Proyecciones (Antofagasta. On line) | vol. 40, n. 3 (2021) | pp. 721-730.

10.22199/issn.0717-6279-4073

PROYECCIONES
Journal of Mathematics

ISSN 0717-6279 (On line)

A subclass with bi-univalence involving Horadam Polynomials and its coefficient bounds

K. Muthunagai[1] orcid.org/0000-0002-3020-7649
G. Saravanan[2] orcid.org/0000-0002-5706-4174
S. Baskaran[3] orcid.org/0000-0001-8980-9671

[1]Vellore Institute of Technology, School of Advanced Sciences, Chennai, TN, India.
muthunagai@vit.ac.in
[2]Patrician College of Arts and Science, Dept. of Mathematics, Chennai, TN, India.
gsaran825@yahoo.com
[3]Agurchand Manmull Jain College, Dept. of Mathematics, Chennai, TN, India.
sbas9991@gmail.com

Received: 02 April 2020 | Accepted: 19 December 2020

Abstract:

In this research contribution, we have constructed a subclass of analytic bi-univalent functions using Horadam polynomials. Bounds for certain coefficients and Fekete-Szegö inequalities have been estimated.

Keywords: Analytic functions; Bi-univalent functions; Horadam polynomials.

MSC (2020): 30C45, 30C15.

Cite this article as (IEEE citation style):

K. Muthunagai, G. Saravanan, and S. Baskaran, "A subclass with bi-univalence involving Horadam Polynomials and its coefficient bounds", *Proyecciones (Antofagasta, On line)*, vol. 40, no. 3, pp. 721-730, 2021, doi: 10.22199/issn.0717-6279-4073

1. Introduction

Let $\mathbf{f}(z)$ be a normalized analytic function of the form

$$\mathbf{f}(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \mathbf{U}, \mathbf{U} = \{z : z \in C, |z| < 1\} \tag{1.1}$$

and let A be the class of all such functions. Let $S = \{f(z) \in A \ni f(z)$ is univalent in $\mathbf{U}\}$.

For $\mathbf{f_1}(z)$ and $\mathbf{f_2}(z) \in A$, we say that $\mathbf{f_1}(z)$ is subordinate to $\mathbf{f_2}(z)$, if there exists a function namely, Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbf{U}$ such that $\mathbf{f_1}(z) = \mathbf{f_2}(w(z))$ and we write $\mathbf{f_1}(z) \prec \mathbf{f_2}(z)$.

Obtaining sharp bounds for $|a_3 - \eta a_2^2|$ of any compact family of functions is called the Fekete-Szegö problem. In particular when $\eta = 1$, the functional represents Schwarzian derivative and the role of Schwarzian derivative in the theory of Geometric functions is remarkable.

Let σ denote the class of all bi-univalent functions in $\mathbf{U}$. We say that a function $\mathbf{f}(z)$ in S belongs to σ, if both $\mathbf{f}(z)$ and its inverse has an analytic continuation to $|w| < 1$.

Lewin [8] introduced the class of bi-univalent functions in 1967 and gave an estimate for the second coefficient for functions belonging to this class as $|a_2| < 1.51$. His result was improved by Brannan and Clunie [3] to $|a_2| \leq \sqrt{2}$. There is an extensive literature on the estimates of the initial coefficients of bi-univalent functions (see [10, 12, 13, 14]).

Horadam polynomials are generalized Horadam numbers and second order polynomial sequence. Recently, Horzum and Kocer [4], studied the Horadam polynomials $\mathbf{h}_n(x)$, which is given by the following recurrence relation [5]

$$\mathbf{h}_n(x) = \varrho x \mathbf{h}_{n-1}(x) + \varrho \mathbf{h}_{n-2}(x), \mathbf{N} - \{1, 2\}.$$

One can refer to [1, 5, 6, 7, 9], for more details. These polynomials and their generalizations play a vital role in Mathematics, Statistics and Physics. The first few Horadam polynomials are listed below:

$$\mathbf{h}_1(x) = \mathbf{b}, \mathbf{h}_2(x) = \mathbf{a}x, \mathbf{h}_3(x) = \mathbf{a}\varrho x^2 + \mathbf{b}p, \tag{1.2}$$

for some real constants, $\mathbf{b}, \mathbf{a}, \varrho$ and p.

Following figure illustrates the special cases of Horadam polynomials

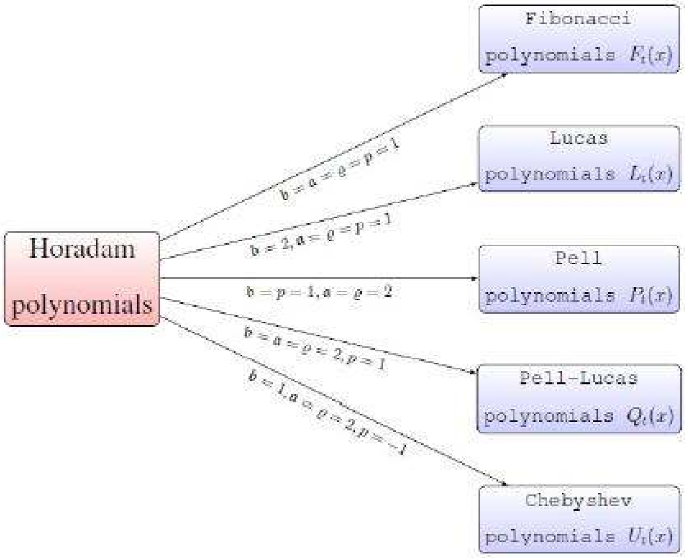

Figure 1:

Theorem 1.1. *Let* $\mathbf{G}(x,z)$ *be the generating function of the Horadam polynomials* $\mathbf{h}_n$*. Then,*

$$\mathbf{G}(x,z) = \sum_{n=1}^{\infty} \mathbf{h}_n(x) z^{n-1} = \frac{\mathbf{b} + (\mathbf{a} - \mathbf{b}\varrho)xz}{1 - \varrho xz - pz^2}$$

Remark 1.1. *[11] Choosing* $\mathbf{b} = 1, \mathbf{a} = \varrho = 2$*,* $p = -1$ *and* $x \longrightarrow \eta$*, in Theorem 1.1, the generating function* $\mathbf{G}(x,z)$ *generates the Chebyshev polynomials* $U_k(\eta)$ *of the second kind, which is given by*

$$U_k(\eta) = (k+1)_2F_1\left(-k, k+2; \frac{3}{2}; \frac{1-\eta}{2}\right) = \frac{sin(k+1)\varphi}{sin\varphi}, (\eta = sin\varphi)$$

in terms of the celebrated Gaussian hypergeometric function ${}_2F_1$.

Definition 1.1. *A function* $\mathbf{f} \in \sigma$ *is said to be in the class*

$$\mathbf{A}_\sigma(\gamma, \delta; x) \quad (\gamma \geq 0; 0 \leq \delta \leq 1; z, w \in \mathbf{U})$$

if

$$(1-\gamma)(1-\delta)\frac{\mathbf{f}(z)}{z} + (\delta + \gamma(1+\delta))\mathbf{f}'(z) + \gamma\delta(z\mathbf{f}''(z) - 2) \prec 1 - \mathbf{b} + \mathbf{G}(x, z)$$

and

$$(1-\gamma)(1-\delta)\frac{\mathbf{g}(w)}{w} + (\delta + \gamma(1+\delta))\mathbf{g}'(w) + \gamma\delta(w\mathbf{g}''(w) - 2) \prec 1 - \mathbf{b} + \mathbf{G}(x, w)$$

where $\mathbf{g} = \mathbf{f}^{-1}$.
By choosing the parameters γ *and* δ *appropriately, the class reduces to known subclasses of bi-univalent functions.*

Example 1.1. *For* $\delta = 0$ *and* $\gamma \geq 1$, $\mathbf{f} \in \sigma$ *is in the class*

$$\mathbf{A}_\sigma(\gamma, 0; x) = \mathbf{D}_\sigma(\gamma; x) \quad (z, w \in \mathbf{U})$$

if

$$(1-\gamma)\frac{\mathbf{f}(z)}{z} + \gamma\mathbf{f}'(z) \prec 1 - \mathbf{b} + \mathbf{G}(x, z)$$

and

$$(1-\gamma)\frac{\mathbf{g}(w)}{w} + \gamma\mathbf{g}'(w) \prec 1 - \mathbf{b} + \mathbf{G}(x, w)$$

where $\mathbf{g} = \mathbf{f}^{-1}$.

Example 1.2. *For* $\delta = 0$ *and* $\gamma = 1$, $\mathbf{f} \in \sigma$ *is in the class*

$$\mathbf{A}_\sigma(1, 0; x) = \mathbf{D}_\sigma(1; x) = \mathbf{H}_\sigma(x) \quad (z, w \in \mathbf{U})$$

if

$$\mathbf{f}'(z) \prec 1 - \mathbf{b} + \mathbf{G}(x, z)$$

and

$$\mathbf{g}'(w) \prec 1 - \mathbf{b} + \mathbf{G}(x, w)$$

where $\mathbf{g} = \mathbf{f}^{-1}$. *The class* $\mathbf{H}_\sigma(x)$ *was investigated and studied by Alamous [2]*

2. Coefficient Bounds

Theorem 2.1. *Let* $\mathbf{f}$ *represented by 1.1 belong to the class* $\mathbf{A}_\sigma(\gamma, t; x)$. *Then,*

$$|a_2| \leq \frac{|\mathbf{a}x|\sqrt{|\mathbf{a}x|}}{\sqrt{|(1+2(\gamma+\delta)+10\gamma\delta)(\mathbf{a}x)^2-(1+(\gamma+\delta)+5\gamma\delta)^2(\mathbf{a}\rho x^2+\mathbf{b}p)|}}$$

and

$$|a_3| \leq \frac{|\mathbf{a}x|}{(1+2(\gamma+\delta)+10\gamma\delta)} + \frac{\mathbf{a}^2x^2}{(1+(\gamma+\delta)+5\gamma\delta)^2}$$

Proof. Consider $\mathbf{f} \in \mathbf{A}_\sigma(\gamma, \delta; x)$. According to Definition 1.1, there exists two analytic functions Θ and Υ with $\Theta(0) = \Upsilon(0) = 0$, $|\Theta(z)| < 1, |\Upsilon(w)| < 1$ for all $z, w \in \mathbf{U}$ such that

$$(1-\gamma)(1-\delta)\frac{\mathbf{f}(z)}{z}+(\delta+\gamma(1+\delta))\mathbf{f}'(z)+\gamma\delta(z\mathbf{f}''(z)-2) \prec 1-\mathbf{b}+\mathbf{G}(x,\Theta(z))$$

and

$$(1-\gamma)(1-\delta)\frac{\mathbf{g}(w)}{w}+(\delta+\gamma(1+\delta))\mathbf{g}'(w)+\gamma\delta(w\mathbf{g}''(w)-2) \prec 1-\mathbf{b}+\mathbf{G}(x,\Upsilon(w)),$$

or equivalently

$$\begin{aligned}(1-\gamma)(1-\delta)\tfrac{\mathbf{f}(z)}{z} + (\delta+\gamma(1+\delta))\mathbf{f}'(z) + \gamma\delta(z\mathbf{f}''(z)-2) \\ = 1 + \mathbf{h}_1(x) - \mathbf{b} + \mathbf{h}_2(x)\Theta(z) + \mathbf{h}_3\Theta^2(z) + \cdots\end{aligned} \tag{2.1}$$

$$\begin{aligned}(1-\gamma)(1-\delta)\tfrac{\mathbf{g}(w)}{w} + (\delta+\gamma(1+\delta))\mathbf{g}'(w) + \gamma\delta(w\mathbf{g}''(w)-2) \\ = 1 + \mathbf{h}_1(x) - \mathbf{b} + \mathbf{h}_2(x)\Upsilon(w) + \mathbf{h}_3\Upsilon^2(w) + \cdots\end{aligned} \tag{2.2}$$

2.1 and 2.2 yield

$$\begin{aligned}(1-\gamma)(1-\delta)\tfrac{\mathbf{f}(z)}{z} + (\delta+\gamma(1+\delta))\mathbf{f}'(z) + \gamma\delta(z\mathbf{f}''(z)-2) \\ = 1 + \mathbf{h}_2(x)u_1z + \left[\mathbf{h}_2(x)u_2 + \mathbf{h}_3(x)u_1^2\right]z^2 + \cdots\end{aligned} \tag{2.3}$$

$$\begin{aligned}(1-\gamma)(1-\delta)\tfrac{\mathbf{g}(w)}{w} + (\delta+\gamma(1+\delta))\mathbf{g}'(w) + \gamma\delta(w\mathbf{g}''(w)-2) \\ = 1 + \mathbf{h}_2(x)v_1w + \left[\mathbf{h}_2(x)v_2 + \mathbf{h}_3(x)v_1^2\right]w^2 + \cdots\end{aligned} \tag{2.4}$$

It is to be noted that if

$$|\Theta(z)| = |u_1z + u_2z^2 + u_3z^3 + \cdots| < 1 \quad (z \in \mathbf{U})$$

and

$$|\Upsilon(w)| = |v_1 w + v_2 w^2 + v_3 w^3 + \cdots| < 1 \quad (w \in \mathbf{U}),$$

then

$$\begin{aligned} |u_k| &\leq 1, \\ |v_k| &\leq 1 \quad (k \in \mathbf{N}). \end{aligned}$$

Comparing the corresponding coefficients in 2.3 and 2.4, we have

$$(1 + (\gamma + \delta) + 5\gamma\delta)\, a_2 = \mathbf{h}_2(x) u_1 \tag{2.5}$$

$$(1 + 2(\gamma + \delta) + 10\gamma\delta)\, a_3 = \mathbf{h}_2(x)\mathbf{u}_2 + \mathbf{h}_3(x) u_1^2 \tag{2.6}$$

$$-\,(1 + (\gamma + \delta) + 5\gamma\delta)\, a_2 = \mathbf{h}_2(x) v_1 \tag{2.7}$$

and

$$(1 + 2(\gamma + \delta) + 10\gamma\delta)\, (2a_2^2 - a_3) = \mathbf{h}_2(x) v_2 + \mathbf{h}_3(x) v_1^2 \tag{2.8}$$

From Eqs. 2.5 and 2.7, we can easily see that

$$u_1 = -v_1 \tag{2.9}$$

$$2\,(1 + (\gamma + \delta) + 5\gamma\delta)^2\, a_2^2 = \mathbf{h}_2^2 [u_1^2 + v_1^2] \tag{2.10}$$

If we add 2.6 to 2.8, we get

$$2\,(1 + 2(\gamma + \delta) + 10\gamma\delta)\, a_2^2 = \mathbf{h}_2(x)(u_2 + v_2) + \mathbf{h}_3(x)(u_1^2 + v_1^2) \tag{2.11}$$

By using 2.10 in the equality 2.11, we have

$$2\left((1 + 2(\gamma + \delta) + 10\gamma\delta)\, \mathbf{h}_2^2(x) - (1 + (\gamma + \delta) + 5\gamma\delta)^2\, \mathbf{h}_3(x)\right) a_2^2 = \mathbf{h}_2^3 (u_2 + v_2) \tag{2.12}$$

which implies

$$|a_2| \leq \frac{|\mathbf{a}x|\sqrt{|\mathbf{a}x|}}{\sqrt{|(1+2(\gamma+\delta)+10\gamma\delta)(\mathbf{a}x)^2 - (1+(\gamma+\delta)+5\gamma\delta)^2 (\mathbf{a}\rho x^2 + \mathbf{b}p)|}}$$

Moreover, if we subtract 2.8 from 2.6 and use 2.9,

$$a_3 - a_2^2 = \frac{\mathbf{h}_2(x)(u_2 - v_2)}{2\left(1 + 2(\gamma+\delta) + 10\gamma\delta\right)} \tag{2.13}$$

Then, in view of 2.10 and 2.13, we have

$$a_3 = \frac{\mathbf{h}_2(x)(u_2 - v_2)}{2(1 + 2(\gamma+\delta) + 10\gamma\delta)} + \frac{\mathbf{h}_2^2(x)(u_1^2 + v_1^2)}{2(1 + (\gamma+\delta) + 5\gamma\delta)^2} \tag{2.14}$$

Applying $\mathbf{h}_2(x)$ and taking modulus, we deduce that

$$|a_3| \leq \frac{|\mathbf{a}x|}{(1 + 2(\gamma+\delta) + 10\gamma\delta)} + \frac{\mathbf{a}^2x^2}{(1 + (\gamma+\delta) + 5\gamma\delta)^2}$$

□

Corollary 2.1. *Let* $\mathbf{f} \in \mathbf{D}_\sigma(\gamma; x)$. *Then*

$$|a_2| \leq \frac{|\mathbf{a}x|\sqrt{|\mathbf{a}x|}}{\sqrt{|(1+2\gamma)(\mathbf{a}x)^2 - (1+\gamma)^2(\mathbf{a}\rho x^2 + \mathbf{b}p)|}}$$

$$|a_3| \leq \frac{|\mathbf{a}x|}{(1 + 2\gamma)} + \frac{\mathbf{a}^2x^2}{(1+\gamma)^2}$$

Corollary 2.2. *Let* $\mathbf{f} \in \mathbf{H}_\sigma(x)$. *Then*

$$|a_2| \leq \frac{|\mathbf{a}x|\sqrt{|\mathbf{a}x|}}{\sqrt{|3(\mathbf{a}x)^2 - 4(\mathbf{a}\rho x^2 + \mathbf{b}p)|}}$$

$$|a_3| \leq \frac{|\mathbf{a}x|}{3} + \frac{\mathbf{a}^2x^2}{4}$$

The class was investigated and studied by Alamous [2].

Corollary 2.3. *For* $\eta \in \left(\frac{1}{2}, 1\right)$, *let the function* $\mathbf{f} \in \mathbf{A}_\sigma(\gamma, \delta; \eta)$ *be of the form 1.1. Then*

$$|a_2| \leq \frac{|2\eta|\sqrt{|2\eta|}}{\sqrt{|4\eta^2(1+2(\gamma+\delta)+10\gamma\delta) - (4\eta^2-1)(1+(\gamma+\delta)+5\gamma\delta)^2|}}$$

$$|a_3| \leq \frac{|2\eta|}{(1 + 2(\gamma+\delta) + 10\gamma\delta)} + \frac{4\eta^2}{(1 + (\gamma+\delta) + 5\gamma\delta)^2}$$

3. Fekete-Szegö Inequality

In this section, for functions belonging to the class $\mathbf{A}_\sigma(\gamma, t; x)$, we have estimated the bounds for the linear functional.

Theorem 3.1. *Let* $\mathbf{f} \in \mathbf{A}_\sigma(\gamma, t; x)$ *and* $\zeta \in \mathbf{R}$. *Then,*

$$\left|a_3 - \zeta a_2^2\right| \leq \begin{cases} \frac{|\mathbf{a}x|}{s_1}, & |\zeta - 1| \leq \left|1 - \frac{s_2^2(\mathbf{a}\rho x^2 + \mathbf{b}p)}{s_1\ \mathbf{a}^2 x^2}\right| \\ \frac{|\zeta-1||\mathbf{a}x|^3}{[s_1 a - s_2^2 \rho]\mathbf{a}x^2 - s_2^2 \mathbf{b}p}, & |\zeta - 1| \geq \left|1 - \frac{s_2^2(\mathbf{a}\rho x^2 + \mathbf{b}p)}{s_1 \mathbf{a}^2 x^2}\right| \end{cases}$$

where,

$$s_1 = 1 + 2(\gamma + \delta) + 10\gamma\delta$$
$$s_2 = 1 + (\gamma + \delta) + 5\gamma\delta$$

Proof. From 2.13, for $\zeta \in \mathbf{R}$, we have

$$a_3 - \zeta a_2^2 = \frac{\mathbf{h}_2(x)(u_2 - v_2)}{2(1+2(\gamma+\delta)+10\gamma\delta)} + (1-\zeta)a_2^2 \tag{3.1}$$

By using 2.12 in 3.1, we have

$$a_3 - \zeta a_2^2 = \frac{\mathbf{h}_2(x)(u_2 - v_2)}{2(1+2(\gamma+\delta)+10\gamma\delta)} + (1-\zeta)\left(\frac{\mathbf{h}_2^3(x)(u_2+v_2)}{2[(1+2(\gamma+\delta)+10\gamma\delta]\mathbf{h}_2^2(x) - (1+\gamma+\delta+5\gamma\delta)^2 \mathbf{h}_3(x)}\right)$$

$$= \mathbf{h}_2(x)\left[\left(\frac{1}{2[1+2(\gamma+\delta)+10\gamma\delta]} + \chi(\gamma,\delta)\right)u_2 + \left(\frac{-1}{2[1+2(\gamma+\delta)+10\gamma\delta]} + \chi(\gamma,\delta)\right)v_2\right]$$

where $\chi(\gamma, \delta) = \frac{(1-\zeta)\mathbf{h}_2^2(x)}{2[(1+2(\gamma+\delta)+10\gamma\delta)\mathbf{h}_2^2(x) - (1+(\gamma+\delta)+5\gamma\delta)^2 \mathbf{h}_3(x)}$

Then, in view of (1.2), we conclude that

$$\left|a_3 - \zeta a_2^2\right| \leq \begin{cases} \frac{|h_2(x)|}{1+2(\gamma+\delta)+10\gamma\delta}, & 0 \leq |\chi(\gamma,\delta)| \leq \frac{1}{2[1+2(\gamma+\delta)+10\gamma\delta]} \\ 2\left|\mathbf{h}_2(x)\right|\ \left|\chi(\gamma,\delta)\right|, & |\chi(\gamma,\delta)| \geq \frac{1}{2[1+2(\gamma+\delta)+10\gamma\delta]} \end{cases}$$

□

Corollary 3.1. *Let* $\mathbf{f} \in \mathbf{D}_\sigma(\gamma; x)$ *and* $\zeta \in \mathbf{R}$. *Then*

$$\left|a_3 - \zeta a_2^2\right| \leq \begin{cases} \frac{|\mathbf{a}x|}{1+2\gamma}, & |\zeta - 1| \leq |1 - \frac{(1+\gamma)^2(\mathbf{a}\varrho x^2+\mathbf{b}p)}{(1+2\gamma)\mathbf{a}^2x^2}|. \\ \frac{|1-\zeta||\mathbf{a}x|^3}{|[(1+2\gamma)\mathbf{a}-(1+\gamma)^2\varrho]\mathbf{a}x^2-[1+\gamma]^2\mathbf{b}p|}, & |\zeta - 1| \geq |1 - \frac{(1+\gamma)^2(\mathbf{a}\varrho x^2+\mathbf{b}p)}{(1+2\gamma)\mathbf{a}^2x^2}|. \end{cases}$$

Corollary 3.2. *Let* $\mathbf{f} \in \mathbf{H}_\sigma(\gamma; x)$ *and* $\zeta \in \mathbf{R}$. *Then*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|\mathbf{a}x|}{3}, & |\zeta - 1| \leq |1 - \frac{4(\mathbf{a}\varrho x^2)}{3\mathbf{a}^2x^2}|. \\ \frac{|\mathbf{a}x|^3|\zeta-1|}{|[3\mathbf{a}-4\varrho]\mathbf{a}x^2-4\mathbf{b}p|}, & |\zeta - 1| \geq |1 - \frac{4(\mathbf{a}\varrho x^2)}{3\mathbf{a}^2x^2}|. \end{cases}$$

Corollary 3.3. *For* $\eta \in \left(\frac{1}{2}, 1\right)$, *let the function* $\mathbf{A}_\sigma(\gamma, \delta; \eta)$ *be of the form 1.1. Then*

$$\left|a_3 - \zeta a_2^2\right| \leq \begin{cases} \frac{|2\eta|}{s_1}, & |\zeta - 1| \leq \left|1 \frac{s_2^2\left(4\eta^2-0\right)}{4s_1\eta^2}\right| \\ \frac{|\zeta-1||2\eta|^3}{|4\left[s_1-s_2^2\right]\eta^2+s_2^2|}, & |\zeta - 1| \geq \left|1 - \frac{s_2^2\left(4\eta^2-1\right)}{4s_1\eta^2}\right| \end{cases}$$

where,

$$s_1 = 1 + 2\left(\gamma + \delta\right) + 10\gamma\delta$$
$$s_2 = 1 + \left(\gamma + \delta\right) + 5\gamma\delta$$

References

[1] A. Akgül and F. Sakar, ”A certain subclass of bi-univalent analytic functions introduced by means of the q-analogue of Noor integral operator and Horadam polynomials”, *Turkish journal of mathematics*, vol. 43, no. 5, pp. 2275-2286, 2019, doi: 10.3906/mat-1905-17

[2] A. G. Alamoush, “Coefficient estimates for certain subclass of bi functions associated the Horadam polynomials”, 2018, *arXiv:* 1812.10589

[3] D. Brannan and J. Clunie, *Aspects of contemporary complex analysis.* London: Academic Press, 1980.

[4] T. Horzum and E. Gökçen Koçer, “On some properties of Horadam polynomials”, *International mathematical forum*, vol. 4, no. 25, pp. 1243-1252, 2009. [On line]. Available: https://bit.ly/2Sw3MGf

[5] A. F. Horadam and J. M. Mahon, ”Pell and Pell-Lucas polynomials”, *The Fibonacci quarterly*, vol. 23, no. 1, pp. 7-20, 1985. [On line]. Available: https://bit.ly/2R137MV

[6] A. Horadam, ”Jacobsthal representation polynomials”, T*he Fibonacci quarterly*, vol. 35, no. 2, pp. 137-148, 1997. [On line]. Available: https://bit.ly/3ez7OWY

[7] T. Koshy, *Fibonacci and Lucas numbers with applications.* New York, NY: Wiley, 2001, doi: 10.1002/9781118033067

[8] M. Lewin, ”On a coefficient problem for bi-univalent functions”, *Proceedings of the American Mathematical Society*, vol. 18, no. 1, pp. 63-63, 1967, doi: 10.1090/s0002-9939-1967-0206255-1

[9] A. Lupas, ”A Guide of Fibonacci and Lucas polynomials”, *Octogon Mathematical Magazine*, vol. 7, no. 1, pp. 2-12, 1999.

[10] G. Saravanan and K. Muthunagai, “Estimation of upper bounds for initial coefficients and Fekete-Szegö inequality for a subclass of analytic bi-univalent functions”, in *Applied mathematics and scientific computing*, B. Rushi Kumar, R. Sivaraj, B. S. R. V. Prasad, M. Nalliah, and A. Subramanyam Reddy, Eds. Basel: Birkhäuser, 2019, pp. 57–65, doi: 10.1007/978-3-030-01123-9_7

[11] G. Szegö, *Orthogonal polynomials*, 4th ed. Providence, RI: AMS, 1975.

[12] R. Vijaya, T. Sudharsan and S. Sivasubramanian, ”Coefficient estimates for certain subclasses of biunivalent functions defined by convolution”, *International journal of analysis,* vol. 2016, Art. ID. 6958098, 2016, doi: 10.1155/2016/6958098

[13] Q. Xu, Y. Gui, and H. Srivastava, ”Coefficient estimates for a certain subclass of analytic and bi-univalent functions”, *Applied mathematics letters,* vol. 25, no. 6, pp. 990-994, 2012, doi: 10.1016/j.aml.2011.11.013

[14] S. Yalçin, K. Muthunagai and G. Saravanan, ”A subclass with bi-univalence involving (p,q)- Lucas polynomials and its coefficient bounds”, *Boletín de la Sociedad Matemática Mexicana*, vol. 26, pp. 1015-1022, doi: 10.1007/s40590-020-00294-z