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A study on M/G/1 retrial G - queue with two phases of service, immediate feedback and working vacations

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Abstract. In this paper, we discuss about the steady state behaviour of M/G/1 retrial queueing system with two phases of services and immediate feedbacks under working vacation policy where the regular busy server is affected due to the arrival of negative customers. Upon arrival if the customer finds the server busy, breakdown or on working vacation it enters an orbit; otherwise the customer enters into the service area immediately. After service completion, the customer is allowed to make finite number of immediate feedback. The feedback service also consists of two phases. At the service completion epoch of a positive customer, if the orbit is empty the server goes for a working vacation. The server works at a lower service rate during working vacation (WV) period. Using the supplementary variable technique, we found out the steady state probability generating function for the system and in orbit. System performance measures and reliability measures are discussed. Finally, some numerical examples are presented to validate the analyticalresults.

1. Introduction

In queueing theory, there have been significant contributions to retrial queues and vacation queues. From Ke et al. [12] andArtalejo and Gomex-Corral [3], we can find general models and results in retrial and vacation queues. Retrial queues are characterized by the fact that an arriving customer, who finds the server busyjoins the retrial group and request for service after some time. Such queues have wide applications which are used to model many problems in telephone switching systems, telecommunication network and retail shopping queues. For more reference, see bibliography on retail queues in Artalejo [2].

The concept of arrival of positive and negative customers in the queueing system created more interest and it has been studied by many due to its application in industries, computers, manufacturing and network systems. Such queues (G-Queues) were first introduced by Gelenbe [8] to model neural networks. The positive customers arrive and receive service in regular manner into the system, whereas the arrival of negative customers neither receive service nor join the orbit instead they destroy and remove the positive customer in service and cause breakdown to the server and service channel will fail for a short interval of time, also this customer arrive only at the service time of the positive customers. Krishnakumar et al. [13], Gao and Wang [7], Peng et al. [17], Wu and Lian. [27], Rajadurai et al. [18,20], have discussed about retrial queueing models with the presence of G-queues. The readers may refer the bibliography paper on queueing system with G-networks by Do [5] for detailed analysis.

One of the additional features that have been discussed widely in retrial queueing literature is the feedback customers. If the customer is not satisfied with the service, then the service can be provided repeatedly to the customer for many reasons. For instance, messages are sent from the source to the destination via packets. The packets are transmitted through router. The packets that are transmitted may be returned and it can be re-transmitted repeatedly until it reaches the desired destination. This type of retransmission is called feedback. Ke and Chang [11], Krishnakumar et al. [13], Varalakshmi et al. [24] have discussed the concept of feedback where the customer joins the tail of the queue to receive feedback service. Kalidas and Kasturi [10] provided a different approach to this aspect, that is the customer who desires to receive another service, directly enters in to the service station without being waiting in the queue. This concept is referred as immediate feedback. Varalakshmi et al. [25] analyzed a single server retrial queue with two phase service and server breakdown where the customer can get finite number of immediate feedback under Bernoulli schedule.

A considerable amount of work has been done in the past in queueing system models with vacations and is successfully implemented in various analysis problems such as production systems, computer systems and communication systems. During vacation the server completely stops providing service to the customer and the server simply takes a break like being checked for maintenance. During working vacation, the server gives service to the customer but in lower rate. This queueing system has subsequent application in the field of mailing service, file transfer and network service. M/M/1 working vacation queue has been introduced by Servi and Finn [23]. This was extended to M/G/1 working vacation queue by Wu and Takagi [26]. Further Arivudainambi et al. [1] analyzed an M/G/1 retrial queue with single working vacation policy. Furthermore at the end of lower speed service rate, if there are customers in the system, the server can end the vacation and come back to normal busy service state. This concept is referred as vacation interruption. Some of the authors like Rajadurai et al. [19, 21], Gao et al. [6], Li and Song [14], Liu and Tian [15], Zhang and Hou [28] have analyzed queueing models with working vacations and vacation interruptions. For recent survey and literature on working vacation queuing models the readers may refer Chandrasekaran et al. [4]

In this paper, we extended the work of Arivudainambiet al. [1] and Zhang and Liu [29] by incorporating the concept of G-queues with immediate feedback. So far, very few authors have studied the concept of retrial queue with negative arrival, working vacation and interruption. This motivates the author's to develop two phase service retrial G-queue with immediate feedback and working vacation policy by using supplementary variable technique though moost work with the concept of retrial queue and working vacation are by using matrix geometry analysis.

The rest of the paper is as follows. The mathematical model considered is described and practical justification of our model is given in Section 2. In section 3, the stability condition for our model is analysed. In section 4, the steady-state joint distribution of the server state and the number of customers in the system and in orbit are obtained. System performance measures and reliability measures are obtained in Section 5. Some special cases are considered in Section 6. In Section 7, the analytical results are validated numerically on the system performance for various paramaters. Finally, Section 8 concludes the paper.

2. Model description

2.1. Description

In this section, we consider a single server retrial queueing system with two phases of service and immediate feedback under working vacation policy where the regular busy server is subjected to breakdown due to the arrival of negative customers.

2.1.1. Arrival Pattern. The positive customers arrive into the system according to a Poisson process with rate λ.

2.1.2. Retrial rule. If an arriving positive customer finds the server idle, then the customers enter in the service station immediately to begin his service. Otherwise, the customer will join the group of blocked customers called orbit and request for service repeatedly till he finds the server free.

2.1.3. The two phase of service. A single server provides two phases of service for all the positive customers (or retrial). As soon as the first phase of service (*FPS*) the customer receives second phase of service (*SPS*). The service time follows general distribution in both the phases with its random variable $S_{i,b}$, distribution function $S_{i,b}(t)$ and LST $S_{i,b}^{*}(v)$.And the first moment is given by $S_{i,b}^{*'}(\delta) = \int_0^\infty xe^{-\delta x}dS_{i,b}(x)$where i=1,2.

*2.1.4. .The removal rule.*The arrival of negative customers into the system follows Poisson process with rate δ. These customers arrive only during the regular service time of the positive customers. The arrival of negative customers will vanish if the server is idle or repair or on working vacation. Their arrival will remove the positive customers being in service forever and causes breakdown to the server. As soon as breakdown occurs, the server is sent for repair immediately.

2.1.5. The repair time. After the repair process, the server is treated as good as new. The servers repair time (G) is assumed to be arbitrarily distributed with the distribution function $G(t)$ and LST $G^*(v)$.

2.1.6. Immediate feedback. At the end of regular service completion of each customer, if the customer is not satisfied with the service, the customer can get second round of service by entering into FPS with probability $\alpha_1$or leave(go out) the system with probability $1\text{-}\alpha_1$. After completion of feedback service, the customer may again go in for a third round of service by entering in to *FPS* followed by *SPS* with probability α_2 or leave the system with the probability $1\text{-}\alpha_2$. This can continue till the customer obtains m rounds of service after which the customer must leave the system. The next customer in the orbit gets into service only if the preceding customer completes all the feedback rounds successfully.

2.1.7. Working vacation process. At the service completion epoch, if the orbit becomes empty the server goes for a working vacation and the vacation time follows exponential distribution with the parameter θ. During WV period, the server provides service at lower rate if any customers arrive. At a service completion epoch in the vacation period if any customers are seen in the orbit, the server will stop the vacation and comes back to the normal busy period which is referred as vacation interruption. Otherwise, the server continues the vacation. At the completion epoch of the vacation, if any customer is found in the orbit, the server switches to regular service else, the server goes for another vacation. The service time during working vacation follows general random variable S_v with distribution function$S_v(t)$; LST$S_v^*(x)$ and the first moment is given by$S_v^{*'}(\theta) = \int_0^\infty xe^{-\theta x}dS_v(x)$. During working vacation, the server provides single phase service and no feedback is allowed.

The inter arrival time, retrial time, service time, working vacation time and repair time are assumed to be independent of each other.

2.2. Practical Justification of the proposed model

This model has a potential application in the area of computer processing system. In a computer processing system, the messages (customers) are sent to the destination through processor (server) for processing and if the processor is busy the messagesare stored in buffer (orbit). The working mail server may be damaged by virus (negative customers), andthe messages are lost at the node. After completion of message processing, the internet service may demand the same service to the processor (the immediate feedback) for finite number of times, if any failures occur in pervious process. To improve the performance of computer, if all the messages are processed and no new messages are seen, the processor startsto perform some maintenance jobs, such as virus scan (working vacations). During this period, if any messages arrive, the processor processes it in slower rate to economize the cost. At the end of completion of maintenance, the processor checks for the messages; if no message is in the system then the processor may decide to go for another maintenance work (multiple working vacations).Other applications are in SimpleMail Transfer Protocol (SMTP) mail system and telephone consultation of medical service systems.

3. Stability condition

The necessary and sufficient condition for the stability of the system is carried out in this section. We analyze the embedded Markov chain's ergodicity at departure or vacation or repair epoch.
In the steady state, we assume that $R(0)=0$, $R(\infty)=1$, $S_{i,b}(0)=0$, $S_{i,b}(\infty)=1$(where $i=1,2$), $S_v(0)=0$, $S_v(\infty)=1$), $G(0)=0$, $G(\infty)=1$ are continuous at $x = 0$. So that the function $a(x),\ \mu_{i,b}(x),\ \mu_v(x)$ *and* $\xi(x)$ are the conditional completion rates for repeated attempts, regular service (on FPS and SPS), lower speed service and repair respectively (for $i = 1, 2$).

$$a(x)dx = \frac{dR(x)}{1-R(x)},\ \mu_{i,b}(x)dx = \frac{dS_{i,b}(x)}{1-S_{i,b}(x)},\ \mu_v(x)dx = \frac{dS_v(x)}{1-S_v(x)},\ \xi(x)dx = \frac{dG(x)}{1-G(x)}.$$

Further, Let $R^0(t), S^0_{1,b}(t), S^0_{2,b}(t), S^0_v(t), G^0(t)$ be the elapsed retrial time, elapsed regular service time in *FPS*, elapsed regular service time in *SPS*, elapsed working vacation time and elapsed repair time respectively at time t. we introduce the random variables,

$$U(t)=\begin{cases} 0, & \text{if the server is idle during working vaction period,} \\ 1, & \text{if the server is idle during regular busy period,} \\ 2, & \text{if the server is busy and in regular busy period in FPS at time t,} \\ 3, & \text{if the server is busy and in regular busy period in SPS at time t,} \\ 4, & \text{if the server is busy during working vacation period at time t,} \\ 5, & \text{if the server is under repair at time t.} \end{cases}$$

Thus, the system state at time t can be expressed by means of bivariate Markov process $\{U(t), N(t); t \geq 0\}$ where $U(t)$ denotes the state of the server (0,1,2,3,4,5) depending on the server is idle on normal busy and working vacation period, busy on regular service in *FPS,SPS* and working vacation period and under repair respectively. $N(t)$ corresponds to the number of customers in the orbit at time t. If $U(t)=1$ and $N(t)>0$, then $R^0(t)$ represent the elapsed retrial time, if $U(t)=2$ and $N(t)\geq 0$ then $S^0_{1,b}(t)$ corresponds to the elapsed service time of the customer being served in *FPS* during regular busy period. If $U(t)=3,$ and $N(t)\geq 0$ then $S^0_{2,b}(t)$ corresponds to the elapsed service time of the customer being served in *SPS* during regular busy period. If $U(t)=4,$ and $N(t)\geq 0$, then $S^0_v(t)$ represents the elapsed time of the customer being served in lower rate service period. If $U(t)=5,$ and $N(t)\geq 0$, then $G^0(t)$ represents elapsed time of the server being repaired.

Let $\{t_n;\ n\in N\}$ be a sequence at which either service completion (normal or working vacation) times or repair termination times. Then the sequence of random variables $Y_n = \{C(t_n+), N(t_n+)\}$ forms an embedded Markov chain for the queueing system.

Theorem 3.1. The embedded Markov chain $\{Y_n;\ n\in N\}$ is ergodic if and only if $\rho < R^*(\lambda)$ for our system to be stable, where $\rho = \frac{\lambda}{\delta}\left(1-\Phi(1)+\delta\left(1-H_b(1)\right)\Phi_1(1)E(G)\right)$

Proof. To prove the sufficient condition of ergodicity, we use Foster's criterion (see Pakes [16]) which states that an irreducible and aperiodic Markov chain $\{Y_n;\ n\in N\}$ is an is ergodic if there exists a nonnegative function $g(k), k\in N$ and ε > 0, such that mean drift $\psi_k = E[g(z_{n+1}) - g(z_n) / z_n = k]$ is finite for all $k\in N$ and $\psi_k \le -\varepsilon$ for all $k\in N$, except perhaps for a finite number k's. In our case, we consider the function $f(k) = k$. Then we have

$$\psi_k = \begin{cases} \rho - 1, & k = 0, \\ \rho - R^*(\lambda), & k = 1, 2... \end{cases}$$

Obviously the inequality $\rho < R^*(\lambda)$ is the sufficient condition for Ergodicity.

To examine the necessary condition as noted in Sennot et al. [22], if the Markov chain $\{Y_n;\ n \ge 1\}$ satisfies Kaplan's condition, namely, $\psi_k<\infty$ for all$k\ge 0$ and there exits $k_0\in$ N such that $\psi_k\ge 0$ for $k\ge k_0$. Notice that, in our case, Kaplan's condition is satisfied because there is a j such that $m_{ik}= 0$ for$k<i -j$ and $i> 0$, where $M = (m_{kj})$ is the one step transition matrix of $\{Z_n;\ n\in N\}$. Then, $\rho \ge R^*(\lambda)$ implies the non-ergodicity of the Markov chain.□

4. Steady state probability analysis

The steady state distribution of the system under consideration is studied in this section.

4.1. The steady state equations

We assume that the stability condition is fulfilled. For the process $\{N(t),\ t\ge 0\}$, we define the limiting probabilities $P_0(t) = P\{C(t)=0,\ N(t)=0\}$ and the limiting probability densities (for $0 \le j \le m$-1)

$$\psi_n(x,t)dx = P\Big\{C(t)=1,\ N(t)=n,\ x \le R^0(t) < x+dx\Big\},\ \text{for } t\ge 0,\ x\ge 0 \text{ and } n\ge 1,$$

$$Q_{j_b,n}(x,t)dx = P\Big\{C(t)=2, N(t)=n,\ x \le S^0_{1,b}(t) < x+dx\Big\},\ \text{for } t\ge 0,\ x\ge 0 \text{ and } n\ge 0,$$

$$P_{j_b,n}(x,t)dx = P\Big\{C(t)=3,\ N(t)=n,\ x \le S^0_{2,b}(t) < x+dx\Big\},\ \text{for } t\ge 0,\ x\ge 0 \text{ and } n\ge 0,$$

$$\Omega_{v,n}(x,t)dx = P\Big\{C(t)=4,\ N(t)=n,\ x \le S^0_v(t) < x+dx\Big\},\ \text{for } t\ge 0,\ x\ge 0 \text{ and } n\ge 0,$$

$$R_n(x,t)dx = P\Big\{C(t)=5,\ N(t)=n,\ x \le G^0(t) < x+dx\Big\},\text{for } t\ge 0,\ x\ge 0 \text{ and } n\ge 0.$$

We assume that the stability condition is fulfilled in the sequel and so that we can set $t\ge 0$, $x\ge 0$, (for $0 \le j \le m$-1)

$$P_0 = \lim_{t\to\infty} P_0(t),\ \psi_n(x) = \lim_{t\to\infty}\psi_n(x,t),\ Q_{j_b,n}(x) = \lim_{t\to\infty} Q_{j_b,n}(x,t),\ P_{j_b,n}(x) = \lim_{t\to\infty} P_{j_b,n}(x,t),\ \Omega_{v,n}(x) = \lim_{t\to\infty}\Omega_{v,n}(x,t),$$
$$R_n(x) = \lim_{t\to\infty} R_n(x,t).$$

By the method of supplementary variable technique, we easily obtain the system of governing equations of this model (for $0\le j\le m$-1 and i=1,2).

$$(\lambda+\theta)P_0 = \left[\sum_{l=0}^{m-2}\bar{\alpha}_{l+1}\int_0^\infty P_{l_b,0}(x)\mu_{2,b}(x)dx + \int_0^\infty P_{m-1_b,0}(x)\mu_{2,b}(x)dx\right] + \int_0^\infty R_0(x)\xi(x)dx + \int_0^\infty \Omega_{v,0}(x)\mu_v(x)dx + \theta P_0 \quad (1)$$

$$\frac{d\psi_n(x)}{dx} + \big(\lambda + a(x)\big)\psi_n(x) = 0,\ n\ge 1 \quad (2)$$

$$\frac{dQ_{j_b,n}(x)}{dx} + \big(\lambda + \mu_{1,b}(x) + \delta\big)Q_{j_b,n}(x) = \lambda Q_{j_b,n-1}(x) \quad (3)$$

$$\frac{dP_{j_b,n}(x)}{dx} + \big(\lambda + \mu_{2,b}(x) + \delta\big)P_{j_b,n}(x) = \lambda P_{j_b,n-1}(x) \quad (4)$$

$$\frac{d\Omega_{v,n}(x)}{dx} + \big(\lambda + \mu_v(x) + \theta\big)\Omega_{v,n}(x) = \lambda\Omega_{v,n-1}(x) \quad (5)$$

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$$\frac{dR_n(x)}{dx}+\left(\lambda+\xi(x)\right)R_n(x) = \lambda R_{n-1}(x)\ n\geq 1 \tag{6}$$

The steady state boundary conditions at *x=0* are

$$\psi_n(0) = \left[\sum_{l=0}^{m-2}\bar{\alpha}_{l+1}\int_0^\infty P_{l_b,n}(x)\mu_{2,b}(x)dx+\int_0^\infty P_{m-1_b,n}(x)\mu_{2,b}(x)dx\right]+\int_0^\infty \Omega_{v,n}(x)\mu_v(x)dx+\int_0^\infty R_n(x)\xi(x)dx, \tag{7}$$

$$Q_{0_b,n}(0) = \int_0^\infty \psi_{n+1}(x)a(x)dx+\lambda\int_0^\infty \psi_n(x)dx+\theta\int_0^\infty \Omega_{v,n}(x)dx \tag{8}$$

$$Q_{j_b,n}(0) = \alpha_j\int_0^\infty P_{j-1_b,n}(x)\mu_{2,b}(x)dx,\ \mathrm{j}=1,2,3,...,m-1 \tag{9}$$

$$P_{j_b,n}(0) = \int_0^\infty Q_{j_b,n}(x)\mu_{1,b}(x)dx,\mathrm{j=0,1,2,...,m\text{-}1} \tag{10}$$

$$\Omega_{v,n}(0) = \begin{Bmatrix}\lambda P_0,\ n=0\\ 0,\ n\geq 1\end{Bmatrix} \tag{11}$$

$$R_n(0) = \delta\left[\int_0^\infty Q_{j_b,n}(x)dx+\int_0^\infty P_{j_b,n}(x)dx\right] \tag{12}$$

The normalizing condition is

$$P_0+\sum_{n=1}^\infty\int_0^\infty \psi_n(x)dx+\sum_{n=0}^\infty\left(\sum_{j=0}^{m-1}\left(\int_0^\infty Q_{j_b,n}(x)dx+\int_0^\infty P_{j_b,n}(x)dx\right)+\int_0^\infty \Omega_{v,n}(x)dx+\int_0^\infty R_n(x)dx\right)=1 \tag{13}$$

4.2. The steady state solution of this model

The probability generating function technique has been employed to obtain the steady state solution for the queueing model considered. In order to solve the system equations, let us define the following generating functions for $|z|\leq 1$, (for $0\leq j\leq m$-1).

$$\psi(x,z)=\sum_{n=1}^\infty \psi_n(x)z^n\ ;\ \psi(0,z)=\sum_{n=1}^\infty \psi_n(0)z^n\ ;\ Q_{j_b}(x,z)=\sum_{n=0}^\infty Q_{j_b,n}(x)z^n;\ Q_{j_b}(0,z)=\sum_{n=0}^\infty Q_{j_b,n}(0)z^n; P_{j_b}(x,z)=\sum_{n=0}^\infty P_{j_b,n}(x)z^n;$$

$$P_{j_b}(0,z)=\sum_{n=0}^\infty P_{j_b,n}(0)z^n;\ \Omega_v(x,z)=\sum_{n=0}^\infty \Omega_{v,n}(x)z^n\ ;\ \Omega_v(0,z)=\sum_{n=0}^\infty \Omega_{v,n}(0)z^n\ ;R(x,z)=\sum_{n=0}^\infty R_n(x)z^n\ ;\ R(0,z)=\sum_{n=0}^\infty R_n(0)z^n$$

On multiplying equations (1)-(12) by $z^n$and summing over *n*, (*n* = 0,1,2...) (for $0\leq j\leq m$-1 and *i=1,2*). We obtain the following equations

$$\frac{\partial\psi(x,z)}{\partial x}+\left(\lambda+a(x)\right)\psi(x,z) = 0 \tag{14}$$

$$\frac{\partial Q_{j_b}(x,z)}{\partial x}+\left(\lambda(1-z)+\mu_{1,b}(x)+\delta\right)Q_{j_b}(x,z) = 0 \tag{15}$$

$$\frac{\partial P_{j_b}(x,z)}{\partial x}+\left(\lambda(1-z)+\mu_{2,b}(x)+\delta\right)P_{j_b}(x,z) = 0 \tag{16}$$

$$\frac{\partial\Omega_v(x,z)}{\partial x}+\left(\lambda(1-z)+\mu_v(x)+\theta\right)\Omega_v(x,z) = 0 \tag{17}$$

$$\frac{\partial R(x,z)}{\partial x}+\left(\lambda(1-z)+\xi(x)\right)R(x,z) = 0 \tag{18}$$

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$$\psi(0,z) = \sum_{l=0}^{m-2} \overline{\alpha_{l+1}} \int_0^\infty P_{l_b}(x,z)\mu_{2,b}(x)dx + \int_0^\infty P_{m-1_b}(x,z)\mu_{2,b}(x)dx + \int_0^\infty \Omega_v(x,z)\mu_v(x)dx + \int_0^\infty R(x,z)\xi(x)dx - \lambda P_0 \quad (19)$$

$$Q_{0_b}(0,z) = \frac{1}{z}\int_0^\infty \psi(x,z)a(x)dx + \lambda\int_0^\infty \psi(x,z)dx + \theta\int_0^\infty \Omega_v(x,z)dx \quad (20)$$

$$Q_{j_b}(0,z) = \alpha_j \int_0^\infty P_{j-1_c}(x,z)\mu_{2,b}(x)dx \quad (21)$$

$$P_{j_b}(0,z) = \int_0^\infty Q_{j_b}(x,z)\mu_{1,b}(x)dx \quad (22)$$

$$\Omega_v(0,z) = \lambda P_0 \quad (23)$$

$$R(0,z) = \delta\left[\int_0^\infty Q_{j_b}(x,z)dx + \int_0^\infty P_{j_b}(x,z)dx\right] \quad (24)$$

The solutions of the equations (14)-(18),is given by

$$\psi(x,z) = \psi(0,z)[1-R(x)]e^{-\lambda x} \quad (25)$$

$$Q_{j_b}(x,z) = Q_{j_b}(0,z)[1-S_{1,b}(x)]e^{-A_b(z)x} \quad (26)$$

$$P_{j_b}(x,z) = P_{j_b}(0,z)[1-S_{2,b}(x)]e^{-A_b(z)x} \quad (27)$$

$$\Omega_v(x,z) = \Omega_v(0,z)[1-S_v(x)]e^{-A_v(z)x} \quad (28)$$

$$R(x,z) = R(0,z)[1-G(x)]e^{-b(z)x} \quad (29)$$

where $A_b(z) = \delta + b(z), A_v(z) = \theta + b(z), b(z) = \lambda(1-z)$

Inserting equations (25) and (28) in equation (20) and solving, we get,

$$Q_{0_b}(0,z) = \frac{R(z)}{z}\psi(0,z) + \lambda P_0 V(z) \quad (30)$$

where $R(z) = R^*(\lambda) + z\left(1-R^*(\lambda)\right)$ and $V(z) = \frac{\theta\left(1-S_v^*\left(A_v(z)\right)\right)}{A_v(z)}$

Substituting equation (22) in (21) and after some simplification we get,

$$Q_{j_b}(0,z) = \left(\prod_{j=0}^{m-1}\alpha_j\right)\left(H_b(z)\right)^{m-1} Q_{0_b}(0,z) \quad (31)$$

Using the equations (26), (27) in (21)-(22) and substituting in equation (24), we get

$$R(0,z) = \delta\sum_{j=0}^{m-1}\frac{\left(1-H_b(z)\right)}{A_b(z)}Q_{j_b}(0,z) \quad (32)$$

Inserting equations (26)-(29), (30), (32) in (19) and making some manipulation, we finally obtain

$$\psi(0,z) = \frac{\lambda z P_0}{Dr(z)}\left\{\begin{array}{l}\left[\Phi(z)A_b(z) + \delta(1-H_b(z))\Phi_1(z)G^*(b(z))\right]V(z) \\ +A_b(z)\left(S_v^*(A_v(z))-1\right)\end{array}\right\} \quad (33)$$

where $Dr(z) = zA_b(z) - \left[\Phi(z)A_b(z) + \delta(1-H_b(z))\Phi_1(z)G^*(b(z))\right]R(z)$

$$H_b(z) = S_{1,b}^*(A_b(z))S_{2,b}^*(A_b(z))$$

$$\Phi(z) = \bar{\alpha}_1 H_b(z) + \sum_{l=1}^{m-1} \bar{\alpha}_{l+1}(\alpha_1\alpha_2...\alpha_l)\left(H_b(z)\right)^{l+1}$$

$$\Phi_1(z) = 1 + \alpha_1 H_b(z) + \alpha_1\alpha_2\left(H_b(z)\right)^2 + ... + (\alpha_1\alpha_2...\alpha_{m-1})\left(H_b(z)\right)^{m-1}$$

Using the equation (33) in (30) and (31), we get

$$Q_{j_b}(0,z) = \left(\prod_{j=0}^{m-1}\alpha_j\right)\left(H_b(z)\right)^j \frac{\lambda P_0 A_b(z)\Big(R(z)\big(S_v^*(A_v(z))-1\big) + zV(z)\Big)}{Dr(z)} \tag{34}$$

Using the equations (22), (26)and (34,) we get

$$P_{j_b}(0,z) = \left(\prod_{j=0}^{m-1}\alpha_j\right)\left(H_b(z)\right)^j \frac{\lambda P_0 A_b(z)\Big(R(z)\big(S_v^*(A_v(z))-1\big) + zV(z)\Big)}{Dr(z)} S_{1,b}^*(A_b(z)) \tag{35}$$

Inserting equation (34) in (32), we get

$$R(0,z) = \delta\sum_{j=0}^{m-1}\left(1-H_b(z)\right)\left(\prod_{j=0}^{m-1}\alpha_j\right)\left(H_b(z)\right)^j \frac{\lambda P_0\Big(R(z)\big(S_v^*(A_v(z))-1\big) + zV(z)\Big)}{Dr(z)} \tag{36}$$

Using the equations (23) and (33)-(36) in (25)-(29), we get the following limiting probability generating functions results $\psi(x,z), Q_{j_b}(x,z), P_{j_b}(x,z), \Omega_v(x,z), R(x,z)$.

Theorem 4.1. The marginal probability distributions of the number of customers in the orbit when the server is idle, busy on *FPS*, busy on *SPS*, on working vacation and under repair are given by

$$\psi(z) = zP_0\left(1-R^*(\lambda)\right)Nr(z)\Big/Dr(z) \tag{37}$$

where $Nr(z) = I(z)V(z) + A_b(z)\left(S_v^*\left(A_v(z)\right)-1\right)$

$$Dr(z) = zA_b(z) - I(z)R(z)$$

$$I(z) = \Phi(z)A_b(z) + \delta(1-H_b(z))\Phi_1(z)G^*(b(z))$$

$$Q_{j_b}(z) = \lambda P_0\left(\prod_{j=0}^{m-1}\alpha_j\right)\left(H_b(z)\right)^j\left[zV(z) + R(z)\left(S_v^*\left(A_v(z)\right)-1\right)\right]\left(1-S_{1,b}^*\left(A_b(z)\right)\right)\Big/Dr(z) \tag{38}$$

$$P_{j_b}(z) = \lambda P_0\left(\prod_{j=0}^{m-1}\alpha_j\right)\left(H_b(z)\right)^j\left[zV(z) + R(z)\left(S_v^*\left(A_v(z)\right)-1\right)\right]S_{1,b}^*\left(A_b(z)\right)\left(1-S_{2,b}^*\left(A_b(z)\right)\right)\Big/Dr(z) \tag{39}$$

$$\Omega_v(z) = \frac{\lambda P_0 V(z)}{\theta} \tag{40}$$

$$R(z) = \sum_{j=0}^{m-1}\frac{\delta\lambda P_0\left(\prod_{j=0}^{m-1}\alpha_j\right)(1-H_b(z))\left(H_b(z)\right)^j\left[zV(z) + R(z)\left(S_v^*\left(A_v(z)\right)-1\right)\right]\left(1-G^*\left(b(z)\right)\right)}{Dr(z)b(z)} \tag{41}$$

where $$\mathrm{P}_0 = \frac{R^*(\lambda)-\rho}{\lambda/\delta\left(1-S_v^*(\theta)\right)\left[\frac{\delta}{\theta} + \omega\left(1-\frac{\lambda}{\theta} - R^*(\lambda)\right) + \tau\right] + R^*(\lambda) - \rho}$$

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$$\rho = \frac{\lambda}{\delta}\left(1-\Phi(1)+\delta\left(1-H_b(1)\right)\Phi_1(1)E(G)\right);\omega = 1-\Phi(1)+\delta\left(1-H_b(1)\right)\Phi_1(1)E(G)$$
$$\tau = \sum_{j=0}^{m-1}\left(1-H_b(1)\right)\left(\prod_{j=0}^{m-1}\alpha_j\right)\left(H_b(1)\right)^j\left(\frac{\lambda}{\theta}+R^*(\lambda)\right)\left(1+\delta E(G)\right)$$

Proof. Integrating the equations $\psi(x,z), Q_{j_b}(x,z), P_{j_b}(x,z), \Omega_v(x,z), R(x,z)$ with respect to x, we define the PGF's as

$$\psi(z)=\int_0^\infty \psi(x,z)dx; Q_{j_b}(z)=\int_0^\infty Q_{j_b}(x,z)dx; P_{j_b}(z)=\int_0^\infty P_{j_b}(x,z)dx; \Omega_v(z)=\int_0^\infty \Omega_v(x,z)dx; R(z)=\int_0^\infty R(x,z)dx;$$

To determine the probability that the server is idle (P_0) we use the normalizing condition. Thus by letting $z=1$ in (37)-(41) and applying l'Hopital's rule whenever necessary we get

$$P_0+\psi(1)+\Omega_v(1)+R(1)+\sum_{j=0}^{m-1}\left(Q_{j_b}(1)+P_{j_b}(1)\right)=1. \ \square$$

Theorem 4.2 The PGF of number of customers in the system and orbit size at stationary point of time is

$$S(z) = \frac{P_0\left[Nr_s(z)\right]}{b(z)\times Dr(z)} \tag{42}$$

and
$$O(z) = \frac{P_0\left[Nr_0(z)\right]}{b(z)\times Dr(z)} \tag{43}$$

where

$$Nr_s(z)=\left\{\begin{array}{l} b(z)\left\{\begin{array}{l}\left(zA_b(z)-I(z)R^*(\lambda)\right)(1+\frac{\lambda V(z)}{\theta}) \\ +z(1-R^*(\lambda))\left(I(z)\left[V(z)\left(1-\frac{\lambda}{\theta}\right)-1\right]+A_b(z)\left(S_v^*(A_v(z))-1\right)\right)\end{array}\right\} \\ +\left[\sum_{j=0}^{m-1}\lambda\left(\prod_{j=0}^{m-1}\alpha_j\right)(1-H_b(z))\left(H_b(z)\right)^j\times\left[zV(z)+R(z)\left(S_v^*\left(A_v(z)\right)-1\right)\right]\left(\delta(1-G^*(b(z)))+b(z)\right)\right]\end{array}\right\}$$

$$Dr(z)= zA_b(z)-\left[\Phi(z)A_b(z)+\delta(1-H_b(z))\Phi_1(z)G^*(b(z))\right]R(z)$$

$$Nr_o(z)=\left\{\begin{array}{l} b(z)\left\{\begin{array}{l}\left(zA_b(z)-I(z)R^*(\lambda)\right)(1+\frac{\lambda V(z)}{\theta}) \\ +z(1-R^*(\lambda))\left(I(z)\left[V(z)\left(1-\frac{\lambda}{\theta}\right)-1\right]+A_b(z)\left(S_v^*(A_v(z))-1\right)\right)\end{array}\right\} \\ +\left[\sum_{j=0}^{m-1}\lambda\left(\prod_{j=0}^{m-1}\alpha_j\right)(1-H_b(z))\left(H_b(z)\right)^j\times\left[zV(z)+R(z)\left(S_v^*\left(A_v(z)\right)-1\right)\right]\left(\delta(1-G^*(b(z)))+zb(z)\right)\right]\end{array}\right\}$$

Proof. The PGF of the number of customers in the system *S(z)* and in the orbit *O(z)* are obtained by

$$S(z)=P_0+\psi(z)+\Omega_v(z)+R(z)+\sum_{j=0}^{m-1}z\left(Q_{j_b}(z)+P_{j_b}(z)\right) \quad O(z)=P_0+\psi(z)+\Omega_v(z)+R(z)+\sum_{j=0}^{m-1}\left(Q_{j_b}(z)+P_{j_b}(z)\right)$$

substituting the equations (37) – (41) in the above results then the equations (42) and (43) can be deduced by direct calculation. □

5. System performance measures

In this section, the steady state probabilities when the system is in different states, the mean orbit (L_q)/system (L_s) size and reliability analysis are analysed respectively.

5.1. Mean system size and orbit size

If the system is in steady state condition

i. The mean orbit size (L_q) can obtained by differentiating equation (43) with respect to z and evaluating at $z = 1$

$$L_q = \lim_{z\to 1}\frac{d}{dz}O(z) = \frac{P_0}{V^*(\lambda)}\left[\frac{Nr_o'''(1)Dr''(1) - Nr_o''(1)Dr'''(1)}{3\left(Dr''(1)\right)^2}\right]$$

ii. The mean system size (L_s) can obtained by differentiating equation (42) with respect to z and evaluating at $z = 1$

$$L_s = \lim_{z\to 1}\frac{d}{dz}S(z) = \frac{P_0}{V^*(\lambda)}\left[\frac{Nr_s'''(1)Dr''(1) - Nr_s''(1)Dr'''(1)}{3\left(Dr''(1)\right)^2}\right]$$

5.2. System state probabilities

From equations (37)- (41) by letting $z=1$ and applying l'Hopital's rule whenever necessary, we get

i. The steady state probability that the server is idle during the retrial time

$$\psi = \psi(1) = \frac{\lambda\left(1-R^*(\lambda)\right)\left(1-S_v^*(\theta)\right)\left(1-\Phi(1)+\delta\left(1-H_b(1)\right)\Phi_1(1)E(G)+\dfrac{\delta}{\theta}\right)}{\delta R^*(\lambda)-\lambda\left(1-\Phi(1)+\delta\left(1-H_b(1)\right)\Phi_1(1)E(G)\right)}$$

ii. The probability that the server is normal busy in *FPS*

$$Q_{j_b} = Q_{j_b}(1) = \frac{\lambda P_0\left(\prod_{j-0}^{m-1}\alpha_j\right)\left(H_b(1)\right)^j\left(1-S_v^*(\theta)\right)\left(1-S_{1,b}^*(\delta)\right)\left(\dfrac{\lambda}{\theta}+R^*(\lambda)\right)}{\delta R^*(\lambda)-\lambda\left(1-\Phi(1)+\delta\left(1-H_b(1)\right)\Phi_1(1)E(G)\right)}$$

iii. The probability that the server is normal busy in *SPS*

$$P_{j_b} = P_{j_b}(1) = \frac{\lambda P_0\left(\prod_{j-0}^{m-1}\alpha_j\right)\left(H_b(1)\right)^j\left(1-S_v^*(\theta)\right)\left(S_{1,b}^*(\delta)\right)\left(1-S_{2,b}^*(\delta)\right)\left(\dfrac{\lambda}{\theta}+R^*(\lambda)\right)}{\delta R^*(\lambda)-\lambda\left(1-\Phi(1)+\delta\left(1-H_b(1)\right)\Phi_1(1)E(G)\right)}$$

iv. The probability that the server is on working vacation is

$$\Omega_v = \Omega_v(1) = \frac{\lambda P_0\left(1-S_v^*(\theta)\right)}{\theta}$$

v. The probability that the server is under repair is

$$R = R(1) = \sum_{j=0}^{m-1} \delta\lambda P_0 \frac{(1-H_b(1))\left(\prod_{j-0}^{m-1}\alpha_j\right)(H_b(1))^j\left(1-S_v^*(\theta)\right)\left(\frac{\lambda}{\theta}+R^*(\lambda)\right)E(G)}{\delta R^*(\lambda)-\lambda\left(1-\Phi(1)+\delta\left(1-H_b(1)\right)\Phi_1(1)E(G)\right)}$$

5.3. Reliability measures

For an unreliable queueing system, the reliability measure will give the knowledge that is required for the mastery of the system. To justify and validate the analytical results of this model, the availability measure and failure frequency of the sever is as follows

i. The availability of the server in steady state (A),that is the probability that the server is either serving a positive customer or is in idle period such that it is given by

$$A = 1-\lim_{z=1} R(z) = 1-R(1)$$

$$A = 1-\sum_{j=0}^{m-1}\lambda \frac{(1-H_b(1))\left(\prod_{j-0}^{m-1}\alpha_j\right)(H_b(1))^j\left(1-S_v^*(\theta)\right)\left(\frac{\lambda}{\theta}+R^*(\lambda)\right)E(G)}{\lambda/\delta\left(1-S_v^*\left(A_v(z)\right)\right)\left[\frac{\delta\left(1+R^*(\lambda)\right)}{\theta}+\omega\left(1-\frac{\lambda}{\theta}\right)+\tau\right]+R^*(\lambda)-\rho}$$

ii. The failure frequency of the server in steady state is given by

$$F = \delta\sum_{j=0}^{m-1}\left(Q_{j_b}(1)+P_{j_b}(1)\right) = \sum_{j=0}^{m-1}\frac{\lambda\left(1-H_b(1)\right)\left(\prod_{j-0}^{m-1}\alpha_j\right)(H_b(1))^j\left(1-S_v^*(\theta)\right)\left(\frac{\lambda}{\theta}+R^*(\lambda)\right)}{\lambda/\delta\left(1-S_v^*\left(A_v(z)\right)\right)\left[\frac{\delta\left(1+R^*(\lambda)\right)}{\theta}+\omega\left(1-\frac{\lambda}{\theta}\right)+\tau\right]+R^*(\lambda)-\rho}$$

6. Special cases

Case (i):*No retrial,No immediate feedback, No negative arrival and Single phase of service*

Let $R^*(\lambda)\to 1; \alpha_j = \mu_{2,b} = \delta = 0$; This model reduces to M/G/1 queue with working vacation policy. The result coincides with Zhang and Hou [28].

$$S(z) = \frac{\left(1-\lambda\beta^{(1)}\right)}{\left(1-\lambda\beta^{(1)}S_v^*(\theta)+\frac{\lambda}{\theta}(1-S_v^*(\theta)\right)}\frac{\left\{\begin{matrix}(1-z)\left\{\left(z-\left(S_{1,b}^*(A_b(z))\right)\right)(1+\frac{\lambda V(z)}{\theta})\right\}\\ +z\left[(1-S_{1,b}^*(A_b(z)))\left[zV(z)+\left(S_v^*\left(A_v(z)\right)-1\right)\right]\right]\end{matrix}\right\}}{(1-z)\left(z-S_{1,b}^*(A_b(z))\right)}$$

Case (ii): *No immediate feedback, No negative arrival, No vacation interruption, Single working vacation and Single phase of service*

Let $\alpha_j = \mu_{2,b} = \delta = \theta = 0$; we obtain an M/G/1 retrial queue with Single working vacation. The following result agrees with Arivudainambi et al. [1].

$$S(z) = \frac{S_v^*(\lambda)\left(R^*(\lambda) - \lambda\beta^{(1)}\right)}{\lambda E(S_v) - R^*(\lambda)S_v^*(\lambda)}\left\{\frac{S_{1,b}^*(A_b(z))\left[(1-z)S_v^*(\lambda)R^*(\lambda) + R(z)\left(S_v^*(A_v(z)) - 1\right)\right]}{\left(z - S_{1,b}^*(A_b(z))R(z)\right)S_v^*(\lambda)}\right\}$$

Case (iii): *No immediate feedback, No negative arrival and Single phase of service*
Let $\alpha_j = \mu_{2,b} = \delta = 0$; our model reduces to an M/G/1 retrial queue with working vacations. The result agrees with the result of Gao et al. [6].

$$S(z) = P_0 \frac{\left\{(1-z)\left\{\begin{array}{l}\left(z - R(z)\left(S_{1,b}^*(A_b(z))\right)\right)(1 + \frac{\lambda V(z)}{\theta}) \\ +z(1 - R^*(\lambda))\left(V(z)S_{1,b}^*(A_b(z)) + \left(S_v^*(A_v(z)) - 1\right)\right)\end{array}\right\} + (1 - S_{1,b}^*(A_b(z)))\left[zV(z) + R(z)\left(S_v^*(A_v(z)) - 1\right)\right]\right\}}{(1-z)\left(z - R(z)S_{1,b}^*(A_b(z))\right)}$$

Case (iv): *No immediate feedback, No negative arrival, No working vacation and Single phase of service*
Let $\alpha_j = \mu_{2,b} = \delta = \theta = \mu_v = 0$;we obtain an M/G/1 retrial queue with general retrial times The following result agrees with Gomez-Corral [9].

$$K(z) = \left\{\frac{\left[R^*(\lambda) - \lambda E(S_{1,b})\right]S_{1,b}^*\left[A_b(z)\right]\left[z-1\right]}{z - R(z)S_{1,b}^*\left[A_b(z)\right]}\right\}$$

7. Numerical illustration

In this section, we present some numerical examples to study the effect of various parameters on the system performance measures of our model. The arbitrary values to the parameters are so chosen such that they satisfy the stability condition. The retrial time, service time, working vacation time and repair time are taken general distributions, where Exponential - $f(x) = \upsilon e^{-\upsilon x}$, Erlangian of order two- $f(x) = \upsilon^2 x e^{-\upsilon x}$, Hyper exponential- $f(x) = c\upsilon e^{-\upsilon x} + (1-c)\upsilon^2 x e^{-\upsilon^2 x}$).

Table 1 shows that when immediate feedback probability for (α_1) increases, the probability that server is idle (P_0) decreases, then the mean orbit size (L_q) increases and probability that server is busy with feedback customer (Q)also increases. Table 2shows that when negative arrival rate (δ) increases, the mean orbit size(L_q) increases, probability that server is idle during retrial time(P) increases and the servers failure frequency (F) also increases. Table 3 shows that when lower speed service rate (μ_v) increases, the probability that server is idle (P_0) increases, then the mean orbit size (L_q) decreases and probability that server is on working vacation (Ω)also decrease for the values of λ= 0.5; δ = 1.5; μ= 8; μ_v= 5; a = 3; r = 0.5; θ= 3; c=0.7.

Table 1: The effect of immediate feedback probability (α_1) on P_0,L_q and Q

feedback probability	*Exponential*			*Erlang-2 stage*			*Hyper-Exponential*		
α_1	P_0	L_q	Q	P_0	L_q	Q	P_0	L_q	Q
0.10	0.8852	0.0726	0.0038	0.7070	0.2499	0.0123	0.8750	0.0779	0.0033
0.20	0.8805	0.0748	0.0077	0.6889	0.2698	0.0249	0.8709	0.0795	0.0066
0.30	0.8758	0.0770	0.0116	0.6702	0.2917	0.0379	0.8668	0.0812	0.0100

0.40	0.8710	0.0792	0.0156	0.6510	0.3158	0.0513	0.8626	0.0828	0.0134
0.50	0.8660	0.0816	0.0197	0.6313	0.3425	0.0650	0.8584	0.0845	0.0169

Table 2: The effect of negative arrival rate (δ) on Lq, P and F

negative arrival rate	*Exponential*			*Erlang-2 stage*			*Hyper-Exponential*		
Δ	L_q	P	F	L_q	P	F	L_q	P	F
0.30	0.3433	0.0901	0.0050	0.8031	0.3687	0.0162	0.4600	0.1448	0.0077
0.40	0.3671	0.0903	0.0087	0.8983	0.3722	0.0284	0.4922	0.1448	0.0135
0.50	0.3906	0.0905	0.0135	0.9934	0.3757	0.0439	0.5241	0.1449	0.0209
0.60	0.4137	0.0907	0.0193	1.0881	0.3790	0.0624	0.5558	0.1450	0.0297
0.70	0.4364	0.0909	0.0260	1.1827	0.3822	0.0839	0.5872	0.1450	0.0399

Table 3: The effect of lower speed service rate (μ_v) on P_0, L_q and Ω

Vacation distribution	*Exponential*			*Erlang-2 stage*			*Hyper-Exponential*		
μ_v	P_0	L_q	Ω	P_0	L_q	Ω	P_0	L_q	Ω
2.00	0.8016	0.0907	0.0802	0.5540	0.3588	0.0776	0.8007	0.0924	0.0897
3.00	0.8290	0.0872	0.0691	0.5818	0.3537	0.0727	0.8244	0.0895	0.0790
4.00	0.8498	0.0842	0.0607	0.6077	0.3481	0.0682	0.8432	0.0868	0.0706
5.00	0.8660	0.0816	0.0541	0.6313	0.3425	0.0641	0.8584	0.0845	0.0637
6.00	0.8791	0.0794	0.0488	0.6525	0.3371	0.0604	0.8710	0.0826	0.0581

For the effect of the parameters on the system measures, three dimensional graphs are illustrated in Figure 1 and Figure 3. In Figure 1, the surface displays an downward trend as expected for increasing the value of arrival rate (λ) and negative arrival rate (δ) against the idle probability (P_0). Figure 2 show that the mean orbit size (L_q) decreases for increasing the value of the lower service rate (μ_v) and regular service rate (μ_b).

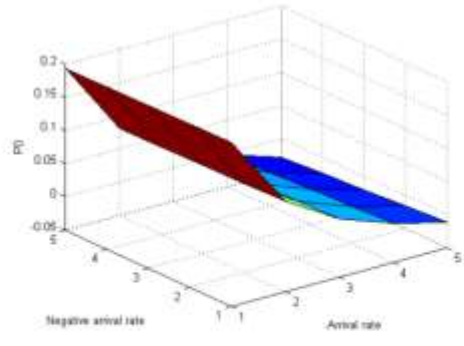

Figure 1. P_0 versus λ and δ

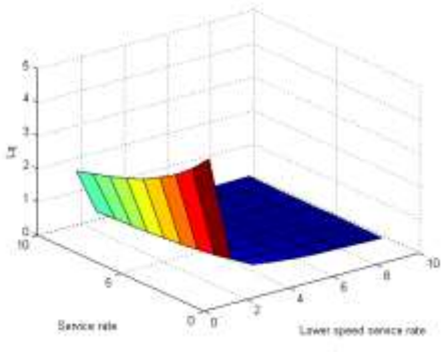

Figure 2. L_q versus μ_v andμ_b

8. Conclusion

In the foregoing analyses, a single server retrial G- queue with two phases of service and immediate feedback under working vacation policy is investigated. By using the method of supplementary variable technique and probability generating function approach, the probability generating functions for the numbers of customers in the system and in its orbit when it is free, busy or on working vacation, under repair are derived. Some system's performance measures and reliability measures are discussed. The explicit expressions for the average queue length of orbit and system have been obtained. Numerical examples are presented to study the impact of various parameters on the system performance. The analytical results that are validated numerically may be useful in many real-life situations such as WWW server, e-mail system, call centres, telecommunication networks, telephone switching system, etc. to design the outputs. The introduction of immediate feedback in presence of retrial G-queues and multiple working vacations is the novelty of this investigation. Our suggested model has practical real-life application in computer processing system which processes the messages through processor.

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