined by $\mathcal{Q} = \mathcal{K} \cup \mathcal{L} \cup \mathcal{M}$.

Let the hesitant fuzzy soft set $(\tilde{\mathcal{A}},\ \mathcal{K})$ describes the mapping of security services with the processing delay parameter. The hesitant fuzzy soft set $(\tilde{\mathcal{B}}, \mathcal{L})$ describes the mapping of security services with the CPU usage parameter and the hesitant fuzzy soft set $(\tilde{\mathcal{C}}, \mathcal{M})$ describes the mapping of security services with the memory overhead parameter. The tabular representation of all three fuzzy soft sets is shown in Fig. 1.

The problem here is selecting an optimal security service from the set of given security services with respect to input parameter set $\mathcal{R}$ by an observer.

Let the input parameter set be $\mathcal{R} = \{r_1 = k_1 \wedge l_1 \wedge m_1,\ r_2 = k_1 \wedge l_3 \wedge m_4,\ r_3 = k_2 \wedge l_2 \wedge m_2,\ r_4 = k_2 \wedge l_4 \wedge m_4,\ r_5 = k_3 \wedge l_3 \wedge m_3,\ r_6 = k_3 \wedge l_4 \wedge m_2,\ r_7 = k_4 \wedge l_4 \wedge m_2\}$.

Step 1: The resultant hesitant fuzzy soft set $(\tilde{\mathcal{D}}, \mathcal{R})$ is computed by applying the AND operation (in accordance with Definition 6) to hesitant fuzzy sets $(\tilde{\mathcal{A}},\ \mathcal{K})$, $(\tilde{\mathcal{B}}, \mathcal{L})$, $(\tilde{\mathcal{C}}, \mathcal{M})$ as shown in Table 3.

Step 2: Calculate the optimal normal decision object.

$$\mathbb{K} = \sum_{i=1}^{|\mathcal{R}|} \frac{\max\{\tilde{\mathcal{D}}(r_i)\}}{r_i}, \qquad r_i \in \mathcal{R}$$

Where

$\max\{\tilde{\mathcal{D}}(r_i)\} = \{h_{ij} \mid max_{h_{ij} \in \tilde{\mathcal{D}}(r_i)(v_j)}\{scr(h_{ij})\}\},\ v_j \in X$

$\max\{\tilde{\mathcal{D}}(r_1)\}=$

$$\{h_{1j} \mid max_{h_{1j} \in \tilde{\mathcal{D}}(r_1)(v_j)}\{scr\left(\tilde{\mathcal{D}}(r_1)(v_1)\right), scr\left(\tilde{\mathcal{D}}(r_1)(v_2)\right), scr\left(\tilde{\mathcal{D}}(r_1)(v_3)\right), scr\left(\tilde{\mathcal{D}}(r_1)(v_4)\right), scr\left(\tilde{\mathcal{D}}(r_1)(v_5)\right)\}\}$$
$$= \{h_{1j} \mid max_{h_{1j} \in \tilde{\mathcal{D}}(r_1)(v_j)}\{scr(\{0.3, 0.4\}), scr(\{0.2, 0.4\}), scr(\{0.4\}), scr(\{0.2, 0.3\}), scr(\{0.3, 0.4\})\}\}$$

Since, according to the assumption of Meimei Xia et al. [33] and Definition 3, the lengths of all HFEs should be the same for computing score function, in Table 3, we make the lengths of all HFEs equal to 3 through the addition of the maximum value in each HFE, e.g., $\tilde{\mathcal{D}}(r_1)(v_1) = \{0.3, 0.4\} = \{0.3, 0.4, 0.4\}$. Now, we have

$$\max\{\tilde{\mathcal{D}}(r_1)\} = \{h_{1j} \mid max_{h_{1j} \in \tilde{\mathcal{D}}(r_1)(v_j)}\{scr(\{0.3, 0.4, 0.4\}), scr(\{0.2, 0.4, 0.4\}), scr(\{0.4, 0.4, 0.4\}), scr(\{0.2, 0.3, 0.3\}), scr(\{0.3, 0.4, 0.4\})\}\}$$
$$= \{h_{1j} \mid max_{h_{1j} \in \tilde{\mathcal{D}}(r_1)(v_j)}\{0.37,\ 0.33,\ 0.40,\ 0.27, 0.37\}\}$$

Since, in the expression above, the maximum score is 0.40, which corresponds to $\tilde{\mathcal{D}}(r_1)(v_3)$, we have

i = 1 ,j = 3 ,$h_{1j} = h_{13}$ and ,$\max\{\tilde{\mathcal{D}}(r_1)\} = h_{13}$.

Hence, $\max\{\tilde{\mathcal{D}}(r_1)\} = h_{13} = \tilde{\mathcal{D}}(r_1)(v_3) = \{0.4\}$

Similarly, we have

$$\max\{(\tilde{\mathcal{D}}(r_2))\} = \{0.4, 0.5, 0.6\},$$
$$\max\{(\tilde{\mathcal{D}}(r_3))\} = \{0.5, 0.6\},$$
$$\max\{(\tilde{\mathcal{D}}(r_4))\} = \{0.4, 0.6, 0.8\},$$
$$\max\{(\tilde{\mathcal{D}}(r_5))\} = \{0.5, 0.6, 0.7\},$$
$$\max\{(\tilde{\mathcal{D}}(r_6))\} = \{0.5, 0.6\},$$
$$\max\{(\tilde{\mathcal{D}}(r_7))\} = \{0.4, 0.5\}.$$

Thus, we obtain the following optimal normal decision object:

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	k_1	k_2	k_3	k_4
v_1	{0.3, 0.5}	{0.2, 0.4}	{0.4, 0.6}	{0.2, 0.3}
v_2	{0.2, 0.4}	{0.6, 0.7}	{0.5, 0.7}	{0.5, 0.6}
v_3	{0.5}	{0.8, 0.9}	{0.6, 0.8}	{0.3, 0.5}
v_4	{0.6, 0.7}	{0.3, 0.5}	{0.7, 0.9}	{0.3, 0.5}
v_5	{0.5, 0.6}	{0.4, 0.6}	{0.3}	{0.4, 0.5}

	l_1	l_2	l_3	l_4	l_5
v_1	{0.3, 0.4}	{0.5, 0.6}	{0.3, 0.5}	{0.4, 0.6}	{0.7}
v_2	{0.6, 0.7}	{0.6, 0.7}	{0.7, 0.8}	{0.3, 0.5}	{0.3, 0.4}
v_3	{0.4}	{0.7, 0.8}	{0.8, 0.9}	{0.6, 0.8}	{0.6, 0.8}
v_4	{0.5, 0.6}	{0.4, 0.5}	{0.5, 0.7}	{0.6, 0.8}	{0.6, 0.8}
v_5	{0.7, 0.8}	{0.4, 0.5}	{0.8}	{0.6, 0.7}	{0.4, 0.6}

	m_1	m_2	m_3	m_4
v_1	{0.5, 0.6}	{0.3, 0.4}	{0.7, 0.8}	{0.6, 0.8}
v_2	{0.3, 0.5}	{0.5, 0.6}	{0.2, 0.4}	{0.3, 0.5}
v_3	{0.5, 0.7}	{0.2, 0.4}	{0.4, 0.6}	{0.4, 0.9}
v_4	{0.2, 0.3}	{0.5, 0.6}	{0.6, 0.7}	{0.4, 0.6}
v_5	{0.3, 0.4}	{0.6, 0.9}	{0.5, 0.6}	{0.2, 0.3}

Fig.1. Illustration of hesitant fuzzy soft set $(\tilde{\mathcal{A}}, \mathcal{K})$, $(\tilde{\mathcal{B}}, \mathcal{L})$, $(\tilde{\mathcal{C}}, \mathcal{M})$ in table format (Example 4)

Table 3. Illustration of hesitant fuzzy soft set $(\tilde{\mathcal{D}}, \mathcal{R})$ in table format

	r_1	r_2	r_3	r_4	r_5	r_6	r_7
v_1	{0.3, 0.4}	{0.3, 0.5}	{0.2, 0.3, 0.4}	{0.2, 0.4}	{0.3, 0.4, 0.5}	{0.3, 0.4}	{0.2, 0.3}
v_2	{0.2, 0.4}	{0.2, 0.3, 0.4}	{0.5, 0.6}	{0.3, 0.5}	{0.2, 0.4}	{0.3, 0.5}	{0.3, 0.5}
v_3	{0.4}	{04, 0.5}	{0.2, 0.4}	{0.4, 0.6, 0.8}	{0.4, 0.6}	{0.2, 0.4}	{0.2, 0.4}
v_4	{0.2, 0.3}	{0.4, 0.5, 0.6}	{0.3, 0.4, 0.5}	{0.3, 0.4, 0.5}	{0.5, 0.6, 0.7}	{0.5, 0.6}	{0.3, 0.5}
v_5	{0.3, 0.4}	{0.2, 0.3}	{0.4, 0.5}	{0.2, 0.3}	{0.3}	{0.3}	{0.4, 0.5}

$$\mathbb{K} = \frac{\{0.4\}}{r_1} + \frac{\{0.4{,}0.5{,}0.6\}}{r_2} + \frac{\{0.5{,}0.6\}}{r_3} + \frac{\{0.4{,}0.6{,}0.8\}}{r_4} + \frac{\{0.5{,}0.6{,}0.7\}}{r_5} + \frac{\{0.5{,}0.6\}}{r_6} + \frac{\{0.4{,}0.5\}}{r_7}$$

Step 3: In this step, soft hesitant fuzzy upper approximation $\overline{\mathcal{D}}(\mathbb{K})$ and soft hesitant fuzzy lower approximation $\underline{\mathcal{D}}(\mathbb{K})$ of optimal normal decision object $\mathbb{K}$ with respect to $(X, \mathcal{Q}, \tilde{\mathcal{D}}^{-1})$ are computed, as shown in Table 4.

Step 4: Score functions $scr(\overline{\mathcal{D}}(\mathbb{K})(v_i))$ and $scr(\underline{\mathcal{D}}(\mathbb{K})(v_i))$ of values $\overline{\mathcal{D}}(\mathbb{K})(v_i)$ and $\underline{\mathcal{D}}(\mathbb{K})(v_i)$ are then calculated, respectively, in accordance with Definition 3. For calculation, we make the lengths of all HFEs the same in Table 3. Then choice value σ_i is defined by adding $scr(\overline{\mathcal{D}}(\mathbb{K})(v_i))$ and $scr(\underline{\mathcal{D}}(\mathbb{K})(v_i))$, as shown in Table 4.

As shown in Table 4, service v_2 has the highest choice value. Therefore, the composite service maker will select it as the optimal service. It can also be observed since $\tilde{\mathcal{D}}^{-1}$ is an arbitrary relation; therefore, according to Theorem 3 and Definition 11, the condition $\underline{\mathcal{D}}(\mathbb{K}) \subseteq \overline{\mathcal{D}}(\mathbb{K})$ does not hold in Table 4.

C. Comparison with the existing solution for multi-criteria decision making based on hesitant fuzzy

In this section, we validate the effectiveness of SHFRS in multi-criteria decision making. For this, we compare our method of decision making with the other existing solution.

For comparison, Example 4 is solved with the help of the hesitant fuzzy soft set-based algorithm proposed by Fuqiang Wang et al. [23]. The Procedural steps of solution of Example 4 by using the hesitant fuzzy soft set-based algorithm are as follows:

Step 1: Since Fuqiang Wang et al. [23] did not discuss the problem of multi-observer hesitant fuzzy information, we consider the hesitant fuzzy soft set $(\tilde{\mathcal{D}}, \mathcal{R})$ shown in Table 3 to be an input hesitant fuzzy soft set.

Step 2: The induced fuzzy soft set $\Delta_{\tilde{\mathcal{D}}} = (\tilde{\mathcal{T}}, \mathcal{R})$ is computed as shown in Table 5.

Step 3: Input a threshold fuzzy set. Here, the threshold fuzzy set is chosen by applying the middle-level decision rule to $\Delta_{\tilde{\mathcal{D}}} = (\tilde{\mathcal{T}}, \mathcal{R})$. That is,

$$\acute{md}_{\Delta_{\tilde{\mathcal{D}}}} = \{(r_1, 0.35), (r_2, 0.394), (r_3, 0.416), (r_4, 0.406), (r_5, 0.472), (r_6, 0.40), (r_7, 0.386)\}.$$

Step 4: For $\acute{md}_{\Delta_{\tilde{\mathcal{D}}}}$, the mid-level soft set $M(\Delta_{\tilde{\mathcal{D}}}: \acute{md}_{\Delta_{\tilde{\mathcal{D}}}})$ is computed as shown in Table 6.

Step 5: In Table 6, the choice value σ_i of each service $v_i \in X$ is computed.

Step 6: The security service with the highest choice value is chosen as optimal security service. Nonetheless, it can be easily seen in Table 6 that there are multiple security services (v_2, v_3, v_4, v_5) for which the choice value is 4. Therefore, it is difficult to decide which security service is the optimal security

Table 4. The results of the decision algorithm (Example 4)

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	$\underline{\mathcal{D}}(\mathbb{K})$	$\overline{\mathcal{D}}(\mathbb{K})$	$scr(\underline{\mathcal{D}}(\mathbb{K}))$	$scr(\overline{\mathcal{D}}(\mathbb{K}))$	σ
v_1	{0.5, 0.6, 0.7}	{0.3, 0.4, 0.5}	0.60	0.40	1.00
v_2	{0.5, 0.6}	{0.5, 0.6}	0.566	0.566	1.132
v_3	{0.4, 0.5, 0.6}	{0.4, 0.6, 0.8}	0.50	0.60	1.10
v_4	{0.4, 0.5, 0.6}	{0.5, 0.6, 0.7}	0.50	0.60	1.10
v_5	{0.5, 0.6}	{0.4, 0.5}	0.566	0.466	1.032

service in universe X. As described in algorithm [23], however, in the case of similar highest choice value σ_i for two or more objects, any one of them can be randomly selected as the optimal decision object. Therefore, we select v_2 as optimal security service.

Let us consider Step 3 again and input another threshold fuzzy set. This time, the threshold fuzzy set is chosen by applying the top-level decision rule to $\Delta_{\tilde{\mathcal{D}}} = (\tilde{\mathcal{T}}, \mathcal{R})$. That is,

$$t\acute{p}_{\Delta_{\tilde{\mathcal{D}}}} = \{(r_1, 0.40), (r_2, 0.50), (r_3, 0.57), (r_4, 0.60), (r_5, 0.60), (r_6, 0.57), (r_7, 0.47)\}.$$

For $\acute{p}_{\Delta_{\tilde{\mathcal{D}}}}$, the top-level soft set $T(\Delta_{\tilde{\mathcal{D}}}: t\acute{p}_{\Delta_{\tilde{\mathcal{D}}}})$ is computed as shown in Table 7. It can be easily seen in Table 7 that only one security service, v_4, has highest choice value. Therefore, it is selected as the optimal security service.

From the procedure above, two results are obtained: one is v_2 (in the case of mid-level soft set), and the other is v_4 (in the case of top-level soft set). It shows that there is ambiguity in the decision results, and it cannot be decided whether object v_2 or object v_4 is optimal. We can easily conclude that the final result of the algorithm proposed by Fuqiang Wang et al. [23] is not unique and dependent on the threshold fuzzy set to a certain extent. Note, however, that our proposed solution to the multi-criteria decision making problem does not depend on any input threshold fuzzy set. Therefore, the final result of our solution is unique or free from ambiguity. A brief comparison of the proposed solution with the solution proposed by Fuqiang Wang et al. [23] is shown in Table 8.

Table 5. Illustration of induced fuzzy soft set $\Delta_{\tilde{\mathcal{D}}} = (\tilde{\mathcal{T}}, \mathcal{R})$ in table format (Example 4)

	r_1	r_2	r_3	r_4	r_5	r_6	r_7
v_1	0.37	0.43	0.30	0.33	0.40	0.37	0.27
v_2	0.33	0.30	0.57	0.43	0.33	0.43	0.43
v_3	0.40	0.47	0.34	0.60	0.53	0.33	0.33
v_4	0.27	0.50	0.40	0.40	0.60	0.57	0.43
v_5	0.37	0.27	0.47	0.27	0.50	0.30	0.47

Table 6. Illustration of mid-level soft set $M(\Delta_{\tilde{\mathcal{D}}}: m\acute{d}_{\Delta_{\tilde{\mathcal{D}}}})$ in table format (Example 4)

	r_1	r_2	r_3	r_4	r_5	r_6	r_7	Choice value
v_1	1	1	0	0	0	0	0	2
v_2	0	0	1	1	0	1	1	4
v_3	1	1	0	1	1	0	0	4
v_4	0	1	0	0	1	1	1	4
v_5	1	0	1	0	1	0	1	4

Table 7. Illustration of top-level soft set $T(\Delta_{\tilde{\mathcal{D}}}: t\acute{p}_{\Delta_{\tilde{\mathcal{D}}}})$ in table format (Example 4)

	r_1	r_2	r_3	r_4	r_5	r_6	r_7	Choice value
v_1	0	0	0	0	0	0	0	0
v_2	0	0	1	0	0	0	0	1
v_3	1	0	0	1	0	0	0	2
v_4	0	1	0	0	1	1	0	3
v_5	0	0	0	0	0	0	1	1

Table 8. A Comparative study of solutions for multi-criteria decision making based on hesitant fuzzy

Feature / Solution	Additional input required	Ambiguity in decision result	Ranking methodology	Multi-source aggregation (multi-observer hesitant fuzzy information)	Applied theory
Fuqiang Wang et al. [23]	Yes	Yes	Choice value, which is calculated by using level soft set.	Not discussed	Hesitant fuzzy soft set
Haidong Zhang et al. [31]	Yes	Yes	Choice value, which is calculated by using score function of upper and lower approximation.	Not discussed	Hesitant fuzzy rough set
Proposed solution	No	No	Choice value, which is calculated by using score function of upper and lower approximation.	AND as a min operator	SHFRS

V. Conclusion

In this paper, we studied the hesitant fuzzy set to solve the multi-criteria decision making problem of optimal security service selection for FMEC. This study has made three new contributions in the area of hesitant fuzzy theory and FMEC. First, we proposed an innovative extension of the hesitant fuzzy rough set theory by fusing it with the hesitant fuzzy soft set, which is known as SHFRS. Second, we introduced a novel solution to multi-criteria decision making problems based on our proposed SHFRS. Finally, the problem of selecting optimal security services for FMEC is solved by using SHFRS based multi-criteria decision making. A practical example of optimal security service selection for FMEC was given, showing the validity of the proposed SHFRS and its application to multi-criteria decision making problems. Our findings suggest that the proposed SHFRS-based multi-criteria decision making solutions can be used in FMEC as a selection module that selects the optimal security services among several available security services, and it can efficiently handle dynamically varying security services with the mobile user's requirements.

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