Available online at www.sciencedirect.com
ScienceDirect
Journal of Discrete Algorithms 6 (2008) 11–19
JOURNAL OF DISCRETE ALGORITHMS
www.elsevier.com/locate/jda

An efficient representation of Benes networks and its applications

Paul D. Manuel [a], Mostafa I. Abd-El-Barr [a], Indra Rajasingh [b], Bharati Rajan [b,*]

[a] *Department of Information Science, Kuwait University, Safat 13060, Kuwait*
[b] *Department of Mathematics, Loyola College, Chennai 600 034, India*

Received 28 November 2005; received in revised form 16 July 2006; accepted 14 August 2006

Available online 1 November 2006

Abstract

The most popular bounded-degree derivative network of the hypercube is the butterfly network. The Benes network consists of back-to-back butterflies. There exist a number of topological representations that are used to describe butterfly—like architectures. We identify a new topological representation of butterfly and Benes networks.

The minimum metric dimension problem is to find a minimum set of vertices of a graph $G(V, E)$ such that for every pair of vertices u and v of G, there exists a vertex w with the condition that the length of a shortest path from u to w is different from the length of a shortest path from v to w. It is *NP*-hard in the general sense. We show that it remains *NP*-hard for bipartite graphs. The algorithmic complexity status of this *NP*-hard problem is not known for butterfly and Benes networks, which are subclasses of bipartite graphs. By using the proposed new representations, we solve the minimum metric dimension problem for butterfly and Benes networks. The minimum metric dimension problem is important in areas such as robot navigation in space applications.

Keywords: Butterfly network; Benes network; Bipartite graphs; Minimum metric dimension problem; *NP*-hard problem

1. Introduction and background

We represent networks as undirected graphs whose nodes represent processors and whose edges represent inter-processor communication links. The set V of nodes of an r-dimensional butterfly correspond to pairs $[w, i]$, where i is the dimension or level of a node $(0 \leqslant i \leqslant r)$ and w is an r-bit binary number that denotes the row of the node. Two nodes $[w, i]$ and $[w', i']$ are linked by an edge if and only if $i' = i + 1$ and either:

1. w and w' are identical, or
2. w and w' differ in precisely the ith bit.

The edges in the network are undirected. An r-dimensional butterfly is denoted by $BF(r)$. An *r-dimensional Benes network* has $2r + 1$ levels, each level with 2^r nodes. The level 0 to level r nodes in the network form an r-dimensional butterfly. The middle level of the Benes network is shared by these butterflies [3]. An r-dimensional Benes is denoted by $B(r)$. Fig. 2 shows a $B(3)$ network.

* Corresponding author.
E-mail addresses: p.manuel@cfw.kuniv.edu (P.D. Manuel), rajanbharati@rediffmail.com (B. Rajan).

1570-8667/$ – see front matter
doi:10.1016/j.jda.2006.08.003

The minimum metric dimension (MMD) problem is *NP*-hard [1,2] for general graphs and it has several applications in robotics and image processing [2]. This *NP*-hard problem has not been studied for butterfly and Benes networks. The butterfly and Benes networks form a subclass of bipartite graphs. In this paper we show that the MMD problem remains *NP*-hard for bipartite graphs and it is polynomially solvable for butterfly and Benes networks. Since Benes networks are back-to-back butterflies, we concentrate only on Benes network. The corresponding results for butterfly networks will be mentioned as corollaries.

2. Proposed Diamond representations of butterfly and Benes networks

In this section, we discuss about representations of butterfly and Benes networks. The proposed representations of butterfly and Benes are shown in Figs. 1(b) and 3. To avoid confusion between the two representations in Fig. 1, the representation in Fig. 1(a) will be called a *normal representation* of butterfly and the representation in Fig. 1(b) will be called a *diamond representation* of butterfly. Similar terminologies are applied for Benes networks. See Figs. 2 and 3.

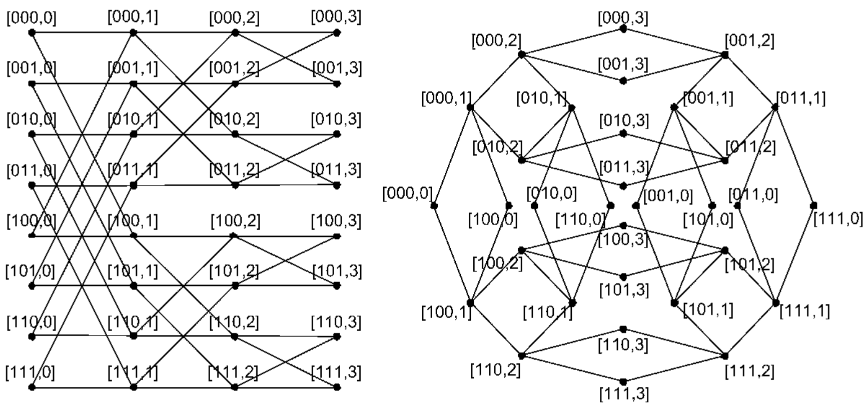
Definition 1. The proposed Diamond representation of butterfly network is defined as follows: Two $(r-1)$-dimensional butterfly networks $BF(r-1)$ form mirror images with respect to an array of level 0 nodes. The level 0 nodes are the vertices belonging to chordless 4-cycles in the diamond formation bridging the two $(r-1)$-dimensional butterfly networks $BF(r-1)$. Each 4-cycle is drawn as a diamond.

This representation provides a structural visualization, an in-depth understanding about the cyclic properties and the organization of spanning trees of butterfly and Benes networks. See Figs. 3 and 1(b). The following lemma on Benes and butterfly network is straightforward from the Diamond representation given in Figs. 3 and 1(b).

Lemma 1. [3,6] *The Benes and butterfly networks are bipartite.*

Even though the Benes network consists of back-to-back butterflies, there is a subtle structural difference between Benes and butterfly. The removal of level 0 nodes of $BF(r)$ leaves two disjoint copies of $BF(r-1)$. In the same way, the removal of level r nodes of $BF(r)$ leaves two disjoint copies of $BF(r-1)$. This recursive structure can be viewed in another way. The removal of level 0 nodes and level r nodes (nodes of degree 2) of $BF(r)$ leaves 4 disjoint copies of a $BF(r-2)$. However the removal of level 0 nodes and level $2r$ nodes (nodes of degree 2) of $B(r)$ leaves 2 disjoint copies of a $B(r-1)$. In other words, the butterfly has dual symmetry, which the Benes does not have. See Figs. 1(b) and 3.

Lemma 2. *The normal and diamond representations of Benes are isomorphic.*



(a) (b)

Fig. 1. (a) Normal representation of butterfly $BF(3)$. (b) Diamond representation of butterfly $BF(3)$.

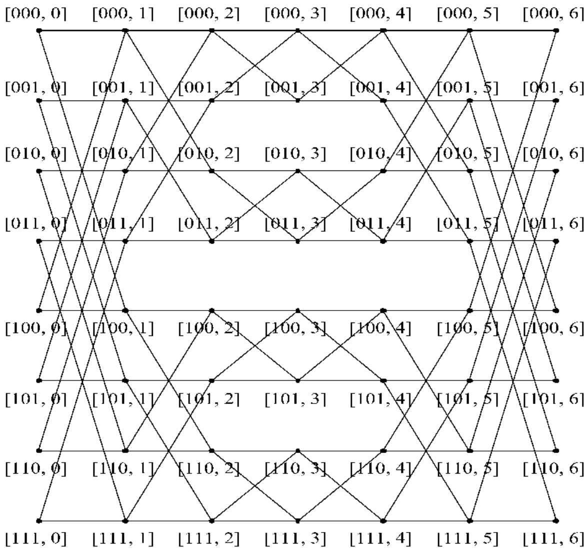


Fig. 2. Normal representation of Benes $B(3)$.

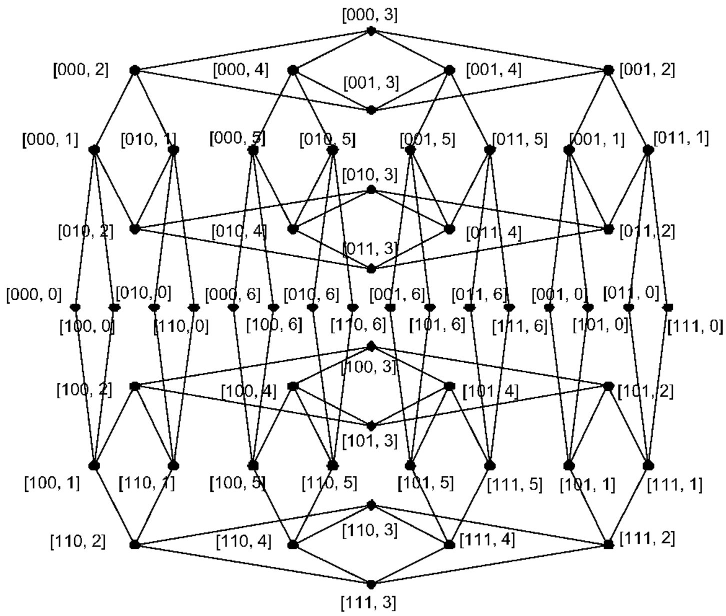


Fig. 3. Diamond representation of Benes $B(3)$.

Proof. We apply induction on the dimension of Benes $B(r)$. Both 1-dimensional Normal representation and 1-dimensional Diamond representation are cycles of length 4. Let us assume that $(k-1)$-dimensional Normal representation and $(k-1)$-dimensional Diamond representation of Benes are isomorphic.

Now let us show that the k-dimensional Normal representation and the k-dimensional Diamond representation of Benes are isomorphic. Remove nodes of degree 2 (level 0 nodes and level $2k$ nodes) from both k-dimensional Normal representation and k-dimensional Diamond representation. By induction hypothesis, the resultant Benes networks are isomorphic. The nodes of degree 2 (level 0 nodes and level $2k$ nodes) of both k-dimensional Normal representation and k-dimensional Diamond representation are organized in the same way as follows: The level 0 nodes $[0u_2 \ldots u_k, 0]$ and $[1u_2 \ldots u_k, 0]$ are adjacent to level 1 nodes $[0u_2 \ldots u_k, 1]$ and $[1u_2 \ldots u_k, 1]$ respectively. In the same way, level $2k$ nodes $[0u_2 \ldots u_k, 2k]$ and $[1u_2 \ldots u_k, 2k]$ are adjacent to level $(2k-1)$ nodes $[0u_2 \ldots u_k, 2k-1]$ and $[1u_2 \ldots u_k,$

$2k-1$] respectively. Moreover, these nodes form a chordless cycle of length 4. These 4-cycles are edge disjoint with the rest of the graph. Hence both Normal representation and Diamond representation are isomorphic. □

Corollary 3. *The Normal and Diamond representations of butterfly are isomorphic.*

3. Minimum metric dimension problem

A *metric basis* for a graph $G(V,E)$ is a set $W \subseteq V$ such that for each pair of vertices u and v of $V \setminus W$, there is a vertex $w \in W$ such that $d(u,w) \neq d(v,w)$. A *minimum metric basis* is a metric basis of minimum cardinality. The members of a minimum metric basis are called *landmarks* and the cardinality of a minimum metric basis is called *minimum metric dimension*. The *minimum metric dimension* (MMD) *problem* is to find a minimum metric basis. The minimum metric dimension problem is *NP*-hard for general graphs [1,2]. This problem is also called navigation problem due to its application of robot navigation in space [2]. Khuller et al. [2] describe the application of this problem in the field of computer science. This problem has been studied for trees, multi-dimensional grids [2], and Petersen graphs [5]. Surprisingly, there is not much relevant work in the literature. The algorithmic complexity status of MMD problem is not known to even simple graphs such as co-graphs, interval graphs, Cayley graphs etc.

To our knowledge, the MMD problem has not been investigated for butterfly and Benes networks. In this paper, we solve this problem for Benes and butterfly networks. It is the first result of this kind. Using the diamond representation of Benes, we identify a minimum metric basis of Benes networks $B(r)$ and we prove that the minimum metric dimension of $B(r)$ is $3(2^{r-1})$. Benes and butterfly networks are bipartite graphs. In this paper, the complexity status of the MMD problem is narrowed down to the fact that the MMD problem is *NP*-hard for bipartite graphs and it is polynomially solvable for Benes and butterfly networks, which are subclasses of bipartite graphs.

3.1. MMD problem is polynomially solvable for Benes and butterfly

The following observation is important for the construction of a minimum metric basis of Benes networks.

Lemma 4. *Let $B(r)$ denote an r-dimensional Benes network. Then*

(i) *Any metric basis W of $B(r)$ has either* $[0u_2\dots u_k, 0]$ *or* $[1u_2\dots u_k, 0]$.
(ii) *Any metric basis W of $B(r)$ has either* $[0u_2\dots u_k, 2r]$ *or* $[1u_2\dots u_k, 2r]$.
(iii) *Any metric basis W of $B(r)$ has either the node* $[u_1u_2\dots u_{k-1}0, r]$ *or* $[u_1u_2\dots u_{k-1}1, r]$.

Proof. (i) The nodes $[0u_2\dots u_k, 0]$, $[0u_2\dots u_k, 1]$, $[1u_2\dots u_k, 0]$, and $[1u_2\dots u_k, 1]$ form a 4-cycle C of $B(r)$. Moreover, nodes $[0u_2\dots u_k, 0]$ and $[1u_2\dots u_k, 0]$ of C are of degree 2. Let x be a node of $B(r)$. If a shortest path between x and $[0u_2\dots u_k, 0]$ traverses $[0u_2\dots u_k, 1]$, then a shortest path between x and $[1u_2\dots u_k, 0]$ also traverses the same node $[0u_2\dots u_k, 1]$. Thus

$$d(x, [0u_2\dots u_k, 0]) = d(x, [1u_2\dots u_k, 0])$$

for any node x of $B(r)$. Therefore, any metric basis W of $B(r)$ has either $[0u_2\dots u_k, 0]$ or $[1u_2\dots u_k, 0]$.

The proofs of (ii) and (iii) are similar. □

Corollary 5. *Let $BF(r)$ denote an r-dimensional butterfly network. Then*

(i) *Any metric basis W of $BF(r)$ has either* $[0u_2\dots u_k, 0]$ *or* $[1u_2\dots u_k, 0]$.
(ii) *Any metric basis W of $BF(r)$ has either the node* $[u_1u_2\dots u_{k-1}0, r]$ *or* $[u_1u_2\dots u_{k-1}1, r]$.

By Lemma 4, any metric basis W of $B(r)$ has at least $\frac{1}{2}(2^r) + \frac{1}{2}(2^r) + \frac{1}{2}(2^r)$ nodes. This provides a simple lower bound for the metric dimension of $B(r)$.

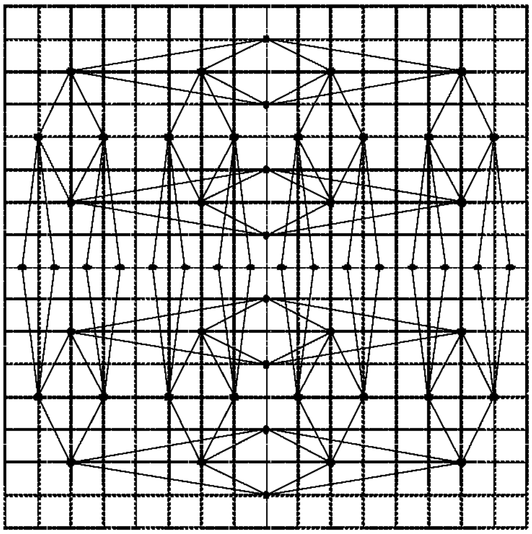


Fig. 4. $B(3)$ on grid.

Lemma 6 *(Lower bound). Any metric basis W of $B(r)$ has at least $3(2^{r-1})$ nodes.*

Corollary 7 *(Lower bound). Any metric basis W of $BF(r)$ has at least 2^r nodes.*

The diamond representation of Benes can be embedded into a square grid. See Fig. 4. Two nodes u and v are said to be *horizontal*, if they are in the same row of the grid. Two nodes u and v are said to be *vertical*, if they are in the same column of the grid. For example, in Fig. 3, the nodes $[000, 2]$ and $[110, 2]$ are vertical nodes whereas nodes $[000, 1]$ and $[000, 5]$ are horizontal nodes.

Here is an important observation. By removing all the nodes of level r, the Benes $B(r)$ is partitioned into four $(r-1)$-dimensional butterflies $BF_1(r-1)$, $BF_2(r-1)$, $BF_3(r-1)$, and $BF_4(r-1)$. See Fig. 5. The butterfly $BF_1(r-1)$ comprises of nodes $\{[u_1u_2\dots u_{r-1}0, t] : u_1u_2\dots u_{r-1}$ is any binary sequence and $0 \leqslant t \leqslant (r-1)\}$. The butterfly $BF_2(r-1)$ comprises of nodes $\{[u_1u_2\dots u_{r-1}0, t] : u_1u_2\dots u_{r-1}$ is any binary sequence and $(r+1) \leqslant t \leqslant 2r\}$. The butterfly $BF_3(r-1)$ comprises of nodes $\{[u_1u_2\dots u_{r-1}1, t] : u_1u_2\dots u_{r-1}$ is any binary sequence and $(r+1) \leqslant t \leqslant 2r\}$. The butterfly $BF_4(r-1)$ comprises of nodes $\{[u_1u_2\dots u_{r-1}1, t] : u_1u_2\dots u_{r-1}$ is any binary sequence and $0 \leqslant t \leqslant r-1\}$.

Lemma 8. *Let $M_0 = \{[0w_2\dots w_r, 0], [0w_2\dots w_r, 2r] : w_2\dots w_r$ is any binary sequence$\}$ and $M_1 = \{[w_1w_2\dots w_{r-1}0, r] : w_1w_2\dots w_{r-1}$ is any binary sequence$\}$. Then $M_0 \cup M_1$ is a metric basis of $B(r)$.*

Proof. Let $W = M_0 \cup M_1$. Let u and v be two arbitrary nodes of $V \setminus W$ of $B(r)$. Say $u = [u_1u_2\dots u_r,\ j]$ and $v = [v_1v_2\dots v_r,\ k]$. If one of the nodes u and v is at level 0, level r, or level $2r$, then it is possible to find w of W such that $d(v, w) \neq d(u, w)$. Thus, in the rest of proof, we assume that both nodes u and v are not at level 0, level r, or level $2r$. There are three possible cases for the nodes u and v.

1. u and v are vertical.
2. u and v are horizontal.
3. u and v are neither vertical nor horizontal.

Case 1 (*u and v are vertical*). When two nodes u and v are vertical, they are of the same level. That is, $u = [u_1u_2\dots u_r,\ j]$ and $v = [v_1v_2\dots v_r,\ j]$. Let us consider a subcase where u and v are the vertices of $BF_1(r-1)$. That is, $j < r$ and $u_r = v_r = 0$. Consider the landmark $w = [u_1u_2\dots u_{r-1}0,\ r]$. Without loss of generality, let us assume that $u_1u_2\dots u_r < v_1v_2\dots v_r$. Let ℓ be the smallest index such that $u_1u_2\dots u_\ell = v_1v_2\dots v_\ell$ and $u_{\ell+1} \neq v_{\ell+1}$. A shortest path between $v = [v_1v_2\dots v_r,\ j]$ and $w = [u_1u_2\dots u_{r-1}0,\ r]$ is

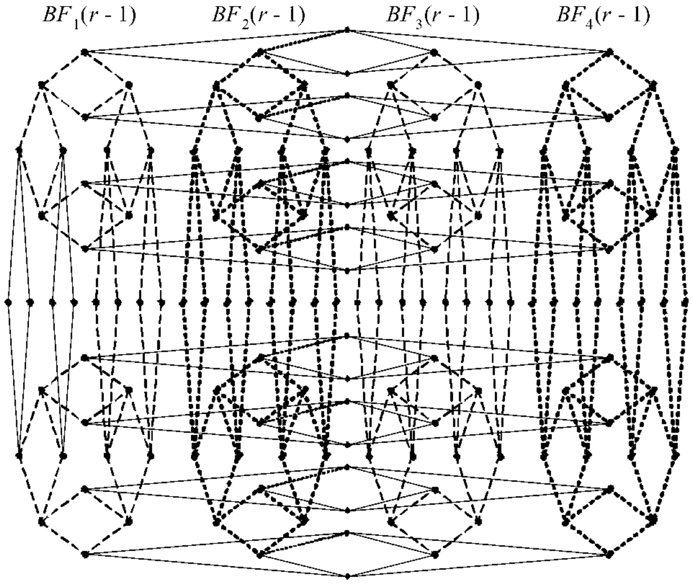


Fig. 5. $BF_1(r-1)$, $BF_2(r-1)$, $BF_3(r-1)$, and $BF_4(r-1)$ of $B(r)$ are marked in different styles.

$$[v_1v_2\ldots v_\ell v_{\ell+1}v_{\ell+2}\ldots v_r, j],\quad [v_1v_2\ldots v_\ell v_{\ell+1}v_{\ell+2}\ldots v_r, j-1],\quad \ldots$$
$$[v_1v_2\ldots v_\ell v_{\ell+1}v_{\ell+2}\ldots v_r, \ell],\quad [v_1v_2\ldots v_\ell u_{\ell+1}v_{\ell+2}\ldots v_r, \ell+1],\quad \ldots$$
$$[v_1v_2\ldots v_\ell u_{\ell+1}u_{\ell+2}\ldots u_r, j],\quad [v_1v_2\ldots v_\ell u_{\ell+1}u_{\ell+2}\ldots u_r, j+1]\quad \ldots$$
$$[v_1v_2\ldots v_\ell u_{\ell+1}u_{\ell+2}\ldots u_r, r].$$

By assumption $u_1u_2\ldots u_\ell = v_1v_2\ldots v_\ell$ and $u_r = 0$. It follows therefore that $[v_1v_2\ldots v_\ell u_{\ell+1}u_{\ell+2}\ldots u_r,\ r] = [u_1u_2\ldots u_{r-1}0,\ r]$. This shortest path between v and w traverses u. Therefore, $d(v,w) \neq d(u,w)$. The other subcases are similar.

Case 2 (*u and v are horizontal*). This is similar to case 1. However, the corresponding landmark is $[0u_2\ldots u_r,\ 0]$ or $[0u_2\ldots u_r, 2r]$.

Case 3 (*u and v are neither horizontal nor vertical*). Let $u = [u_1u_2\ldots u_r,\ j]$ and $v = [v_1v_2\ldots v_r,\ k]$. Let us consider a subcase where u and v are the vertices of $BF_1(r-1)$. That is, $j < r$, $k < r$, and $u_r = v_r = 0$. Let us consider first the case where $j \neq k$ (the case $j = k$ is similar). Let us assume that $j > k$. Consider a landmark $w = [u_1u_2\ldots u_{r-1}0,\ r]$. It is possible to verify that $d(w,u) = r - j$. All the level ℓ nodes of $B(r)$ form an independent set for each ℓ. Thus, a shortest path from w at level r to v at level k traverses some vertex at level t, $k \leqslant t \leqslant r$. Thus, $d(w,v) \geqslant r-k$. Since $j > k$, we have $r-k > r-j$ and thus $d(v,w) \neq d(u,w)$. The other subcases are similar. See Figs. 8 and 9 for verification. □

Corollary 9. *Let* $M_0 = \{[0w_2\ldots w_r, 0] : w_2\ldots w_r$ *is any binary sequence*$\}$ *and* $M_1 = \{[w_1w_2\ldots w_{r-1}0,\ r] : w_1w_2\ldots w_{r-1}$ *is any binary sequence*$\}$. *Then* $M_0 \cup M_1$ *is a metric basis of* $BF(r)$.

Theorem 10. *The minimum metric dimension problem is polynomially solvable for Benes and butterfly networks.*

3.2. NP-hardness of the MMD problem for bipartite graphs

The minimum metric dimension problem is *NP*-hard for general graphs [1,2]. We now show that the problem of finding the minimum metric dimension of an arbitrary bipartite graph is *NP*-hard. The basic idea of this proof is due to S. Khuller, B. Ragavachari, and A. Rosenfeld. We give a sketch of the construction here. The reader will find the proof in [4].

The problem is clearly in *NP*. We give the *NP*-hardness proof by a reduction from 3-SAT. Consider an arbitrary input to 3-SAT, a formula F with n variables and m clauses. Let the variables be $x_1, x_2\ldots x_n$ and the clauses be

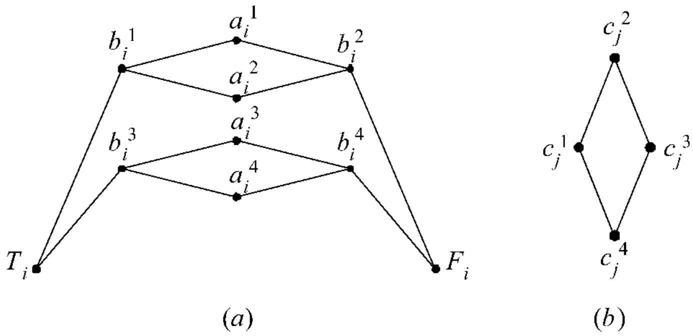


Fig. 6. (a) Variable gadget of x_i. (b) Clause gadget of C_j.

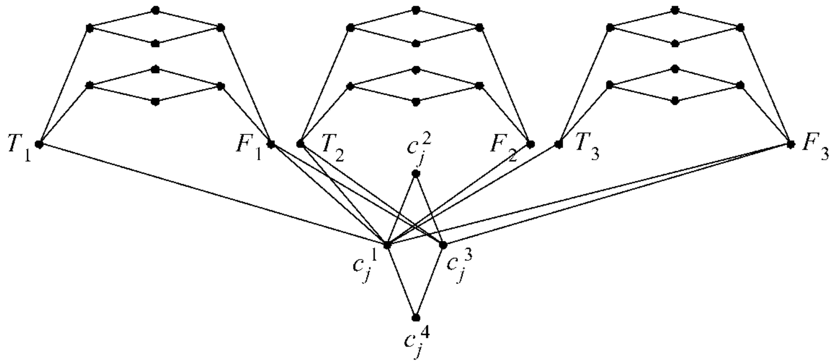


Fig. 7. Clause $C_j = x_1 \vee \bar{x}_2 \vee x_3$.

$C_1, C_2 \ldots C_m$. Without loss of generality, we assume that for every i, $1 \leqslant i \leqslant n$, there exists j, $1 \leqslant j \leqslant m$ such that C_j contains either x_i or $\bar{x}_i$. In other words, we assume that every literal x_i is in some clause C_j. Now onwards x_l, $1 \leqslant l \leqslant n$ is called a positive literal and $\bar{x}_l$ is called a negative literal. For each variable x_i we construct a variable gadget as shown in Fig. 6(a). The nodes T_i and F_i are the "true" and "false" ends of the gadget. The gadget is attached to the rest of the graph only through these nodes.

Suppose $C_j = y_{i_1} \vee y_{i_2} \vee y_{i_3}$, where $y_{i_k} = x_{i_k}$ or $\bar{x}_{i_k}$, $1 \leqslant k \leqslant 3$, is a literal in clause C_j. For each such clause C_j we construct a clause gadget as shown in Fig. 6(b). We now show the connections between the clause and variable gadgets. If a variable x_i occurs as a positive literal in clause C_j, we add the edges (T_i, c_j^1), (F_i, c_j^1) and (F_i, c_j^3). If it occurs in C_j as a negative literal, we add the same edges, except we replace (F_i, c_j^3) by (T_i, c_j^3). We call these truth-testing edges. Fig. 7 shows the truth testing edges added to the clause $C_j = x_1 \vee \bar{x}_2 \vee x_3$.

Thus the graph G that is constructed from the formula F with n variables and m clauses has $10n + 4m$ nodes. The edges of G are variable gadget edges, clause gadget edges, and truth testing edges. It is clear that given F, G can be easily constructed in polynomial time. Since there is no odd cycle, G is a bipartite graph.

Lemma 11. *G is a bipartite graph.*

We shall now prove that F is satisfiable if and only if the metric dimension of G is exactly $3n + m$.

Lemma 12. *Let x_i be an arbitrary variable in F. Then any metric basis must contain at least one of $\{a_i^1, a_i^2\}$, at least one of $\{a_i^3, a_i^4\}$ and at least one of $\{b_i^1, b_i^2, b_i^3, b_i^4\}$.*

Lemma 13. *Let C_j be an arbitrary clause in F. Then any metric basis must contain at least one of $\{c_j^2, c_j^4\}$.*

Corollary 14. *The metric dimension of G is at least $3n + m$.*

Lemma 15. *If F is satisfiable, the metric dimension of G is $3n + m$.*

Lemma 16. *If the metric dimension of G is $3n + m$, then F is satisfiable.*

Lemmas 15 and 16 together complete the reduction from 3-SAT to the metric dimension problem for bipartite graphs. This completes the proof of the following theorem.

Theorem 17. *The MMD problem is NP-hard for bipartite graphs.*

4. Conclusion

Even though this paper focuses on Benes networks, all the results are applicable to butterfly too. We solve the MMD problem for Benes and butterfly networks. We also show that the MMD problem is *NP*-hard for bipartite graphs. The Benes and butterfly networks are bipartite. Thus we narrow down the gap between the polynomial classes and *NP*-hard classes of the MMD problem.

Though wrapped butterfly is a butterfly-like architecture, it is not straightforward to extend these results to wrapped butterfly. The MMD problem remains open for other fixed interconnection networks such as hypercube, shuffle exchange, star, pancake, De Bruijn and torus architectures. The *NP*-hard problems such as achromatic number problem and minimum crossing number problem [1] are open for Benes and butterfly networks. It is interesting to see whether these problems can be solved using this representation.

Appendix A

Figs. 8 and 9 show the Normal and Diamond representations of Benes $B(4)$, respectively.

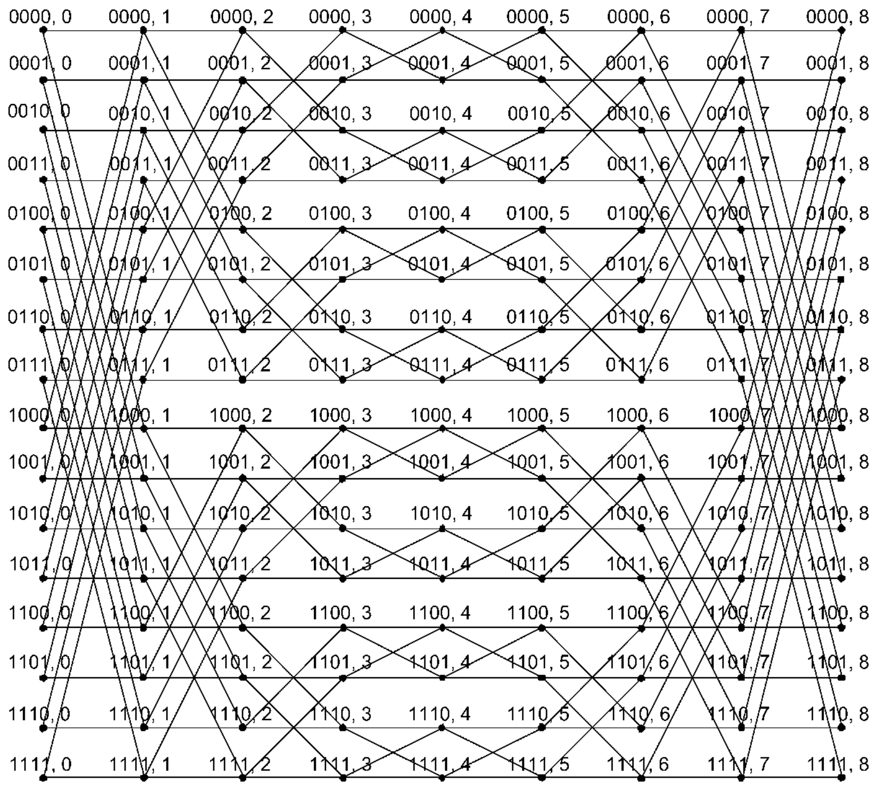


Fig. 8. Normal representation of Benes $B(4)$.

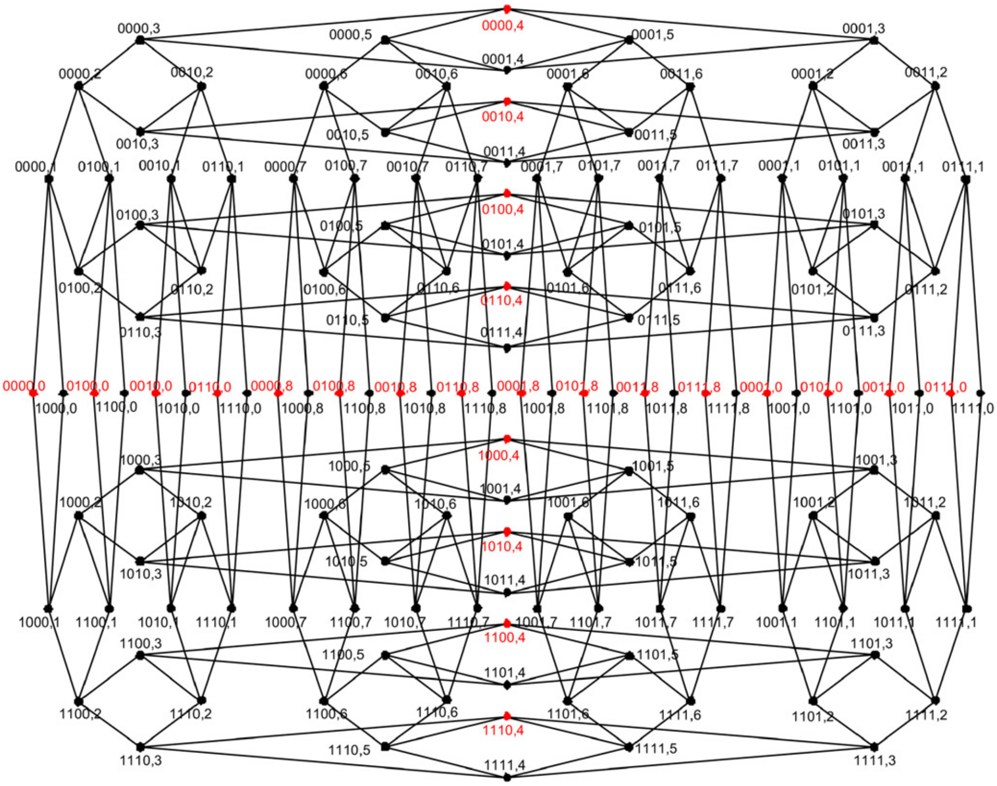


Fig. 9. Benes $B(4)$. The nodes in red colour form a minimum metric basis (for interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article).

References

[1] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP—Completeness, W.H. Freeman and Company, 1979.

[2] S. Khuller, B. Ragavachari, A. Rosenfeld, Landmarks in graphs, Discrete Applied Mathematics 70 (3) (1996) 217–229.

[3] T.F. Leighton, Introduction to Parallel Algorithms and Architecture: Arrays, Trees, Hypercubes, Morgan Kaufmann Publishers, ISBN 1-55860-117-1, 1992.

[4] P.D. Manuel, M.I. Abd-El-Barr, I.Rajasingh, B. Rajan, An efficient representation of Benes networks and its applications, in: Proc. of the Sixteenth Australasian Workshop on Combinatorial Algorithms, September 18–21, Australia, 2005, pp. 217–230.

[5] B. Rajan, I. Rajasingh, J.A. Cynthia, P.D. Manuel, On minimum metric dimension, in: The Indonesia–Japan Conference on Combinatorial Geometry and Graph Theory, Bandung, Indonesia, September 13–16, 2003.

[6] J. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, ISBN 1-4020-0020-0, 2001.