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An exponentially fitted tridiagonal finite difference method for singularly perturbed differential-difference equations with small shift

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Abstract This paper deals with the singularly perturbed boundary value problem for a linear second order differential-difference equation of convection-diffusion type. In the numerical treatment of such type of problems, first we use Taylor's approximation to tackle the term containing the small shift. A fitting parameter has been introduced in a tridiagonal finite difference method and is obtained from the theory of singular perturbations. Thomas algorithm is used to solve the tridiagonal system. The method is analysed for convergence. Several numerical examples are solved to demonstrate the applicability of the method. Graphs are plotted for the solutions of these problems to illustrate the effect of small shift on the boundary layer solution.

1. Introduction

In this paper we study a linear singularly perturbed differential-difference equation which contains a negative shift in the convection term

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x) \text{ on } 0 < x < 1, \qquad (1)$$

subject to the interval and boundary conditions

$$y(x) = \phi(x), \; x \leqslant 0, \quad y(1) = \beta, \qquad (2)$$

where ε is a small parameter, $0 < \varepsilon \ll 1$ and δ is a shift parameter satisfying $(\varepsilon - \delta a(x)) > 0$ for all $x \in [0,1], a(x) \geqslant M > 0$, $b(x), f(x)$ and $\phi(x)$ are sufficiently smooth functions and β is a constant. For a function $y(x)$ to be a smooth solution to the problem (1–2), it must satisfy (1) and (2), be continuous on $[0,1]$ and be continuously differentiable on $(0,1)$. It is assumed that $b(x) \leqslant -\theta < 0$ where θ is a positive constant. For small values of ε the function $y(x)$ has a boundary layer near $x = 0$.

Singular perturbation problems arise very frequently in fluid mechanics, fluid dynamics, elasticity, aerodynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, WKB problems, the modelling of steady and

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unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, magneto-hydrodynamics duct problems at high Hartman numbers, etc. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. So, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. An over view of some existence and uniqueness results and applications of singularly perturbed equations may be obtained in [1–4].

It is well known that for small values of ε, standard numerical methods for solving such problems are unstable and do not give accurate results. Therefore, it is important to develop suitable numerical methods for solving these problems, whose accuracy does not depend on the parameter value ε, i.e., methods that are convergent ε- uniformly. These include fitted finite difference methods, finite element methods using special elements such as exponential elements, and methods which use a priori refined or special non-uniform grids which condense in the boundary layers in a special manner. Various approaches to the design and analysis of approximate numerical methods for singularly perturbed differential equations can be found in [5–8].

In recent years, there has been a growing interest in the numerical study of singularly perturbed differential-difference equations because of their applications in many scientific and technical fields ([9] and references therein). BVPs involving differential-difference equations are used to study signal transmission with time delays in control theory [10], first exit problems in neurobiology [11,12], the study of optically bistable devices [13], in describing the human pupil-light reflex [14], in variety of models for physiological processes or diseases [15,16]. Lange and Miura [17,18] gave an asymptotic approach to solve boundary value problems for second order singularly perturbed differential-difference equations with small shifts. Extensive numerical work had been initiated by M.K. Kadalbajoo and K.K. Sharma in their papers [19–22]. Some numerical aspects of this type of problems with small shifts were considered in [23,24]. Recently, numerical integration method has been developed by Phaneendra et al. [25] and by Reddy et al. [26] for various classes of these problems.

The numerical method presented here comprises a fitted finite difference scheme on a uniform mesh. Briefly, the outline is as follows: In Section 2, we state some important properties of the analytical solution. The finite difference discretization is given in section 3 for boundary value problems with left end boundary layer. Convergence analysis of numerical scheme is discussed in Section 4. In Section 5, we discuss our method for singularly perturbed differential-difference equations with boundary layer on right end of the underlying interval. To demonstrate the efficiency of the proposed method, numerical experiments are carried out for several test problems and the results are given in Section 6. Finally the conclusions are given in the last section.

2. The continuous problem

Here we show some properties of the solution of (1) and (2).

We consider that the shift parameter (δ) is smaller than singular perturbation parameter (ε). Now, to tackle the term containing delay, we use Taylor's series as pointed out by Cunningham ([27], pp. 222) and Tian [28] in his thesis work. Taking the Taylor's series expansion of the term $y'(x-\delta)$, we have

$$y'(x-\delta) \approx y'(x) - \delta y''(x).$$

Thus we have from Eq. (1) the approximating equation

$$(\varepsilon - \delta a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad 0 \leqslant x \leqslant 1, \tag{3}$$

$$y(0) = \phi(0) = \phi_0(\text{say}), \quad y(1) = \beta. \tag{4}$$

Let L be the operator corresponding to Eq. (3), i.e.,

$$L \equiv (\varepsilon - \delta a(x))\frac{d^2}{dx^2} + a(x)\frac{d}{dx} + b(x)I.$$

The operator L satisfies the following continuous minimum principle and stability estimate:

Lemma 1. *Suppose $\pi(x)$ be any sufficiently smooth function satisfying $\pi(0) \geqslant 0$ and $\pi(1) \geqslant 0$. Then $L\pi(x) \leqslant 0$ for all $x \in (0,1)$ implies that $\pi(x) \geqslant 0$ for all $x \in [0,1]$.*

Proof. Let $z \in [0,1]$ be such that $\pi(z) = \min_{x\in[0,1]}\pi(x)$ and assume that $\pi(z) < 0$. Clearly $z \notin \{0,1\}$, therefore $\pi'(z) = 0$ and $\pi''(z) \geqslant 0$. Now we have

$$L\pi(z) = (\varepsilon - \delta a(z))\pi''(z) + a(x)\pi'(z) + b(x)\pi(z) > 0,$$

which contradicts our assumption, therefore we must have $\pi(z) \geqslant 0$ and thus $\pi(x) \geqslant 0 \ \forall x \in [0,1]$.

Now we are able to show the stability of solutions of the continuous problem (3–4). □

Lemma 2. *Let $y(x)$be the solution of the problem (3–4), then we have*

$$\|y\| \leqslant \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|),$$

where $\|.\|$ is the l_∞ norm given by $\|y\| = \max_{x\in[0,1]}|y(x)|$.

Proof. Let us construct the two barrier functions $\psi^\pm$ defined by

$$\psi^\pm(x) = \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|) \pm y(x).$$

Then we have

$$\psi^\pm(0) = \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|) \pm y(0)$$
$$= \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|) \pm \phi_0, \text{ since } y(0) = \phi_0 \geqslant 0,$$

$$\psi^\pm(1) = \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|) \pm y(1)$$
$$= \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|) \pm \beta, \text{ since } y(1) = \beta \geqslant 0,$$

and we have

$$L_\varepsilon\psi^\pm(x) = (\varepsilon - \delta a(x))(\psi^\pm(x))' + a(x)(\psi^\pm(x))' + b(x)\psi^\pm(x)$$
$$= b(x)(\theta^{-1}\|f\| + \max(|\phi_0|, |\beta|)) \pm L_\varepsilon y(x)$$
$$= b(x)(\theta^{-1}\|f\| + \max(|\phi_0|, |\beta|)) \pm f(x).$$

We have $b(x)\theta^{-1} \leqslant -1$, since $b(x) \leqslant -\theta < 0$.

Using this inequality in the above inequality, we get

$$L_\varepsilon \psi^{\pm}(x) \leqslant (-\|f\| \pm f(x)) + b(x)\max(|\phi_0|, |\beta|) \leqslant 0 \quad \forall x \in (0,1), \text{ since } \|f\| \geqslant f(x).$$

Therefore by the minimum principle [5], we obtain $\psi^{\pm}(x) \geqslant 0$ for all $x \in [0,1]$, which gives the required estimate.

Lemma 1 implies that the solution is unique and since the problem under consideration is linear, the existence of the solution is implied by its uniqueness. Further, the boundedness of the solution is implied by Lemma 2. □

Lemma 3. *Let $y(x) = y_0 + z_0$ be the zeroth order approximation to the solution of (3) and (4), where y_0 represents the zeroth order approximate outer solution (i.e., the solution of the reduced problem) and z_0 represents the zeroth order approximate solution in the boundary layer region. Then for a fixed positive integer i,*

$$\lim_{h\to 0} y(ih) \approx y_0(0) + (\phi(0) - y_0(0))\exp\{-a(0)i\,\rho\}$$

where $\rho = \dfrac{h}{\varepsilon - \delta a(0)}$.

Proof. Let $y_0(x)$ be the solution of reduced problem

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \quad y_0(1) = \beta \text{ and}$$

$z_0(t)$ is the solution of the boundary value problem

$$z_0''(t) + a(0)z_0'(t) = 0, z_0(0) = \phi(0) - y_0(0), z_0(\infty) = 0$$

where $t = \dfrac{x}{\varepsilon - \delta M}$.

From the theory of singular perturbations it is well known that the zeroth order asymptotic approximation to the solution of (3) and (4) is (cf. [4]; pp 22-26)

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\phi(0) - y_0(0)) \times \exp\left\{-\int_0^x \left(\frac{a(x)}{\varepsilon - \delta a(x)}\right)dx\right\}.$$

As we are considering the differential equations on sufficiently small sub intervals, the coefficients could be assumed to be locally constant. Hence

$$y(x) \approx y_0(x) + (\phi(0) - y_0(0))\exp\left\{-\left(\frac{a(0)}{\varepsilon - \delta a(0)}\right)x\right\}.$$

So, at the nodal points we have,

$$y(x_i) \approx y_0(x_i) + (\phi(0) - y_0(0))\exp\left\{-\left(\frac{a(0)}{\varepsilon - \delta a(0)}\right)x_i\right\}, \quad i = 0,1,2,\ldots,N.$$

i.e., $y(ih) \approx y_0(ih) + (\phi(0) - y_0(0))\exp\left\{-\left(\dfrac{a(0)}{\varepsilon - \delta a(0)}\right)ih\right\}$.

Therefore

$$\lim_{h\to 0} y(ih) \approx y_0(0) + (\phi(0) - y_0(0))\exp\{-a(0)i\,\rho\} \text{ for } i = 0,1,2,\ldots,N$$

where $\rho = \frac{h}{\varepsilon - \delta a(0)}$ (cf. [5], pp. 93–94). □

3. Exponentially fitted tri-diagonal finite difference method

Classical methods cannot be expected to perform uniformly well over the full range of values of h (mesh parameter) and ε (perturbation parameter). Almost all uniformly convergent schemes involve coefficients containing exponentials. In this section, we describe an exponentially fitted tri-diagonal finite difference method to obtain the approximate solution of the boundary value problem (3–4). At present, the methods which are accurate to $O(h^2)$ are in common use. The principal attractive feature of the second order methods is that the central difference approximations that are used ultimately lead to a tridiagonal matrix problem which may be efficiently solved using a direct elimination method. Methods having truncation errors smaller than $O(h^2)$ traditionally have been developed by the direct inclusion of higher order differences in the approximations at each mesh point. These type of methods produce a system of equations which are not tridiagonal and hence are more difficult to solve than $O(h^2)$ schemes. A balance must be struck between the level of accuracy achieved and the computational efficiency of the scheme.

We divide the interval [0, 1] into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots, x_n = 1$ be the mesh points. Then we have $x_i = ih,\ i = 0,1,2,\ldots,N$. Using central difference formulae, the finite difference representation of equation (3) may be written at a typical mesh point $x_i,\ i = 0,1,2,\ldots,N$, according to

$$\frac{(\varepsilon - \delta a_i)\sigma(\rho)}{h^2}\left\{\boldsymbol{\delta}^2 - \frac{1}{12}\boldsymbol{\delta}^4\right\}y_i + \frac{a_i}{h}\left\{\boldsymbol{\mu\delta} - \frac{1}{6}\boldsymbol{\mu\delta}^3\right\}y_i + b_i y_i = f_i + B_i y_i, \quad 1 \leqslant i \leqslant N-1, \tag{5}$$

where

$$B_i = \frac{-(\varepsilon - \delta a_i)\sigma}{h^2}\left\{\frac{1}{90}\boldsymbol{\delta}^6 + \cdots\right\} - \frac{a_i}{h}\left\{\frac{1}{30}\boldsymbol{\mu\delta}^5 + \cdots\right\} \text{ and } B_i \text{ is } O(h^4) \tag{6}$$

and $a(x_i) = a_i; b(x_i) = b_i; f(x_i) = f_i; y(x_i) = y_i$ and $\boldsymbol{\delta}, \boldsymbol{\mu}$ are usual central difference operators, defined by Fox [29].

The boundary conditions become

$$y_0 = \phi(x_0),\ y_N = \beta \tag{7}$$

Here we have introduced a fitting parameter $\sigma(\rho)$ in a finite difference scheme and it is required to find $\sigma(\rho)$, in such a way that the solution of (5) converges uniformly in ε to the solution of (3–4).

Eq. (5) provides the basis for a fourth order method, however, at this stage, the left hand side of Eq. (5) is not tridiagonal, because it involves the differences $\boldsymbol{\mu\delta}^3 y_i$ and $\boldsymbol{\delta}^4 y_i$. Evidently tridiagonal estimates of $\boldsymbol{\mu\delta}^3 y_i$ and $\boldsymbol{\delta}^4 y_i$ correct to $O(h^5)$ and $O(h^6)$, respectively, are required and these estimates are obtained as follows: (cf. [30])

Differentiating (3) once with respect to x, then using central difference formulae, gives a tridiagonal $O(h^5)$ approximation for $\boldsymbol{\mu\delta}^3 y_i$ as follows:

$$\boldsymbol{\mu\delta}^3 y_i = \frac{h^3}{\varepsilon - \delta a_i}f_i' + \frac{h\delta a_i'}{\varepsilon - \delta a_i}\boldsymbol{\delta}^2 y_i - \frac{h a_i}{\varepsilon - \delta a_i}\boldsymbol{\delta}^2 y_i - \frac{h^2}{\varepsilon - \delta a_i}a_i'\boldsymbol{\mu\delta} y_i - \frac{h^2}{\varepsilon - \delta a_i}b_i\boldsymbol{\mu\delta} y_i - \frac{h^3}{\varepsilon - \delta a_i}b_i' y_i + C_i y_i, \tag{8}$$

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where

$$C_i = \frac{1}{4}\boldsymbol{\mu\delta}^5 + \cdots - \frac{h\delta a'_i}{12(\varepsilon - \delta a_i)}\boldsymbol{\delta}^4 + \cdots + \frac{ha_i}{12(\varepsilon - \delta a_i)}\boldsymbol{\delta}^4 - \cdots + \frac{h^2 a'_i}{6(\varepsilon - \delta a_i)}\boldsymbol{\mu\delta}^3 - \cdots + \frac{h^2 b_i}{6(\varepsilon - \delta a_i)}\boldsymbol{\mu\delta}^3 - \cdots \quad (9)$$

and $a'(x_i) = a'_i; b'(x_i) = b'_i; f'(x_i) = f'_i; y'(x_i) = y'_i$.

Similarly differentiating (3) twice, then using (3) and central difference formulae, gives a $O(h^6)$ approximation for $\boldsymbol{\delta}^4 y_i$ as follows :

$$\boldsymbol{\delta}^4 y_i = \frac{h^4}{\varepsilon - \delta a_i} f''_i - \frac{h^4}{(\varepsilon - \delta a_i)^2}(2\delta a'_i - a_i) f'_i + h^2\left(\frac{(\delta a'_i - a_i)(2\delta a'_i - a_i)}{(\varepsilon - \delta a_i)^2} + \frac{(\delta a''_i - 2a'_i - b_i)}{\varepsilon - \delta a_i}\right)\boldsymbol{\delta}^2 y_i - h^3\left(\frac{(a'_i + b_i)(2\delta a'_i - a_i)}{(\varepsilon - \delta a_i)^2} + \frac{(a''_i + 2b'_i)}{\varepsilon - \delta a_i}\right)\boldsymbol{\mu\delta} y_i - h^4\left(\frac{b'_i(2\delta a'_i - a_i)}{(\varepsilon - \delta a_i)^2} + \frac{b''_i}{\varepsilon - \delta a_i}\right) y_i + D_i y_i, \quad (10)$$

where

$$D_i = \frac{1}{6}\boldsymbol{\delta}^6 - \cdots - \frac{h^2}{12}\left(\frac{(2\delta a'_i - a_i)(\delta a'_i - a_i)}{(\varepsilon - \delta a_i)^2} + \frac{(\delta a''_i - 2a'_i - b_i)}{(\varepsilon - \delta a_i)}\right)\boldsymbol{\delta}^4 + \cdots + \frac{h^3}{6}\left(\frac{(2\delta a'_i - a_i)(a'_i + b_i)}{(\varepsilon - \delta a_i)^2} + \frac{(a''_i + 2b'_i)}{(\varepsilon - \delta a_i)}\right)\boldsymbol{\mu\delta}^3 - \cdots \quad (11)$$

By substituting (8) and (10) in (5) and simplifying, we obtain a finite difference scheme as follows:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i + \Gamma_i y_i; \quad i = 1, 2, \ldots, N-1, \quad (12)$$

where

$$E_i = \sigma - \frac{\sigma h^2 (\delta a'_i - a_i)(2\delta a'_i - a_i)}{12(\varepsilon - \delta a_i)^2} - \frac{\sigma h^2 (\delta a''_i - 2a'_i - b_i)}{12(\varepsilon - \delta a_i)} + \frac{a_i h^2 (a_i - \delta a'_i)}{6(\varepsilon - \delta a_i)^2} - \frac{\sigma h^4 (a'_i + b_i)(2\delta a'_i - a_i)}{24(\varepsilon - \delta a_i)^2} + \frac{\sigma h^3 (a''_i + 2b'_i)}{24(\varepsilon - \delta a_i)} - \frac{ha_i}{2(\varepsilon - \delta a_i)} - \frac{h^3 a_i (a'_i + b_i)}{12(\varepsilon - \delta a_i)^2}$$

$$F_i = 2\sigma - \frac{\sigma h^2 (\delta a'_i - a_i)(2\delta a'_i - a_i)}{6(\varepsilon - \delta a_i)^2} - \frac{\sigma h^2 (\delta a''_i - 2a'_i - b_i)}{6(\varepsilon - \delta a_i)} + \frac{h^2 a_i (a_i - \delta a'_i)}{3(\varepsilon - \delta a_i)^2} - \frac{\sigma h^4 b'_i (2\delta a'_i - a_i)}{12(\varepsilon - \delta a_i)^2} - \frac{\sigma h^4 b''_i}{12(\varepsilon - \delta a_i)} - \frac{h^4 a_i b'_i}{6(\varepsilon - \delta a_i)^2} - \frac{h^2 b_i}{(\varepsilon - \delta a_i)}$$

$$G_i = \sigma - \frac{\sigma h^2 (\delta a'_i - a_i)(2\delta a'_i - a_i)}{12(\varepsilon - \delta a_i)^2} - \frac{\sigma h^2 (\delta a''_i - 2a'_i - b_i)}{12(\varepsilon - \delta a_i)} + \frac{a_i h^2 (a_i - \delta a'_i)}{6(\varepsilon - \delta a_i)^2} - \frac{\sigma h^4 (a'_i + b_i)(2\delta a'_i - a_i)}{24(\varepsilon - \delta a_i)^2} - \frac{\sigma h^3 (a''_i + 2b'_i)}{24(\varepsilon - \delta a_i)} + \frac{ha_i}{2(\varepsilon - \delta a_i)} + \frac{h^3 a_i (a'_i + b_i)}{12(\varepsilon - \delta a_i)^2}$$

$$H_i = \frac{h^2 f_i}{(\varepsilon - \delta a_i)} + \frac{\sigma h^4 f''_i}{12(\varepsilon - \delta a_i)} + \frac{\sigma h^4 (2\delta a'_i - a_i) f'_i}{12(\varepsilon - \delta a_i)^2} + \frac{h^4 a_i f'_i}{6(\varepsilon - \delta a_i)^2}$$

and the leading order error associated with Eq. (12) is

$$\Gamma_i = \frac{(\varepsilon - \delta a_i)\sigma}{360}\boldsymbol{\delta}^6 - \cdots + \frac{ha_i}{120}\boldsymbol{\mu\delta}^5 - \cdots - \frac{h}{144}\left\{h\sigma\left(\frac{(2\delta a'_i - a_i)(\delta a'_i - a_i)}{(\varepsilon - \delta a_i)} + (\delta a''_i - 2a'_i - b_i)\right) + \frac{2a_i(ha_i - \delta a'_i)}{(\varepsilon - \delta a_i)}\right\}\boldsymbol{\delta}^4 + \cdots + \frac{h^3}{72}\left\{\frac{\sigma(a'_i + b_i)(2\delta a'_i - a_i)}{(\varepsilon - \delta a_i)} + \frac{2a_i(a'_i + b_i)}{(\varepsilon - \delta a_i)} + \sigma(a''_i + 2\mathbf{b}'_i)\right\}\boldsymbol{\mu\delta}^3 - \cdots$$

3.1. Calculation of the fitting parameter

We consider $h \propto O(\varepsilon - \delta a_i)$, i.e., $\frac{h}{\varepsilon - \delta a_i}$ is finite.

Now, taking limit as $h \to 0$ in (12), using Lemma 3 and simplifying, we get the constant fitting parameter as

$$\sigma(\rho) = \frac{2a(0)\rho\left[\rho(\delta a'(0) - a(0)) + 3Coth\left(\frac{a(0)\rho}{2}\right)\right]}{12 - \rho^2(\delta a'(0) - a(0))(2\delta a'(0) - a(0))}$$

where $\rho = \dfrac{h}{\varepsilon - \delta a(0)}$. (13)

To obtain the solution of the original problem, we solve the tridiagonal system (12) where σ is given by (13) subject to the boundary conditions (7). We used Thomas algorithm to solve the tridiagonal system.

4. Convergence analysis

The Numerical scheme (12) can be rewritten as

$$(-\sigma + u_i)y_{i-1} + (2\sigma + v_i)y_i + (-\sigma + w_i)y_{i+1} + g_i + T_i = 0, \quad (14)$$

where

$$u_i = \frac{\sigma h^2}{12(\varepsilon - \delta a_i)^2}(\delta a'_i - a_i)(2\delta a'_i - a_i) + \frac{\sigma h^2}{12(\varepsilon - \delta a_i)}(\delta a''_i - 2a'_i - b_i) - \frac{a_i h^2}{6(\varepsilon - \delta a_i)^2}(a_i - \delta a'_i) + \frac{\sigma h^4}{24(\varepsilon - \delta a_i)^2}(a'_i + b_i)(2\delta a'_i - a_i) - \frac{\sigma h^3}{24(\varepsilon - \delta a_i)}(a''_i + 2b'_i) + \frac{a_i h}{2(\varepsilon - \delta a_i)} + \frac{a_i h^3}{12(\varepsilon - \delta a_i)^2}(a'_i + b_i),$$

$$v_i = -\frac{\sigma(\delta a'_i - a_i)(2\delta a'_i - a_i)h^2}{6(\varepsilon - \delta a_i)^2} - \frac{\sigma(\delta a''_i - 2a'_i - b_i)h^2}{6(\varepsilon - \delta a_i)} + \frac{a_i h^2 (a_i - \delta a'_i)}{3(\varepsilon - \delta a_i)^2} - \frac{\sigma b'_i (2\delta a'_i - a_i)h^4}{12(\varepsilon - \delta a_i)^2} - \frac{\sigma b''_i h^4}{12(\varepsilon - \delta a_i)} - \frac{a_i b'_i h^4}{6(\varepsilon - \delta a_i)^2} - \frac{b_i h^2}{(\varepsilon - \delta a_i)},$$

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$$w_i = \frac{\sigma(\delta a_i' - a_i)(2\delta a_i' - a_i)h^2}{12(\varepsilon - \delta a_i)^2} + \frac{\sigma(\delta a_i'' - 2a_i' - b_i)h^2}{12(\varepsilon - \delta a_i)} - \frac{a_i(a_i - \delta a_i')h^2}{6(\varepsilon - \delta a_i)^2} + \frac{\sigma(a_i' + b_i)(2\delta a_i' - a_i)h^4}{24(\varepsilon - \delta a_i)^2} + \frac{\sigma(a_i'' + 2b_i')h^3}{24(\varepsilon - \delta a_i)} - \frac{a_i h}{2(\varepsilon - \delta a_i)} - \frac{a_i(a_i' + b_i)h^3}{12(\varepsilon - \delta a_i)^2},$$

$$g_i = \frac{f_i h^2}{(\varepsilon - \delta a_i)} + \frac{\sigma f_i'' h^4}{12(\varepsilon - \delta a_i)} + \frac{\sigma(2\delta a_i' - a_i)f_i' h^4}{12(\varepsilon - \delta a_i)^2} + \frac{a_i f_i' h^4}{6(\varepsilon - \delta a_i)^2}.$$

Incorporating the boundary conditions $y_0 = \phi(x_0) = \phi(0)$, $y_N = \beta$ we obtain the system of equations in the matrix form as

$$(D + P)y + M + T(h) = 0 \tag{15}$$

where

$$D = [-\sigma, 2\sigma, -\sigma] = \begin{bmatrix} 2\sigma & -\sigma & 0 & \dots & 0 \\ -\sigma & 2\sigma & -\sigma & \dots & 0 \\ 0 & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & \dots & 0 & -\sigma & 2\sigma \end{bmatrix}$$

and $P = [u_i, v_i, w_i] = \begin{bmatrix} v_1 & w_1 & 0 & \dots & 0 \\ u_2 & v_2 & w_2 & \dots & 0 \\ 0 & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & \dots & 0 & u_{N-1} & v_{N-1} \end{bmatrix}$ are tridiagonal matrices of order $N-1$, and $M = [g_1 + (-\sigma + u_1)\phi(0), g_2, g_3, \dots, g_{N-2}, g_{N-1} + (-\sigma + w_{N-1})\beta]^T$.

$T(h) = O(h^6)$ and $y = [y_1, y_2, \dots, y_{N-1}]^T$, $T(h) = [T_1, T_2, \dots, T_{N-1}]^T, 0 = [0, 0, \dots, 0]^T$ are the associated vectors of Eq. (15).

Let $Y = [Y_1, Y_2, \dots, Y_{N-1}]^T \cong y$ which satisfies the equation

$$(D + P)Y + M = 0. \tag{16}$$

Let $e_i = Y_i - y_i, i = 1, 2, \dots, N-1$ be the discretization error so that $E = [e_1, e_2, \dots e_{N-1}]^T = Y - y$.

Subtracting Eq. (15) from Eq. (16) we get

$$(D + P)E = T(h). \tag{17}$$

Let $|a_i| \leqslant C_1, |a_i'| \leqslant C_2, |a_i''| \leqslant C_3, |b_i| \leqslant K_1, |b_i'| \leqslant K_2, |b_i''| \leqslant K_3$

Let $p_{i,j}$ be the $(i,j)^{th}$ element of the matrix P, then

$$|p_{i,i+1}| = |w_i| \leqslant h \left[\begin{matrix} \frac{\sigma(\delta C_2 - C_1)(2\delta C_2 - C_1)h}{12(\varepsilon - \delta C_1)^2} + \frac{\sigma(\delta C_3 - 2C_2 - K_1)h}{12(\varepsilon - \delta C_1)} + \frac{C_1(C_1 - \delta C_2)h}{6(\varepsilon - \delta C_1)^2} + \\ \frac{\sigma(C_2 + K_1)(2\delta C_2 - C_1)h^3}{24(\varepsilon - \delta C_1)^2} + \frac{\sigma(C_3 + 2K_2)h^2}{24(\varepsilon - \delta C_1)} + \frac{C_1}{2(\varepsilon - \delta C_1)} + \frac{C_1(C_2 + K_1)h^2}{12(\varepsilon - \delta C_1)^2} \end{matrix} \right];$$

$$i = 1, 2, \dots, N-2, \tag{18a}$$

$$|p_{i,i-1}| = |u_i| \leqslant h \left[\begin{matrix} \frac{\sigma(\delta C_2 - C_1)(2\delta C_2 - C_1)h}{12(\varepsilon - \delta C_1)^2} + \frac{\sigma(\delta C_3 - 2C_2 - K_1)h}{12(\varepsilon - \delta C_1)} + \frac{C_1(C_1 - \delta C_2)h}{6(\varepsilon - \delta C_1)^2} + \\ \frac{\sigma(C_2 + K_1)(2\delta C_2 - C_1)h^3}{24(\varepsilon - \delta C_1)^2} + \frac{\sigma(C_3 + 2K_2)h^2}{24(\varepsilon - \delta C_1)} + \frac{C_1}{2(\varepsilon - \delta C_1)} + \frac{C_1(C_2 + K_1)h^2}{12(\varepsilon - \delta C_1)^2} \end{matrix} \right];$$

$$i = 2, 3, \dots, N-1. \tag{18b}$$

Thus for sufficiently small h (i.e., as $h \to 0$), we have

$$-\sigma + |p_{i,i+1}| < 0, i = 1, 2, \dots, N-2, \tag{19a}$$

$$-\sigma + |p_{i,i-1}| < 0, i = 2, 3, \dots, N-1. \tag{19b}$$

Hence, the matrix $(D + P)$ is irreducible [31].

Let S_i be the sum of the elements of the i^{th} row of the matrix (D + P), then

$$S_i = \sigma - \frac{\sigma(\delta a_i' - a_i)(2\delta a_i' - a_i)h^2}{12(\varepsilon - \delta a_i)^2} - \frac{\sigma(\delta a_i'' - 2a_i' - b_i)h^2}{12(\varepsilon - \delta a_i)} + \frac{a_i(a_i - \delta a_i')h^2}{6(\varepsilon - \delta a_i)^2} - \frac{\sigma b_i'(2\delta a_i' - a_i)h^4}{12(\varepsilon - \delta a_i)^2} - \frac{\sigma b_i'' h^4}{12(\varepsilon - \delta a_i)} - \frac{a_i b_i' h^4}{6(\varepsilon - \delta a_i)^2} - \frac{b_i h^2}{(\varepsilon - \delta a_i)} + \frac{\sigma(a_i' + b_i)(2\delta a_i' - a_i)h^4}{24(\varepsilon - \delta a_i)^2} + \frac{\sigma(a_i'' + 2b_i')h^3}{24(\varepsilon - \delta a_i)} - \frac{a_i h}{2(\varepsilon - \delta a_i)} - \frac{a_i(a_i' + b_i)h^3}{12(\varepsilon - \delta a_i)^2} \quad \text{for } i = 1,$$

$$S_i = \frac{\sigma(a_i' + b_i)(2\delta a_i' - a_i)h^4}{12(\varepsilon - \delta a_i)^2} - \frac{\sigma b_i'(2\delta a_i' - a_i)h^4}{12(\varepsilon - \delta a_i)^2} - \frac{\sigma b_i'' h^4}{12(\varepsilon - \delta a_i)} - \frac{a_i b_i' h^4}{6(\varepsilon - \delta a_i)^2} - \frac{b_i h^2}{(\varepsilon - \delta a_i)} \quad \text{for } i = 2, 3, \dots, N-2,$$

$$S_i = \sigma - \frac{\sigma(\delta a_i' - a_i)(2\delta a_i' - a_i)h^2}{12(\varepsilon - \delta a_i)^2} - \frac{\sigma(\delta a_i'' - 2a_i' - b_i)h^2}{12(\varepsilon - \delta a_i)} + \frac{a_i(a_i - \delta a_i')h^2}{6(\varepsilon - \delta a_i)^2} - \frac{\sigma b_i'(2\delta a_i' - a_i)h^4}{12(\varepsilon - \delta a_i)^2} + \frac{\sigma(a_i' + b_i)(2\delta a_i' - a_i)h^4}{24(\varepsilon - \delta a_i)^2} - \frac{\sigma b_i'' h^4}{12(\varepsilon - \delta a_i)} - \frac{\sigma(a_i'' + 2b_i')h^3}{24(\varepsilon - \delta a_i)} - \frac{a_i b_i' h^4}{6(\varepsilon - \delta a_i)^2} + \frac{a_i h}{2(\varepsilon - \delta a_i)} - \frac{b_i h^2}{(\varepsilon - \delta a_i)} + \frac{a_i(a_i' + b_i)h^3}{12(\varepsilon - \delta a_i)^2} \quad \text{for } i = N-1.$$

Let

$$C_{1^*} = \min_{1 \leqslant i \leqslant N-1} |a_i|, \quad C_1^* = \max_{1 \leqslant i \leqslant N-1} |a_i|, \quad C_{2^*} = \min_{1 \leqslant i \leqslant N-1} |a_i'|,$$

$$C_2^* = \max_{1 \leqslant i \leqslant N-1} |a_i'|, \quad C_{3^*} = \min_{1 \leqslant i \leqslant N-1} |a_i''|, \quad C_3^* = \max_{1 \leqslant i \leqslant N-1} |a_i''|,$$

$$K_{1^*} = \min_{1 \leqslant i \leqslant N-1} |b_i|, \quad K_1^* = \max_{1 \leqslant i \leqslant N-1} |b_i|, \quad K_{2^*} = \min_{1 \leqslant i \leqslant N-1} |b_i'|,$$

$$K_2^* = \max_{1 \leqslant i \leqslant N-1} |b_i'|, \quad K_{3^*} = \min_{1 \leqslant i \leqslant N-1} |b_i''|, \quad K_3^* = \max_{1 \leqslant i \leqslant N-1} |b_i''|.$$

Then

$$0 < C_{1^*} \leqslant C_1 \leqslant C_1^*, \; 0 < C_{2^*} \leqslant C_2 \leqslant C_2^*, \; 0 < C_{3^*} \leqslant C_3 \leqslant C_3^*,$$
$$0 < K_{1^*} \leqslant K_1 \leqslant K_1^*, \; 0 < K_{2^*} \leqslant K_2 \leqslant K_2^*, \; 0 < K_{3^*} \leqslant K_3 \leqslant K_3^*.$$

For sufficiently small $h, (D + P)$ is monotone [31,32]. Hence $(D + P)^{-1}$ exists and $(D + P)^{-1} \geqslant 0$.

From the error Eq. (17) we have $\|E\| \leqslant \|(D + P)^{-1}\|.\|T\|$.

For sufficiently small h, we have

$$S_i > \frac{h^4}{24(\varepsilon - \delta C_1)^2}[-2\sigma(2\delta C_2 - C_1)K_2 - 2(\varepsilon - \delta C_1)\sigma K_3 - 4C_1 K_2 + \sigma(C_2 + K_1)(2\delta C_2 - C_1)h^4] > \frac{h^4}{24(\varepsilon - \delta C_1)^2} Q_1 \quad \textit{for } i = 1, \tag{20a}$$

where $Q_1 = \lfloor -2\sigma(2\delta C_2 - C_1)K_2 - 2(\varepsilon - \delta C_1)\sigma K_3 - 4C_1 K_2 + \sigma(C_2 + K_1)(2\delta C_2 - C_1)h^4 \rfloor$

$$S_i > \frac{h^4}{12(\varepsilon-\delta a_i)^2}[\sigma(C_2+K_1)(2\delta C_2-C_1)-\sigma(2\delta C_2 - C_1)K_2 - \sigma(\varepsilon-\delta C_1)K_3 - 2C_1K_2] > \frac{h^4}{12(\varepsilon-\delta C_1)^2}Q_2 \quad for\ i=2,3,\ldots,N-2, \tag{20b}$$

where $Q_2 = [\sigma(C_2+K_1)(2\delta C_2 - C_1) \ -\sigma(2\delta C_2 - C_1)K_2 - \sigma(\varepsilon-\delta C_1)K_3 - 2C_1K_2]$

$$S_i > \frac{h^4}{24(\varepsilon-\delta a_i)^2}[-2\sigma K_2(2\delta C_2-C_1)+\sigma(C_2+K_1)(2\delta C_2-C_1) - 2\sigma(\varepsilon-\delta C_1)K_3 - 4C_1K_2] > \frac{h^4}{24(\varepsilon-\delta C_1)^2}Q_3 \quad for\ i=N-1, \tag{20c}$$

where $Q_3 = [-2\sigma K_2(2\delta C_2 - C_1) + \sigma(C_2 \ +K_1)(2\delta C_2 - C_1) - 2\sigma(\varepsilon-\delta C_1)K_3 - 4C_1K_2]$.

Let $(D+P)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(D+P)^{-1}$ and we define

$$\|(D+P)^{-1}\| = \max_{1\leqslant i\leqslant N-1}\sum_{k=1}^{N-1}(D+P)^{-1}_{i,k} \text{ and } \|T(h)\| = \max_{1\leqslant i\leqslant N-1}|T_i|.$$

Since $(D+P)^{-1}_{i,k} \geqslant 0$ and $\sum_{k=1}^{N-1}(D+P)^{-1}_{i,k}\cdot S_k = 1$ for $i=1,2,\ldots,N-1$,

We have,

$$(D+P)^{-1}_{i,k} \leqslant \frac{1}{S_k} < \frac{24(\varepsilon-\delta C_1)^2}{h^4}Q_1 \text{ for } k=1,$$

$$(D+P)^{-1}_{i,k} \leqslant \frac{1}{S_k} < \frac{24(\varepsilon-\delta C_1)^2}{h^4}Q_3 \text{ for } k=N-1.$$

Further $\sum_{k=2}^{N-2}(D+P)^{-1}_{i,k} \leqslant \frac{1}{\min\limits_{2\leqslant k\leqslant N-2} S_k} \leqslant \frac{12(\varepsilon-\delta C_1)^2}{h^4}Q_2$ for $i=1,2,\ldots,N-1$.

From the error Eq. (17), using Eqs. (20a)–(20c) we get

$$\|E\| = \frac{24(\varepsilon-\delta C_1)^2}{h^4}|Q_1+2Q_2+Q_3| \times O(h^6) = O(h^2).$$

This establishes the convergence of the finite difference scheme (12).

5. Right end boundary layer problems

Finally, we discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. When $0 < (\varepsilon - \delta a(x)) \ll 1$, $a(x) \leqslant M < 0, b(x) < 0$ throughout the interval [0, 1], where M is some negative constant, the boundary value problem (3) and (4) displays a boundary layer at $x = 1$.

Lemma 4. *Let $y(x) = y_0 + z_0$ be the zeroth order approximation to the solution of (3) and (4), where y_0 represents the zeroth order approximate outer solution (i.e., the solution of the reduced problem) and z_0 represents the zeroth order approximate solution in the boundary layer region. Then for a fixed positive integer i,*

$$\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1))\exp\left\{a(1)\left(\frac{1}{\varepsilon-\delta a(1)} - i\rho\right)\right\}$$

$$\text{where } \rho = \frac{h}{\varepsilon - \delta a(1)}.$$

Proof. The proof is based on asymptotic analysis (cf. [4]; pp. 22–26), and similar to the proof of Lemma 3. □

Applying the same procedure as in section 2 and using Lemma 4, we will get the tridiagonal system (12) with fitting parameter as

$$\sigma(\rho) = \frac{2a(0)\rho\left[\rho(\delta a'(0) - a(0)) + 3Coth\left(\frac{a(1)\rho}{2}\right)\right]}{12 - \rho^2(\delta a'(0) - a(0))(2\delta a'(0) - a(0))}$$

and it can be solved easily by Thomas Algorithm.

6. Numerical results

To demonstrate the applicability of the method we consider three boundary value problems of singularly perturbed linear differential difference equations exhibiting boundary layer at the left end of the interval [0,1], and two problems exhibiting boundary layer at the right end of the under lying interval. These problems were widely discussed in the literature. Since the exact solutions of the problems for different values of δ are not known, the maximum absolute errors for the examples are calculated using the following double mesh principle

$$E_N = \max_{0\leqslant i\leqslant N}|y_i^N - y_{2i}^{2N}|.$$

The maximum absolute error is tabulated in the form of Tables 1–5 for considered examples with $\delta = (0.5)\varepsilon$. The graphs of the

Table 1 The maximum absolute errors for example 1 when $\delta = 0.5\varepsilon$.

ε	N				
	100	200	300	400	500
2^{-1}	4.4909e−006	1.1368e−006	5.0729e−007	2.8592e−007	1.8319e−007
2^{-2}	4.3667e−005	1.1008e−005	4.9055e−006	2.7630e−006	1.7698e−006
2^{-3}	1.5708e−004	3.9549e−005	1.7618e−005	9.9219e−006	6.3544e−006
2^{-4}	4.0146e−004	1.0096e−004	4.4963e−005	2.5319e−005	1.6215e−005
2^{-5}	9.1663e−004	2.2973e−004	1.0224e−004	5.7552e−005	3.6853e−005
2^{-6}	2.0105e−003	4.9626e−004	2.2015e−004	1.2379e−004	7.9234e−005
2^{-7}	4.5289e−003	1.0566e−003	4.6323e−004	2.5936e−004	1.6562e−004
2^{-8}	1.0878e−002	2.3354e−003	9.8549e−004	5.4323e−004	3.4412e−004
2^{-9}	2.6303e−002	5.5967e−003	2.3168e−003	1.1895e−003	7.3598e−004
2^{-10}	5.7290e−002	1.3587e−002	5.4773e−003	2.8384e−003	1.7066e−003

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Table 2 The maximum absolute errors for example 2 when $\delta = 0.5\varepsilon$.

ε	N				
	100	200	300	400	500
2^{-1}	4.8349e−006	1.2130e−006	5.3973e−007	3.0378e−007	1.9448e−007
2^{-2}	1.1017e−005	2.7675e−006	1.2319e−006	6.9346e−007	4.4402e−007
2^{-3}	2.3287e−005	5.8531e−006	2.6059e−006	1.4671e−006	9.3941e−007
2^{-4}	4.7328e−005	1.1891e−005	5.2939e−006	2.9804e−006	1.9085e−006
2^{-5}	9.5223e−005	2.3852e−005	1.0614e−005	5.9743e−006	3.8253e−006
2^{-6}	1.9381e−004	4.7941e−005	2.1278e−005	1.1966e−005	7.6586e−006
2^{-7}	4.0513e−004	9.7643e−005	4.2924e−005	2.4052e−005	1.5367e−005
2^{-8}	1.0492e−003	2.0427e−004	8.8473e−005	4.9008e−005	3.1117e−005
2^{-9}	2.4101e−003	5.3064e−004	6.2934e−004	1.0256e−004	6.4726e−005
2^{-10}	5.3382e−003	1.2222e−003	6.2225e−004	2.6683e−004	1.6907e−004

Table 3 The maximum absolute errors for example 3 when $\delta = 0.5\varepsilon$.

ε	N				
	100	200	300	400	500
2^{-1}	1.9655e−006	4.9026e−007	2.1772e−007	1.2242e−007	7.8321e−008
2^{-2}	1.9179e−006	4.7581e−007	2.1092e−007	1.1850e−007	7.5775e−008
2^{-3}	8.3118e−007	2.1843e−007	9.8653e−008	5.5936e−008	3.5971e−008
2^{-4}	1.0923e−005	2.7568e−006	1.2293e−006	6.9264e−007	4.4374e−007
2^{-5}	3.9166e−005	9.8357e−006	4.3813e−006	2.4672e−006	1.5801e−006
2^{-6}	1.0907e−004	2.7250e−005	1.2118e−005	6.8177e−006	4.3662e−006
2^{-7}	2.6973e−004	6.6690e−005	2.9444e−005	1.6573e−005	1.0595e−005
2^{-8}	6.7411e−004	1.5238e−004	6.6895e−005	3.7323e−005	2.3755e−005
2^{-9}	2.0469e−003	3.5612e−004	1.5042e−004	8.1365e−005	5.1417e−005
2^{-10}	6.7449e−003	1.0583e−003	3.8111e−004	1.8324e−004	1.1370e−004

Table 4 The maximum absolute errors for example 4 when $\delta = 0.5\varepsilon$.

ε	N				
	100	200	300	400	500
2^{-1}	7.6419e−006	1.9149e−006	8.5173e−007	4.7929e−007	3.0680e−007
2^{-2}	1.9827e−005	4.9725e−006	2.2124e−006	1.2452e−006	7.9717e−007
2^{-3}	3.2339e−005	8.1145e−006	3.6108e−006	2.0324e−006	1.3013e−006
2^{-4}	9.0164e−005	2.2669e−005	1.0093e−005	5.6824e−006	3.6386e−006
2^{-5}	2.6106e−004	6.5660e−005	2.9238e−005	1.6462e−005	1.0541e−005
2^{-6}	6.0174e−004	1.5124e−004	6.7341e−005	3.7916e−005	2.4280e−005
2^{-7}	1.3071e−003	3.2652e−004	1.4521e−004	8.1723e−005	5.2322e−005
2^{-8}	2.8233e−003	6.8815e−004	3.0442e−004	1.7099e−004	1.0938e−004
2^{-9}	1.5352e−002	1.4658e−003	6.3557e−004	3.5429e−004	2.2579e−004
2^{-10}	1.5922e−002	7.6345e−003	1.3748e−003	7.4749e−004	4.7012e−004

Table 5 The maximum absolute errors for example 5 when $\delta = 0.5\varepsilon$.

ε	N				
	100	200	300	400	500
2^{-1}	2.0941e−006	5.2369e−007	2.3277e−007	1.3093e−007	8.3798e−008
2^{-2}	7.5810e−006	1.8971e−006	8.4341e−007	4.7449e−007	3.0369e−007
2^{-3}	2.3392e−005	5.8605e−006	2.6064e−006	1.4666e−006	9.3882e−007
2^{-4}	6.0192e−005	1.5093e−005	6.7148e−006	3.7789e−006	2.4192e−006
2^{-5}	1.3524e−004	3.3922e−005	1.5094e−005	8.4949e−006	5.4386e−006
2^{-6}	2.8392e−004	7.1183e−005	3.1671e−005	1.7825e−005	1.1412e−005
2^{-7}	5.8053e−004	1.4514e−004	6.4545e−005	3.6321e−005	2.3252e−005
2^{-8}	1.1856e−003	2.9339e−004	1.3020e−004	7.3213e−005	4.6853e−005
2^{-9}	6.4097e−003	5.9656e−004	2.6278e−004	1.4735e−004	9.4174e−005
2^{-10}	5.3993e−003	3.3651e−003	5.3829e−004	2.9911e−004	1.9027e−004

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solution of the considered examples for different values of shift parameter are plotted in Figs. 1–10 to examine the effect of small shift on the boundary layer behaviour of the solution. Our results are compared with the results given in [17,20,21]. It has been observed that the proposed method gives high accurate numerical results and higher order of convergence than the upwind finite difference scheme.

The numerical rate of convergence for all the examples have been calculated by the formula

$$R_N = \frac{\log|E_N/E_{2N}|}{\log 2}$$

and it has been observed that for all the examples cited below $R_N \approx 2$.

Example 1 (17, pp. 254). $\varepsilon y''(x) + y'(x-\delta) + y(x) = 0$, subject to the interval and boundary conditions $y(x) = 1; -\delta \leqslant x \leqslant 0, y(1) = 1$.

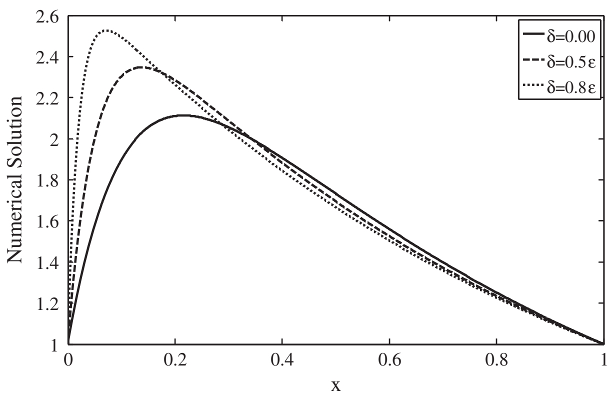

Figure 1 The numerical solution of example 1 with $\varepsilon = 0.1$.

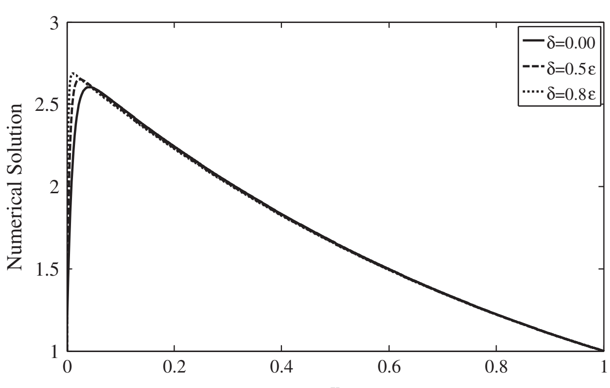

Figure 2 The numerical solution of example 1 with $\varepsilon = 0.01$.

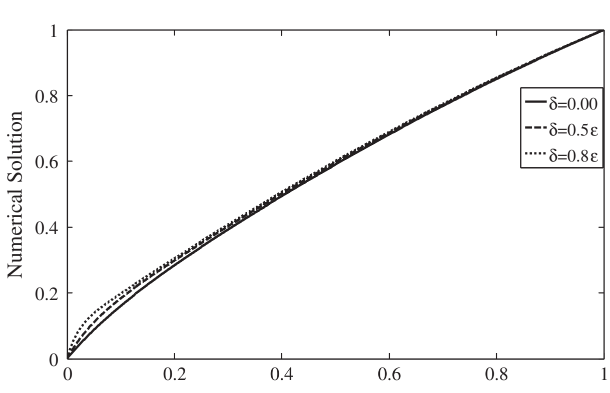

Figure 3 The numerical solution of example 2 with $\varepsilon = 0.1$.

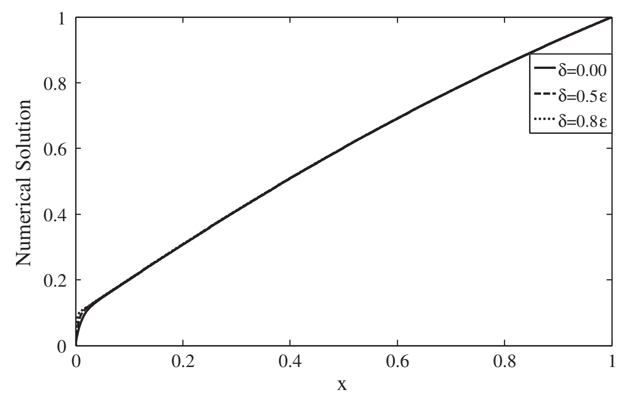

Figure 4 The numerical solution of example 2 with $\varepsilon = 0.01$.

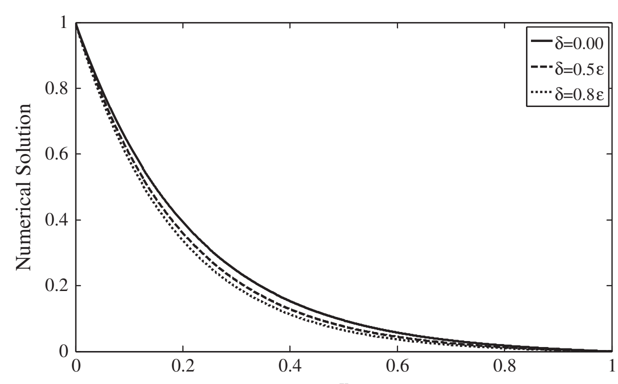

Figure 5 The numerical solution of example 3 with $\varepsilon = 0.1$.

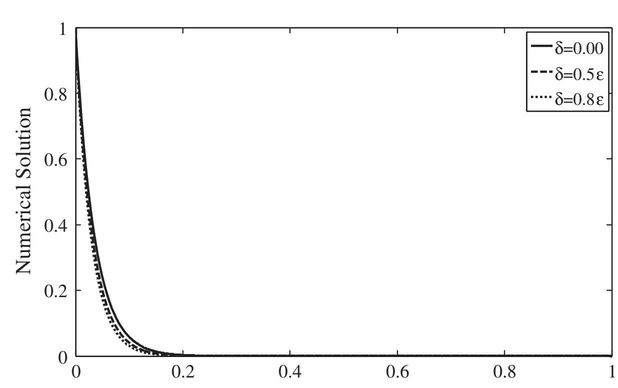

Figure 6 The numerical solution of example 3 with $\varepsilon = 0.01$.

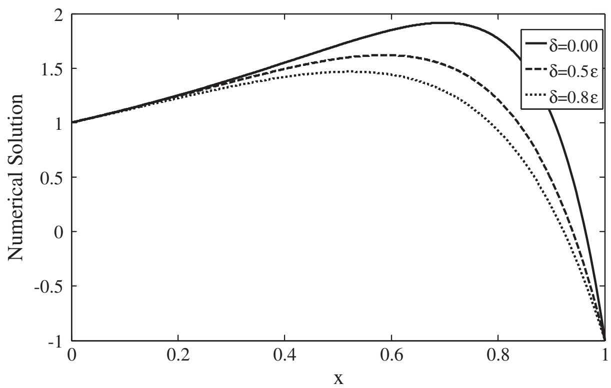

Figure 7 The numerical solution of example 4 with $\varepsilon = 0.1$.

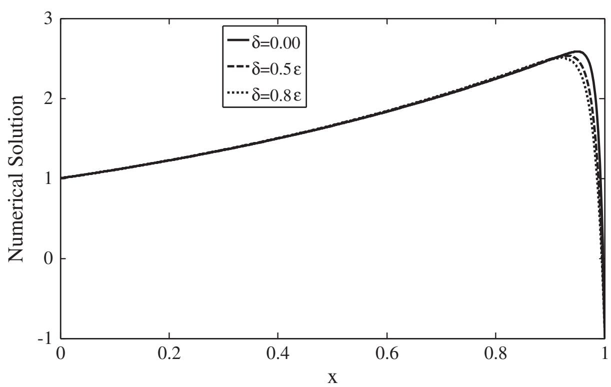

Figure 8 The numerical solution of example 4 with $\varepsilon = 0.01$.

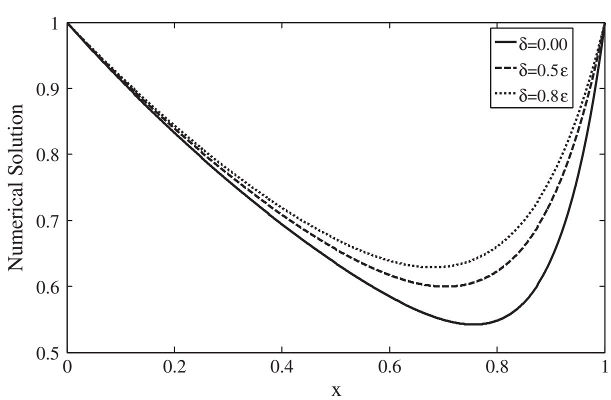

Figure 9 The numerical solution of example 5 with $\varepsilon = 0.1$.

Example 2. $\varepsilon y''(x) + (1+x)y'(x-\delta) - e^{-x}y(x) = 1$, subject to the interval and boundary conditions $y(x) = 0; -\delta \leqslant x \leqslant 0$, $y(1) = 1$.

Example 3 (20, pp. 699). $\varepsilon y''(x) + 0.25y'(x-\delta) - y(x) = 0$, subject to the interval and boundary conditions $y(x) = 1; -\delta \leqslant x \leqslant 0, y(1) = 0$.

Example 4 (20, pp. 707). $\varepsilon y''(x) - y'(x-\delta) + y(x) = 0$, subject to the interval and boundary conditions $y(x) = 1; -\delta \leqslant x \leqslant 0, y(1) = -1$.

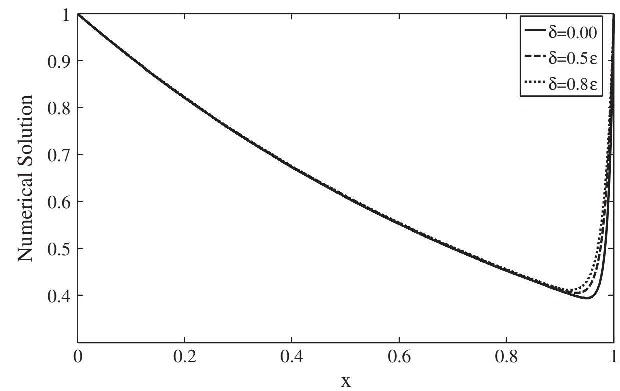

Figure 10 The numerical solution of example 5 with $\varepsilon = 0.01$.

Example 5 (21, pp. 26). $\varepsilon y''(x) - e^{x}y'(x-\delta) - y(x) = 0$, subject to the interval and boundary conditions $y(x) = 1; -\delta \leqslant x \leqslant 0$, $y(1) = 1$.

7. Conclusions

In this paper an exponentially fitted tridiagonal finite difference method is presented for solving boundary value problems for singularly perturbed differential–difference equations containing a small negative shift. The method is developed for problems with shift parameter smaller than the perturbation parameter. The method is shown to have almost second order parameter uniform convergence. An extensive amount of computational work has been carried out to demonstrate the proposed method and to show the effect of shift parameter on the boundary layer behaviour of the solution. The maximum absolute error is tabulated in the form of Tables 1–5 for the considered examples. The graphs of the solution of the considered examples for different values of delay are plotted in Figs. 1–10 to examine the effect of shift on the boundary layer of the solution. From the figures, we observed that as the shift parameter increases, thickness of the layer decreases in the case when the solution exhibits layer behaviour on the left side while it increases in the case when the solution exhibits boundary layer behaviour on the right side of the interval. On the basis of the numerical results of a variety of examples, it is concluded that the present method offers significant advantage for the linear singularly perturbed differential difference equations.

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