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An unreliable $M^{[X]}$/G/1 retrial Queue with multi optional stages of service and delay in repair

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Abstract: Unreliable vacation retrial queue and multi stages of service delay in repair is studied.After completion of the i^{th} $(i=1,2,...k)$ stage of service, the unit may have the option to choose $(i+1)^{th}$ stage of service with probability θ_i, or with p_i may join into orbit to give feedback or may leave the station with probability $q_i = 1 - p_i - \theta_i,\ (i = 1, 2, ...k - 1)$ and $q_i = 1 - p_i,\ (i = k)$ .After service completion if the orbit has no units, server takes avacation.During repair, the unit waiting in the system to complete the remaining service (delay time) is discussed.We analyzed the system using the method of supplementary variable. Simulation results are given using MATLAB.

1. Introduction

Retrial queueing system with vacations is very useful while dealing with real time situations. The survey on retrial queues by Artelijo et al.[1],Artalejo[2], [3]and Falin et al. [7] is followed to frame this work.Wanget al. [13] have studied the retrial queueing system with single server and second optional services. Recently, Salehiradat al. [11] and Bagyamet al. [4] have discussed about Bernoulli feedback.

Service station breakdowns are very common in queueing systems. Keet al. [9], Choudhury et al. [6] discussed, about two phases of service batch retrial queueing pattern and delaying repair. Chen et al. [5] analyzed the breakdowm queues.Wang et al. [13] and Zhang M et al. [14] discussed the vacations in queueing system.Krishnakumar et al. [10] surveyed a queueing systems.

This paper finds applications in communications oriented systems and in industrial organizations, etc.

2. Characteristics of the model

2.1 Arrival process

Units arriving the system in batches with Poisson arrival rate λ. Let X_k, the number of units in the k^{th} batch, where k =1,2 ,3,… with common distribution $\Pr[X_k = n] = \chi_n$, n= 1,2,3…The PGF (probability generating function) of X *is* $X(z)$.The first and second moments are $E(X)$ and $E(X(X-1))$.

2.2 Retrial process:
If there is no space to wait, one from the arriving unit begins service (if the server is free) and rest are waiting in the orbit. If an arriving batch finds the server either busy or on vacation or breakdown, then the batch joins into an orbit.HereInter-retrial times form an arbitrary distribution $R(x)$ with corresponding Laplace-Stieltijes transform (LST) $R^*(s)$.

2.3 Service process:
Here aserver gives k stages of service. The First Stage Service (FSS) is followed by $i$stages of service. The service time $S_i$for$i=1,2,...k$ has a distribution (general) function $S_i(x)$having LST $S_i^*(s)$ and first and second moments are $E(S_i)$ and $E(S_i^2)$, (i= 1,2,…k).

2.4 Feedback rule:
After completion of i^{th} stage of service the customer may go to $(i+1)^{th}$stage with probability θ_i or may join into the orbit as feedback customer with probability p_i or leaves the system with probability $q_i = 1-\theta_i - p_i$ for i = 1,2,…k-1. If the customer in the last k^{th}stage may join to the orbit with probability p_k or leaves the system with probability $q_k = 1 - p_k$. From this model, the service time or the time required by the customer to complete the service cycle is a random variable S is given by $S = \sum_{i=1}^{k} \Theta_{i-1} S_i$ having the LST $S^*(s) = \prod_{i=1}^{k} \Theta_{i-1} S_i^*(s)$ and the expected value is $E(S) = \sum_{i=1}^{k} \Theta_{i-1} E(S_i)$, where $\Theta_i = \theta_1\theta_2...\theta_i$ and $\Theta_0 = 1.$

2.5 Vacation process:
If the orbit has no units, the server takes a single vacation (simply taking break or secondary job etc.,) of random length V. After finishing the vacation, the server is idle to provide service for primary units or units from the orbit. Here the distribution function $V(x)$ and LST $V^*(s)$ with moments $E(V)$ and $E(V^2)$.

2.6 Breakdownand repair:
The service station may down at any time with Poisson rate α_i where i=1,2,…k during service. The unit on service has to wait to complete the remaining service. This waiting time is taken as delay time. The server continues the service for this unit after the repair process.
Here the waiting time is defined as delay time. The delay time $D_i$has density function $D_i(y)$, Laplace-Stieltijes Transform $D_i^*(s)$ and finite k^{th} moment $E(D_i^k)$ (i=1,2 ,…k and k=1,2). The repair time G_i has the distributions function $G_i(y)$and LST $G_i^*(s)$for (i=1,2,…k).Consider various Probability processes involved in the system are mutually exclusive.

In the steady state, let $R(0)=0, R(\infty)=1, S_i(0)=0, S_i(\infty)=1, i=1,2,...k$ are continuous at x = 0 and $D_i(0)=0, D_i(\infty)=1$, $G_i(0)=0, G_i(\infty)=1$ are continuous at y = 0, $(1 \le i \le k)$ .Let $R^0(t)$, $S_i^0(t)$, $D_i^0(t)$ and $G_i^0(t)$ be the elapsed times for retrial, serviceon i^{th} stage, delayinrepairon i^{th} stage, repair on i^{th} stage, $(1 \le i \le k)$ respectively.Now, a random variable at time t,

$$C(t)=\begin{cases}0, \text{ if the server is idle,}\\ 1, \text{ if the server is busy on } i^{\text{th}} \text{ stage,}\\ 2, \text{ if the server is repair on } i^{\text{th}} \text{ stage,}\\ 3, \text{ if the server is on delaying repair of } i^{\text{th}} \text{ stage,}\\ 4, \text{ if the server is on vacation.}\end{cases}$$

The Markov process $\{C(t), N(t); t \geq 0\}$ describes the system state, where $C(t)$ - the server state and $N(t)$ - the number in orbit at time *t*, the functions $a(x)$, $\mu_i(x)$, $\gamma(x)$, $\eta_i(y)$ and $\xi_i(y)$ are the conditional completion rates for retrial, service, vacation, delayin repair and repair respectively ($1 \leq i \leq k$).

$$a(x)dx = \frac{dR(x)}{1-R(x)}, \mu_i(x)dx = \frac{dS_i(x)}{1-S_i(x)}, \gamma(x)dx = \frac{dV(x)}{1-V(x)}, \eta_i(y)dy = \frac{dD_i(y)}{1-D_i(y)}$$

and $\xi_i(y)dy = \dfrac{dG_i(y)}{1-G_i(y)}$. Define $B_i^* = S_1^* S_2^* ... S_i^*$ and $B_0^* = 1$. The first moment M_{1i} and second moment M_{2i} of B_i^* are given by

$$M_{1i} = \lim_{z \to 1} dB_i^*[A_i(z)] \big/ dz = \sum_{j=1}^{i} \lambda E(X) E(S_j) \left(1 + \alpha_j [E(G_j) + E(D_j)]\right),$$

$$M_{2i} = \lim_{z \to 1} d^2 B_i^*[A_i(z)] \big/ dz^2 = \sum_{j=1}^{i} \left[\begin{array}{l} -M_{1i} \left\{ \begin{array}{l} \lambda E(X(X-1)) + \alpha_j [\lambda E(X(X-1))[E(G_j)+E(D_j)]] \\ -(\lambda E(X))^2 [E(G_j^2) + E(D_j^2)]] \end{array} \right\} \\ \quad + \left(\lambda E(X) E(S_j)\right)^2 E(S_j^2) \left(1 + \alpha_j [E(G_j) + E(D_j)]\right)\right)^2 \end{array} \right],$$

where $A_i(z) = \alpha_i \left(1 - G_i^*(b(z)) D_i^*(b(z))\right)$+b(z) and $b(z) = \lambda\left(1 - X(z)\right)$

Let $\{t_n; n = 1,2,\ldots\}$ be the service period ending time or repair period ending time. In this system, $Z_n = \{C(t_n+), N(t_n+)\}$ forms anembeddedMarkov chain which is ergodic $\Leftrightarrow \rho < 1$, where $\rho = E(X)\left(1 - R^*(\lambda)\right) + \left(\sum_{i=1}^{k} \Theta_{i-1} M_{1i} + \sum_{i=1}^{k} p_i \Theta_{i-1} - \sum_{i=1}^{k-1} \Theta_i M_{1i}\right)$.

3. Steady state probability functions

For the process $\{N(t),\ t \geq 0\}$, define the probabilities at time *t* as,

$P_0(t)$ - *Pr*(the system is empty),

At time *t* and *n* customers in the orbit,

$P_n(x,t)$ - *Pr*(an elapsed retrial time *x* of the retrial customers),

$\Pi_{i,n}(x,t), (1 \leq i \leq k)$ - *Pr*(elapsed service time*x* on i^{th} stage of the customer under service),

$Q_n(x,t)$ - *Pr*(elapsed vacation time *x*of the customer on vacation),

$R_{i,n}(x,y,t), (1 \leq i \leq k)$ - *Pr* (an elapsed times for service is*x*and repair is *y* on i^{th} stage),

$D_{i,n}(x,y,t), (1 \leq i \leq k)$ - *Pr* (elapsed times for serviceis*x* and delayin repairis*y* on i^{th} stage).

The stability condition existsfor*t*≥0, *x*≥0, *y*≥0, *n*≥0 for *i* = 1,2,…*k*.

$$P_0=\lim_{t\to\infty}P_0(t),\ P_n(x)=\lim_{t\to\infty}P_n(x,t),\ \Pi_{i,n}(x)=\lim_{t\to\infty}\Pi_{i,n}(x,t),$$
$$Q_n(x)=\lim_{t\to\infty}Q_n(x,t),\ \Omega_{i,n}(x,y)=\lim_{t\to\infty}\Omega_{i,n}(x,y,t),\ \text{for } t\ge 0. R_{i,n}(x,y)=\lim_{t\to\infty}R_{i,n}(x,y,t),\ \text{for } t\ge 0.$$

3.1 Steady state equations

The following equations are obtained by the supplementary variable technique for (i=1,2,..k).

$$\lambda P_0=\int_0^\infty Q_0(x)\gamma(x)dx. \tag{1}$$

$$\frac{dP_n(x)}{dx}=-\lambda P_n(x)-a(x)P_n(x),\ n\ge 1\ . \tag{2}$$

$$\frac{d\Pi_{i,0}(x)}{dx}=-\lambda\Pi_{i,0}(x)-\alpha_i\Pi_{i,0}(x)-\mu_i(x)\Pi_{i,0}(x)+\int_0^\infty \xi_i(y)R_{i,0}(x,y)dy, n=0. \tag{3}$$

$$\frac{d\Pi_{i,n}(x)}{dx}=-\lambda\Pi_{i,n}(x)-\alpha_i\Pi_{i,n}(x)-\mu_i(x)\Pi_{i,n}(x)+\lambda\sum_{k=1}^{n}\chi_k\Pi_{i,n-k}(x)+\int_0^\infty \xi_i(y)R_{i,n}(x,y)dy, n\ge 1. \tag{4}$$

$$\frac{dQ_0(x)}{dx}+Q_0(x)[\lambda+\gamma(x)]=0,\ n=0. \tag{5}$$

$$\frac{dQ_n(x)}{dx}+[\lambda+\gamma(x)]Q_n(x)=\lambda\sum_{k=1}^{n}\chi_k Q_{n-k}(x),\ n=1,2,... \tag{6}$$

$$\frac{d\Omega_{i,0}(x,y)}{dy}+\Omega_{i,0}(x,y)[\lambda+\xi_i(y)]=0,\ n=0. \tag{7}$$

$$\frac{d\Omega_{i,n}(x,y)}{dy}+\Omega_{i,n}(x,y)[\lambda+\xi_i(y)]=\lambda\sum_{k=1}^{n}\Omega_{i,n-k}(x,y)\chi_k,\ n=1,2,... \tag{8}$$

$$\frac{dR_{i,0}(x,y)}{dy}+R_{i,0}(x,y)[\lambda+\xi_i(y)]=0,\ n=0. \tag{9}$$

$$\frac{dR_{i,n}(x,y)}{dy}+R_{i,n}(x,y)[\lambda+\xi_i(y)]=\lambda\sum_{k=1}^{n}R_{i,n-k}(x,y)\chi_k,\ n\ge 1. \tag{10}$$

Boundary conditions at $x = 0$ and $y = 0$ of the steady state system are

$$P_n(0)=\sum_{i=1}^{k}q_i\int_0^\infty \mu_i(x)\Pi_{i,n}(x)dx+\sum_{i=1}^{k}p_i\int_0^\infty \mu_i(x)\Pi_{i,n-1}(x)dx+\int_0^\infty \gamma(x)Q_n(x)dx,\ n\ge 1\ . \tag{11}$$

$$\Pi_{\mathrm{i},0}(0)=\int_0^\infty a(x)P_1(x)dx+P_0\lambda\chi_1,\ n=0. \tag{12}$$

$$\Pi_{1,n}(0)=\int_0^\infty a(x)P_{n+1}(x)dx+\lambda\sum_{k=1}^{n}\chi_k\int_0^\infty P_{n-k+1}(x)dx+P_0\lambda\chi_{n+1},\ n\ge 1. \tag{13}$$

$$\Pi_{i,n}(0)=\theta_{i-1}\int_0^\infty \mu_{i-1}(x)\Pi_{i-1,n}(x)dx,\ n\ge 1,\ (2\le i\le k). \tag{14}$$

$$Q_0(0)=\sum_{i=1}^{k}q_i\int_0^\infty \mu_i(x)\Pi_{i,0}(x)dx, n=0. \tag{15}$$

$$Q_n(0)=0, n=2,3,... \tag{16}$$

$$\Omega_{i,n}(x,0)=\alpha_i[\Pi_{i,n}(x)]\alpha_i,\ n\geq 1. \tag{17}$$

$$R_{i,n}(x,0)=\int_0^\infty \eta_i(y)\Omega_{i,n}(x,y)dy,\ n\geq 0. \tag{18}$$

The normalizing condition is

$$\left(\begin{array}{l} P_0+\sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx+\ \sum_{n=0}^{\infty}\sum_{i=1}^{k}\int_0^\infty \Pi_{i,n}(x)dx \\ +\sum_{n=0}^{\infty}\sum_{i=1}^{k}\int_0^\infty\int_0^\infty R_{\mathrm{i},n}(x,y)dxdy+\sum_{n=0}^{\infty}\sum_{i=1}^{k}\int_0^\infty\int_0^\infty \Omega_{\mathrm{i},n}(x,y)dxdy+\sum_{n=0}^{\infty}\int_0^\infty Q_n(x)dx \end{array}\right)=1. \tag{19}$$

The above equations are solved by using generating functions. Multiplying (2) to (18) by $\sum_{n=o}^{\infty} z^n$ then,

$$\frac{\partial P(x,z)}{\partial x}=-P(x,z)[\lambda+a(x)]. \tag{20}$$

$$\frac{\partial \Pi_i(x,z)}{\partial x}+[\lambda(1-X(z))+\alpha_i+\mu_i(x)]\Pi_i(x,z)=\int_0^\infty \xi_i(y)R_i(x,y,z)dy. \tag{21}$$

$$\frac{\partial Q(x,z)}{\partial x}+[\lambda(1-X(z))+\gamma(x)]Q(x,z)=0. \tag{22}$$

$$\frac{d\Omega_i(x,y,z)}{dy}+[\lambda(1-X(z))+\xi_i(y)]\Omega_i(x,y,z)=0. \tag{23}$$

$$\frac{dR_i(x,y,z)}{dy}+[\lambda(1-X(z))+\xi_i(y)]R_i(x,y,z)=0. \tag{24}$$

At x = 0 and y = 0,

$$P(0,z)=\sum_{i=1}^{k}\left\{\left(p_i z+q_i\right)\int_0^\infty \Pi_i(x,z)\mu_i(x)dx\right\}+\int_0^\infty Q(x,z)\gamma(x)dx-\lambda P_0-Q_0(0). \tag{25}$$

$$\Pi_1(0,z)=\frac{1}{z}\int_0^\infty a(x)P(x,z)dx+\lambda\frac{X(z)}{z}\int_0^\infty P(x,z)dx+\frac{\lambda X(z)}{z}P_0. \tag{26}$$

$$\Pi_i(0,z)=\theta_{i-1}\int_0^\infty \mu_{i-1}(x)\Pi_{i-1,n}(x)dx,\quad (2\leq i\leq k). \tag{27}$$

$$Q(0,z)=Q_0(0) \tag{28}$$

$$\Omega_i(x,0,z)=\alpha_i[\Pi_i(x,z)]. \tag{29}$$

$$R_i(x,0,z)=\int_0^\infty \eta_i(y)\Omega_i(x,y,z)dy,\ n\geq 0. \tag{30}$$

Solving the equations (20) to (24), it follows that for ($1\leq i\leq k$)

$$P(x,z) = e^{-\lambda x}[1 - R(x)]P(0,z). \tag{31}$$

$$\prod_i (x,z) = e^{-A_i(z)x}[1 - S_i(x)]\prod_i (0,z) \tag{32}$$

$$Q(x,z) = e^{-b(z)x}[1 - V(x)]Q(0,z) \tag{33}$$

$$\Omega_i(x,y,z) = e^{-b(z)y}[1 - D_i(y)]\Omega_i(x,0,z) \tag{34}$$

$$R_i(x,y,z) = e^{-b(z)y}[1 - G_i(y)]R_i(x,0,z), \tag{35}$$

where, $A_i(z) = \alpha_i\left(1 - G_i^*(b(z))D_i^*(b(z))\right) + b(z)$ and $b(z) = \lambda\left(1 - X(z)\right)$.

From (5), $Q_0(x) = Q_0(0)[1 - V(x)]e^{-b\lambda x}$. (36)

Multiplying (36) by $\gamma(x)$ on both sides and integrating with respect to x from 0 to ∞,

from (1) , $Q_0(0) = \dfrac{\lambda P_0}{V^*(\lambda)}$. (37)

From (26) and (31), $\Pi_1(0,z) = \dfrac{P(0,z)}{z}\left[R^*(\lambda)(1 - X(z)) + X(z)\right] + \dfrac{\lambda X(z)}{z}P_0$. (38)

From (32) and (38), $\Pi_i(0,z) = \Theta_{i-1}\Pi_1(0,z)\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right),\ (i = 2,3,...k)$. (39)

Similarly, $\Omega_i(x,0,z) = \alpha_i\Pi_i(0,z)\dfrac{S_i^*\left[A_i(z)\right]}{A_i(z)}$. (40)

From (30) and (34), $R_i(x,0,z) = \Omega_i(x,0,z)D_i^*(b(z))$. (41)

Using (37) and (39) and (33) in (25), then

$$P(0,z) = \sum_{i=1}^{k}\left\{\left(p_i z + q_i\right)\Pi_i(0,z)\left(S_i^*\left[A_i(z)\right]\right)\right\} + Q(0,z)V^*\left[b(z)\right] - \lambda P_0 - \frac{\lambda P_0}{\left[V^*(\lambda)\right]}. \tag{42}$$

$$P(0,z) = \lambda P_0 \times \left\{\frac{X(z)\Sigma + z\left(N(z) - 1\right)}{z - \left[R^*(\lambda)(1 - X(z)) + X(z)\right]\omega}\right\}, \tag{43}$$

where $N(\mathrm{z}) = \dfrac{\left(V^*(b(z)) - 1\right)}{\left[V^*(\lambda)\right]}$,

Using (43) in (26), we get, $\Pi_1(0,z) = \lambda P_0\left\{\dfrac{\left(N(z) - 1\right)\left[R^*(\lambda)(1 - X(z)) + X(z)\right] + X(z)}{z - \left[R^*(\lambda)(1 - X(z)) + X(z)\right]\omega}\right\}$. (44)

$$\Pi_i(0,z)=\lambda P_0\Theta_{i-1}(\mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z)))\left\{\frac{(N(z)-1)\left[R^*(\lambda)(1-X(z))+X(z)\right]+X(z)}{z-\left[R^*(\lambda)(1-X(z))+X(z)\right]\omega}\right\}. \tag{45}$$

$$\Omega_i(x,0,z)=\alpha_i\Theta_{i-1}(\mathrm{B}_i^*(\mathrm{A}_i(z)))\frac{\Pi_1(0,z)}{\mathrm{A}_i(z)}. \tag{46}$$

From (28), we get $Q(0,z)=Q_0(0)=\dfrac{\lambda P_0}{V^*(\lambda)}$ (47)

Using (44) and (40), we get $R_i(x,0,z)=\alpha_i\Theta_{i-1}(\mathrm{B}_i^*(\mathrm{A}_i(z)))D_i^*(b(z))\dfrac{\Pi_1(0,z)}{\mathrm{A}_i(z)}$. (48)

Using Eqn. (31) to Eqn. (35) andEqn.(43)toEqn. (48),

$P(x,z),\Pi_i(x,z),Q(x,z),\ \Omega_i(x,y,z)$ *and* $R_i(x,y,z)$ are obtained under $\rho < 1$ and given below,

$$P(x,z)=\lambda P_0\times\left\{\frac{X(z)\Sigma+z(N(z)-1)}{z-\left[R^*(\lambda)(1-X(z))+X(z)\right]\omega}\right\}(1-R(x))e^{-\lambda x} \tag{49}$$

$$\Pi_i(x,z)=\lambda P_0\left\{\frac{\Theta_{i-1}\left((N(z)-1)\left[R^*(\lambda)(1-X(z))+X(z)\right]+X(z)\right)\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)(1-S_i(x))e^{-A_i(z)x}}{z-\left[R^*(\lambda)(1-X(z))+X(z)\right]\omega}\right\} \tag{50}$$

$$Q(x,z)=\frac{\lambda P_0}{V^*(\lambda)}(1-V(x))e^{-b(z)x} \tag{51}$$

$$\Omega_i(z)=\alpha_i\Theta_{i-1}[1-S_i(x)]e^{-A_i(z)x}[1-D_i(y)]e^{-b(z)y}(\mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z)))\Pi_1(0,z) \tag{52}$$

$$R_i(x,y,z)=\alpha_i\Pi_i(0,z)B_{i-1}^*\left[A_{i-1}(z)\right]D_i^*(b(z))\Pi_1(0,z)[1-S_i(x)]e^{-A_i(z)x}\times[1-G_i(y)]e^{-b(z)y} \tag{53}$$

where, $A_i(z)=\alpha_i\left(1-G_i^*(b(z))D_i^*(b(z))\right)+b(z)$ and $b(z)=\lambda(1-X(z))$.
Next the marginal orbit size distributions due is investigated.

Theorem 3.1.Under$\rho < 1$, the stationary distributions of the numbers in the system when server being idle, busy duringith stage, on vacation, repair on ith stage (for $1\le i\le k$) are given by

$$P(z)=\left(1-R^*(\lambda)\right)P_0\times\left\{\frac{X(z)\Sigma+z(N(z)-1)}{z-\left[R^*(\lambda)(1-X(z))+X(z)\right]\omega}\right\}. \tag{54}$$

$$\Pi_i(z)=\lambda P_0\Theta_{i-1}(\mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z)))\frac{\left(1-S_i^*(A_i(z))\right)}{A_i(z)}\left\{\frac{(N(z)-1)\left[R^*(\lambda)(1-X(z))+X(z)\right]+X(z)}{z-\left[R^*(\lambda)(1-X(z))+X(z)\right]\omega}\right\}. \tag{55}$$

$$Q(z)=\frac{P_0\left(1-V^*(b(z))\right)}{\left[V^*(\lambda)\right](1-X(z))}. \tag{56}$$

$$\Omega_i(z) = \frac{\alpha_i \Theta_{i-1}\left(1 - S_i^*(A_i(z))\right)\left(1 - D_i^*(b(z))\right)}{A_i(z)b(z)} B_{i-1}^*\left[A_{i-1}(z)\right]\Pi_1(0,z). \tag{57}$$

$$R_i(z) = \frac{\alpha_i \Theta_{i-1}\left(1 - S_i^*(A_i(z))\right)\left(1 - G_i^*(b(z))\right)}{A_i(z)b(z)} B_{i-1}^*\left[A_{i-1}(z)\right]D_i^*(b(z))\Pi_1(0,z), \tag{58}$$

where,

$$P_0 = \frac{\left\{1 - E(X)\left(1 - R^*(\lambda)\right) - \omega\right\}}{\left\{\left(1 + \frac{N^1(1)}{E(X)}\right)\left(1 - \left(1 - R^*(\lambda)\right)E(X) - \omega\right) + \sum_{i=1}^{k} \lambda\Theta_{i-1}E(S_i)\left(1 + \alpha_j[E(G_j) + E(D_j)]\right)(E(X) + N'(1) - \left(1 - R^*(\lambda)\right)E(X))\right\}}. \tag{59}$$

Proof.Integrating(49) to (53) with respect to $x$and$y$, defined the followingfor ($1 \leq i \leq k$) $P(z) = \int_0^\infty P(x,z)dx,\ \Pi_i(z) = \int_0^\infty \Pi_i(x,z)dx,\ Q(z) = \int_0^\infty Q(x,z)dx.\ R_i(x,z) = \int_0^\infty R_i(x,y,z)dy,\ R_i(z) = \int_0^\infty R_i(x,z)dx,$ $\int_0^\infty \Omega_i(x,z) = \int_0^\infty \Omega_i(x,y,z)dy,\ \Omega_i(z) = \int_0^\infty \Omega_i(x,z)dx.$ Since,P_0 can be determined using (19). $P_0 + P(1) + Q(1) + \sum_{i=1}^{k}\left(\Pi_i(1) + \Omega_i(1) + R_i(1)\right) = 1$ isobtainedby setting $z = 1$ in (54) to (59).

Theorem 3.2. Under $\rho < 1$, PGF of the system size and orbit size distribution at stationary point of time is

$$K(z) = \frac{Nr(z)}{Dr(z)}, \tag{60}$$

$$where \quad Nr(z) = P_0 \left\{\begin{array}{l} z\left\{\sum_{i=1}^{k} \Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1 - S_i^*(A_i(z))\right)\left((N(z)-1)\left[R^*(\lambda)(1 - X(z)) + X(z)\right] + X(z)\right)\right\} \\ -N(z)\left(z - \left[R^*(\lambda) + X(z)\left(1 - R^*(\lambda)\right)\right]\omega\right) + \left[1 - X(z)\right]\left(\begin{array}{l} z - \left[R^*(\lambda)(1 - X(z)) + X(z)\right]\omega \\ +\left(X(z)\Sigma + z\left(N(z)-1\right)\right)\left(1 - R^*(\lambda)\right)\end{array}\right)\end{array}\right\},$$

$$Dr(z) = \left[1 - X(z)\right]\left(z - \left[R^*(\lambda)(1 - X(z)) + X(z)\right]\Sigma\right),$$

and

$$\omega = \sum_{i=1}^{k} \Theta_{i-1}M_{1i} - \sum_{i=1}^{k} p_i\Theta_{i-1} + \sum_{i=1}^{k-1} \Theta_i M_{1i}.$$

Also

$$H(z) = \frac{NR(z)}{Dr(z)}, \tag{61}$$

$$where \quad NR(z) = P_0 \left\{ \begin{array}{l} \left\{ \sum_{i=1}^{k} \Theta_{i-1} \left(B_{i-1}^{*}\left[A_{i-1}(z)\right]\right)\left(1 - S_i^{*}(A_i(z))\right)\left((N(z)-1)(R^{*}(\lambda) + X(z)\left(1 - R^{*}(\lambda)\right)) + X(z)\right) \right\} \\ -N(z)\left(z - \left[R^{*}(\lambda) + X(z)\left(1 - R^{*}(\lambda)\right)\right]\omega\right) + \left[1 - X(z)\right] \left(\begin{array}{l} z - \left[R^{*}(\lambda) + X(z)\left(1 - R^{*}(\lambda)\right)\right]\omega \\ +\left(X(z)\Sigma + z\left(N(z)-1\right)\right)\left(1 - R^{*}(\lambda)\right) \end{array} \right) \end{array} \right\},$$

Where $P_0$is given in Eq. (59).

Proof. The statement is obtained by using $K(z) = P_0 + P(z) + Q(z) + z\sum_{i=1}^{k}\Pi_i(z) + \sum_{i=1}^{k}\Omega_i(z) + \sum_{i=1}^{k}R_i(z)$ and $H(z) = P_0 + P(z) + Q(z) + \sum_{i=1}^{k}\Pi_i(z) + \sum_{i=1}^{k}\Omega_i(z) + \sum_{i=1}^{k}R_i(z)$.

4. **Performance measures**

Here, the mean numbers in the orbit (L_q), the mean numbers in the system (L_s), the mean waiting time in the system (W_s) and in the queue (W_q) are required to analyze the model.

Theorem 4.1. If the system satisfies $\rho < 1$, then the following probabilities of the server state, that is the server is idle during the retrial, busy during ithstage, on vacation, delaying repair during ithstage and under repair on ithstage respectively are obtained.

$$P = \frac{\left(1 - R^{*}(\lambda)\right)}{\beta_1}\left(E(X) + N'(1) + \omega - 1\right).$$

$$\Pi_i = \sum_{i=1}^{k}\Pi_i = \frac{1}{\beta_1}\sum_{i=1}^{k}\left\{\Theta_{i-1}\lambda E(S_i)\right\}(N'(1) + E(X)R^{*}(\lambda)).$$

$$Q = \frac{1}{\beta_1}\left\{1 - E(X)\left(1 - R^{*}(\lambda)\right) - \omega\right\}\frac{N'(1)}{E(X)}.$$

$$\Omega_i = \sum_{i=1}^{k}\Omega_i = \frac{1}{\beta_1}\sum_{i=1}^{k}\alpha_i E(D_i)\left\{\Theta_{i-1}\lambda E(S_i)\right\}(N'(1) + E(X)R^{*}(\lambda)).$$

$$R_i = \sum_{i=1}^{k}R_i = \frac{1}{\beta_1}\sum_{i=1}^{k}\alpha_i E(G_i)\left\{\Theta_{i-1}\lambda E(S_i)\right\}(N'(1) + E(X)R^{*}(\lambda)).$$

Proof. The statement followed by using

$$P = \lim_{z\to1}P(z), \quad \sum_{i=1}^{k}\Pi_i = \lim_{z\to1}\sum_{i=1}^{k}\Pi_i(z),\ \mathrm{Q} = \lim_{z\to1}\mathrm{Q}(z)\ ,\ \sum_{i=1}^{k}\Omega_i = \lim_{z\to1}\sum_{i=1}^{k}\Omega_i(z)\ and\ \sum_{i=1}^{k}R_i = \lim_{z\to1}\sum_{i=1}^{k}R_i(z).$$

Theorem 4.2. Let L_s, L_q, W_s and W_q be the average system size, average orbit size, average waiting time in the system and average waiting time in the orbit respectively, then under ρ < 1,

$$L_q = P_0\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right],$$

where,

$$Nr_q''(1) = -2\left\{\left\{\sum_{i=1}^{k}\Theta_{i-1}M_{2i}\left(N'(1)+E(X)R^*(\lambda)\right)\right\} - (E(X))^2\left(1-R^*(\lambda)\right) - 2N'(1) + E(X)(2R^*(\lambda)-1)(\omega-1)\right\},$$

$$Dr_q''(1) = -2E(X)\left(1-\rho\right),$$

$$Dr_q''' = 3\left\{E(X)\left[\left(1-R^*(\lambda)\right)\left(E(X(X-1)) + 2E(X)\omega\right) + \tau\right] - E(X(X-1))(1-\rho)\right\},$$

$$Nr_q'''(1) = 3\left\{\begin{array}{l} -\sum_{i=1}^{k}\Theta_{i-1}M_{1i}\left[N''(1) + E(X(X-1)) + E(X)\left(1-R^*(\lambda)\right)(2N'(1)-1)\right] \\ +N'(1)\left[2E(X)\left(1-R^*(\lambda)\right)-1\right]\omega + E(X(X-1))\left[E(X) + R^*(\lambda)N'(1)-1\right] \\ +[N'(1)+E(X(X-1))]\left(1-E(X)\left(1-R^*(\lambda)\right) + \sum_{i=1}^{k}\Theta_{i-1}\left[M_{2i}+2M_{1i}M_{1i-1}\right]-\tau\right) \end{array}\right\},$$

$$\omega = \sum_{i=1}^{k}\Theta_{i-1}M_{1i} - \sum_{i=1}^{k}p_i\Theta_{i-1} + \sum_{i=1}^{k-1}\Theta_i M_{1i},$$

$$\tau = \sum_{i=1}^{k}\Theta_{i-1}M_{2i} + 2\sum_{i=1}^{k}p_i\Theta_{i-1}M_{1i} - \sum_{i=1}^{k-1}\Theta_i M_{2i},$$

$$and \quad \rho = E(X)\left(1-R^*(\lambda)\right) - \omega.$$

$$L_s = P_0\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right],$$

Where, $Nr_s'''(1) = Nr_q'''(1) - 6\sum_{i=1}^{k}\Theta_{i-1}M_{1i}\left(N'(1)+E(X)R^*(\lambda)\right).$

$$W_s = \frac{L_s}{\lambda E(X)} \text{ and } W_q = \frac{L_q}{\lambda E(X)}.$$

Proof: Under ρ < 1, $L_q$is obtained from

$$L_q = \frac{Nr(z)}{Dr(z)} = \lim_{z\to1}\frac{d}{dz}H(z) = H'(1) = P_0\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right].$$

And L_s is obtained from

$$L_s = \frac{Nr(z)}{Dr(z)} = \lim_{z\to1}\frac{d}{dz}K(z) = K'(1) = P_0\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right].$$

$W_s$andW_q are obtained by Little's formula , $L_s = \lambda W_s$ and $L_q = \lambda W_q$.

4.1Special case
Single phase, No retrial, No Vacation and No breakdown, No delaying repair

Let $P\,[X = 1] = 1$, $R^*(\lambda)\to 1$, $P[V = 0] = 1$and $\alpha_i = 0$. Our model can be reduced to multi stage M/G/1 queueing system with Bernoulli feedback. The following results agree with Salehirad and Badamchizadeh [12].

$$K(z) = P_0\left\{\frac{\left(1-S_1^*(A_1(z))\right)+\sum_{i=2}^{k}\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)}{z-\sum_{i=1}^{k}\left\{\left(p_i z+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}}\right\}.$$

5. Numerical illustration

Here, some numerical examples are given using MATLAB. The times for retrial, service vacation and repair respectively are exponentially $f(x)=ve^{-vx}, x>0$ for Erlang-2stage $f(x)=v^2xe^{-vx}, x>0$ and hyper-exponentially $f(x)=cve^{-vx}+(1-c)v^2e^{-v^2x}, x>0$ distributed. And assume the arbitrary values to the parameters satisfies ρ< 1. The computed values of P_0 ,P, Π_i, Q and R_i for (i=1,2,…k) respectively are given in the tables. For the effect o fa, p, γ and ξ_i are retrial rate, feedback probability, vacation rate and repair rate on FSS respectively graphs are given in Figure 1 to 6.

Table 1 indicates that when $E(X)$ increases, then P_0 decreases, L_q and W_q are increasing for the values of λ= 0.5; p_1= 0.2; μ_1 = 15;α_1 = 0.2; ξ_1 = 7; γ=1; k =1;θ_1=0.2;a=5; η_1=7.Table 2 shows that when a increases, then P_0 decreases, L_q and $P$are increasing forλ= 0.2; p_1 = 0.2; μ_1 = 15; α_1 = 0.4; ξ_1 = 7; γ =3; θ_1=0.2; p_2 = 0.3; μ_2 = 10; $\alpha 2$ = 0.6; ξ_2 = 5; η_2 =3; η_1 =4; k =2.; θ_2=0.4; J =2; $E(X)$=1.

Table 1. The effect of Mean batch size $E(X)$ on P_0, Lq and W_q

Retrial distribution	Exponential			Erlang – 2 stage			Hyper – Exponential		
$E(X)$	P_0	L_q	W_q	P_0	L_q	W_q	P_0	L_q	W_q
0.50	0.5585	0.2334	0.4669	0.2882	0.4356	0.8712	0.5634	0.2278	0.4556
0.60	0.5557	0.2398	0.4796	0.2844	0.4680	0.9361	0.5619	0.2313	0.4625
0.70	0.5529	0.2471	0.4943	0.2804	0.5074	1.0148	0.5605	0.2352	0.4703
0.80	0.5500	0.2555	0.5111	0.2761	0.5548	1.1096	0.5591	0.2394	0.4789
0.90	0.5470	0.2651	0.5301	0.2716	0.6115	1.2230	0.5576	0.2442	0.4884

Table 2. The effect of$a$ on P_0, Lq and P

Retrial rate a	Exponential			Erlang – 2 stage			Hyper – Exponential		
a	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
2.00	0.8559	0.1979	0.1212	0.7112	0.8434	0.3935	0.9055	0.1137	0.0768
3.00	0.8565	0.1552	0.0787	0.7161	0.5272	0.2391	0.9057	0.0871	0.0444
4.00	0.8568	0.1365	0.0583	0.7182	0.4162	0.1718	0.9058	0.0769	0.0309
5.00	0.8570	0.1261	0.0463	0.7193	0.3605	0.1341	0.9059	0.0716	0.0237
6.00	0.8571	0.1194	0.0384	0.7200	0.3273	0.1100	0.9059	0.0684	0.0191

Table 3. The effect of p_1 on P_0, L_q and P

Feedback probability	Exponential			Erlang – 2 stage			Hyper – Exponential		
p_1	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
0.10	0.9273	0.0410	0.0345	0.8471	0.1068	0.0806	0.9338	0.0370	0.0315
0.20	0.9226	0.0536	0.0426	0.8350	0.1499	0.1008	0.9296	0.0481	0.0389
0.30	0.9164	0.0717	0.0535	0.8183	0.2203	0.1291	0.9241	0.0641	0.0488
0.40	0.9077	0.0998	0.0686	0.7933	0.3489	0.1711	0.9164	0.0885	0.0625
0.50	0.8947	0.1479	0.0913	0.7522	0.6295	0.2404	0.9050	0.1296	0.0828

Table 4. The effect of (γ) on P_0, L_q and Q

Vacation rate	Exponential			Erlang – 2 stage			Hyper – Exponential		
γ	P_0	L_q	Q	P_0	L_q	Q	P_0	L_q	Q
5.00	0.7707	0.4050	0.1780	0.5083	4.1119	0.2719	0.8086	0.2867	0.1532
6.00	0.7987	0.3716	0.1502	0.5486	3.7027	0.2332	0.8339	0.2625	0.1280
7.00	0.8192	0.3472	0.1299	0.5791	3.3940	0.2040	0.8521	0.2451	0.1099
8.00	0.8348	0.3286	0.1143	0.6030	3.1535	0.1811	0.8658	0.2320	0.0962
9.00	0.8471	0.3140	0.1021	0.6221	2.9612	0.1628	0.8765	0.2210	0.0856

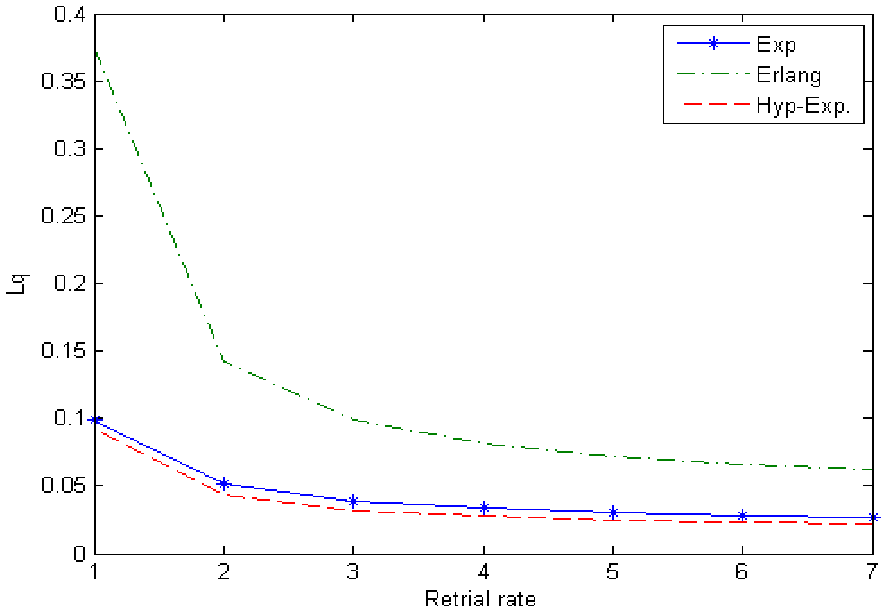


Figure 1. L_q verses a

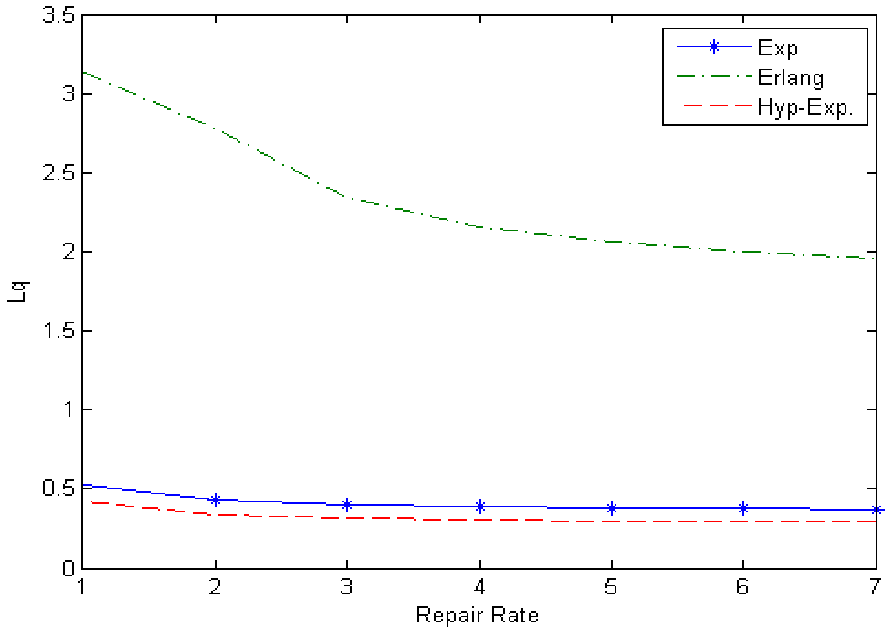


Figure 2. L_q verses ξ_1

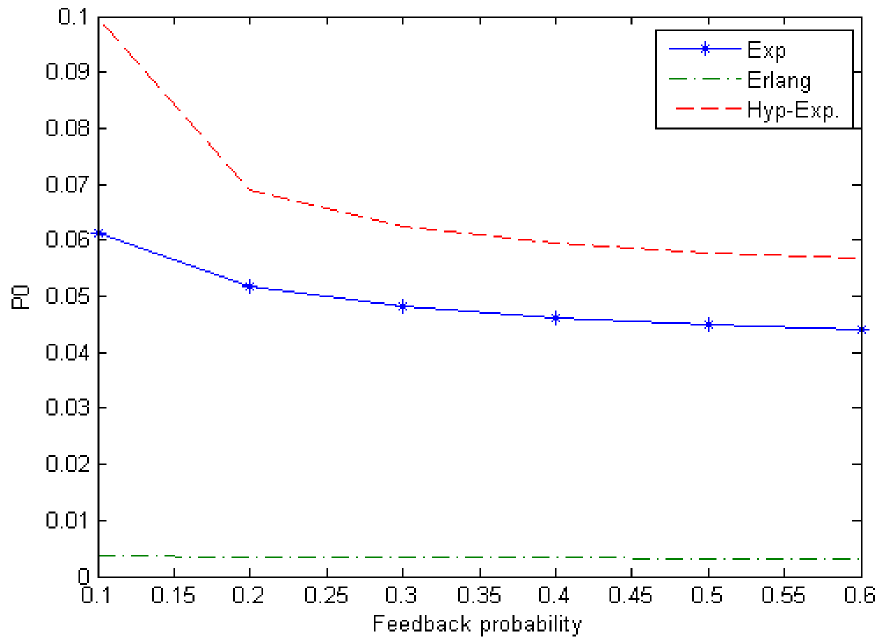


Figure 3. P_0versesη_1

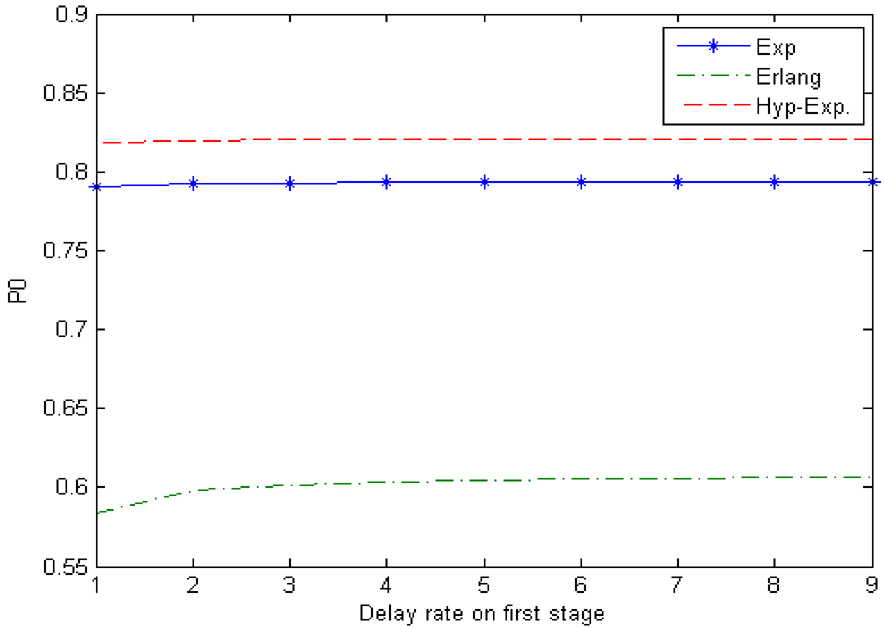
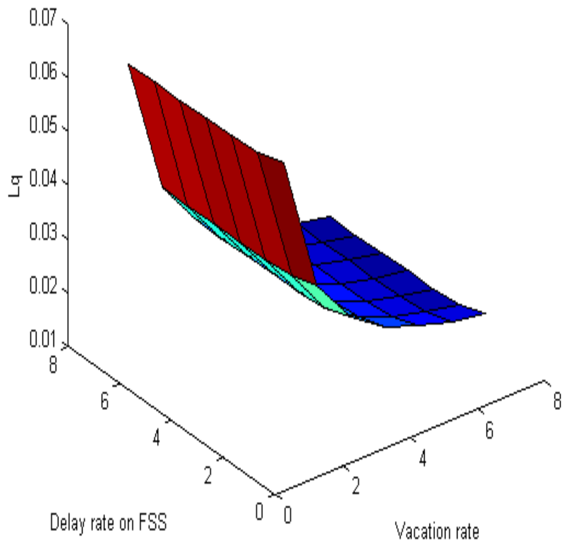


Figure 4. P_0verses p_1



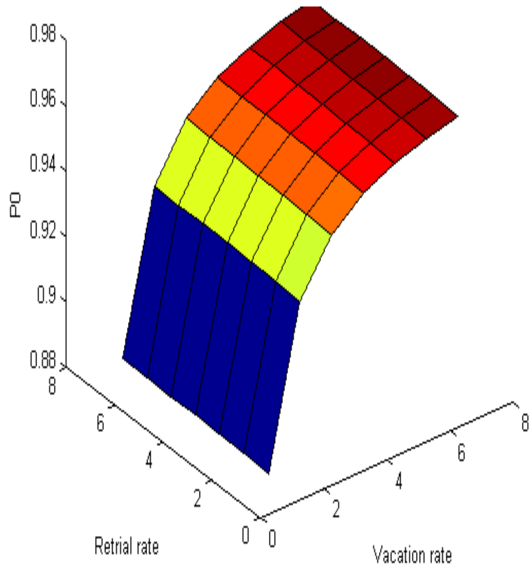


Figure 5. $L_q$versesη_1 *and* γ

Figure 6. P_0verses*a* *and* γ

Table 3 shows that when p_1 increases, then P_0 decreasing, L_q and P are increasing for $\lambda = 0.2$; $\mu_1 = 15$; $\alpha_1 = 0.2$; $\xi_1 = 3$; $\gamma = 5$; $\theta_1 = 0.2$; $\mu_2 = 10$; $\alpha 2 = 0.3$; $\xi_2 = 5$; $\eta_2 = 3$; $\eta_1 = 4$; $k = 1$; $\theta_2 = 0.4$; $E(X)=2$. Table 4 shows that when γincreases, then P_0 -increases, L_q and Q also decreasing for $\lambda = 0.2$; $p_2 = 0.3$; $\mu_1 = 15$; $\alpha_1 = 0.4$; $\xi_1 = 7$; $\theta_1 = 0.2$; $p_1 = 0.2$; $\mu_2 = 7$; $\alpha 2 = 0.6$; $\xi_2 = 5$; $\eta_2 = 3$; $\eta_1 = 4$; $k = 2$; $\theta_2 = 0.4$; $E(X)=1$.

Figure 1 indicates that L_q decreases for a increases. Figure 2 indicates that L_q decreases for $\xi_1$increases. Figure 3 indicates that P_0 increases for ηincreases. Figure 4 indicates that P_0 decreases for p_1 increases. Figure 5 indicates that L_q decreases for increasing the values of γ and η. Figure 6 indicates that P_0 increases for a and γ.

6. Conclusion

Unreliable vacation retrial queue and multi stages of service delay in repair with batch arrival policy are meticulously studied. The PGF of the numbers in the system and orbit are found. The performance measures were obtained. L_s, L_q, W_s and W_q are obtained. The mathematical results are validated by simulation results.

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