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Analytical Solutions for Limit Cycles of the Forced Van Der Pol Duffing Oscillator

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Abstract. We study limit cycle solutions of the forced Van der Pol Duffing oscillator under the condition that the external frequency is equal to the resultant frequency of the nonlinear oscillator. The effects of the damping parameter (μ), the nonlinear term (β) and the magnitude of the external forcing (g) are discussed. We use the homotopy analysis method (HAM) to obtain closed form solutions. We note that without forcing we obtain solutions reported previously in the literature. We minimize the square residual error of this problem to obtain solutions by HAM. Finally, a comparison of numerical and analytical solutions has been carried out.

Keywords: Limit cycle; forced Van der Pol duffing oscillator; homotopy analysis method, square residual error.
PACS: 02.30.Hq

INTRODUCTION

Nonlinear oscillators contain very rich dynamics with respect to different initial conditions and different parameter values. Analytical solutions for these problems has been studied by a number of authors [[1]-[7]]. One nonlinear oscillator namely, the forced Van der Pol Duffing oscillator have been used in describing different phenomena in different areas of science and engineering as seen from the literature [[8]-[12]].

The forced Van der Pol Duffing oscillator is

$$x'' - \mu(1-x^2)x' + \alpha x + \beta x^3 = g\cos(\bar{\omega}t), \quad \mu > 0 \tag{1}$$

where x denotes displacement from the equilibrium position, the prime(s) represent the derivatives of x with respect to t, $\mu > 0$ is the damping parameter, $\bar{\omega}$ is the external frequency, g is amplitude of the external forcing and α, β are constants. All parameters are not necessarily small.

Particular cases of Eq. (1) namely, the Van der Pol oscillator ($\alpha = 1, \beta = 0, g = 0$) and the Van der Pol Duffing oscillator ($\alpha = 1, g = 0$) have been solved analytically in [4] and [5] respectively by Chen and Liu. They have used the homotopy analysis method to obtain closed form limit cycle solutions without external forcing. We know that a limit cycle solution depends on two physical quantities, one is the amplitude of the oscillator and other one is its frequency. Approximate analytical solutions for the forced Van der Pol Duffing oscillator Eq. (1) have been reported by Kimiaeifar et al. [13], but they have not discussed limit cycle behaviour.

In this work, to the best of our knowledge for the first time we study limit cycle solutions of Eq. (1) with forcing under the condition that the external frequency is the same as the resultant frequency of the nonlinear oscillator. We use the homotopy analysis method [[1],[14]] to obtain limit cycle solutions. This method does not require the existence of any perturbation parameter in the problem. We first minimize the square residual error of this problem as described in [15] and [16] . We present plots showing how the frequency of the limit cycle changes as we change the damping parameter μ, the nonlinear term β and the external forcing g. We compare our analytical solution with numerical solutions obtained by NDsolve (Mathematica).

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SOLUTION PROCEDURE AND THE SQUARE RESIDUAL ERROR

We refer the reader to [5] for further details i.e. how we can obtain solutions of Eq. (1) in the case of without forcing (at $g = 0$)? We follow the same methodology as discussed in [5]. The forcing term is an addition in our analysis. Under the transformation $\tau = \omega t$ and $x(t) = x(\tau)$ and $\bar{\omega} = \omega$, Eq.(1) becomes

$$\omega^2 x^{''} - \omega\mu(1-x^2)x^{'} + \alpha x(\tau) + \beta x^3 = g\cos(\tau). \tag{2}$$

The initial conditions are

$$x(0) = a \quad x^{'}(0) = 0 \tag{3}$$

We assume $H(\tau) = 1$ throughout the discussion. With the initial guess $x_0(\tau) = a_0\cos(\tau)$, we obtain the following (upon removal of secular terms):

$$a_0 = \pm 2 \quad \text{and} \quad \omega_0 = \pm\sqrt{\alpha + 3\beta - \frac{g}{2}} \tag{4}$$

On increasing the order of the HAM approximation we obtain the other terms a_m and ω_m for each m. Kindly see [5] for a_m and ω_m.

Let Δ_m is the square residual error of equation (2) at the m^{th} order HAM approximation, then

$$\Delta_m = \int_0^{2\pi} (N[\sum_{i=0}^{m} x_i(\xi)])^2 d\xi \tag{5}$$

One can determine the value of h (the so called convergence control parameter) by solving the following algebraic equation

$$\frac{\partial \Delta_m}{\partial h} = 0. \tag{6}$$

Another way is to plot Δ_m vs. h and choose the h which corresponds to the minimum Δ_m. We refer [15] and [16] for the evaluation of the square residual error described above.

DISCUSSION

On suitable choices (where ω_0 is a real number) of α, β and g, we obtain limit cycle solutions and we explicitly find the frequency of the limit cycle, because secular terms have to be removed for a limit cycle solution. At $g = 0, \alpha = 1$ and $\beta = 0$ we obtain the limit cycle solutions obtained by Chen and Liu in [4]. Similarly at $g = 0$ and $\alpha = 1$ we obtain the limit cycle solutions as obtained by the same author in [5]. For instance putting $\alpha = 1$ and $g = 0$ in (4) the initial approximation

$$\omega_0 = \sqrt{1 + 3\beta}$$

is exactly same as given in [5].

Instead of a proper choice of h we use the optimal value of h by minimizing the square residual error of the problem. Through out the calculation we keep the order of the HAM approximation as 10. Tables [1-3] show the variation of the frequency of the limit cycle as we change μ, β and g, respectively, along with the corresponding square residual error and the optimal h. Figs. [1] and [2] show a comparison between the analytical and numerical solution for $x(\tau)$ and $x'(\tau)$ respectively. Figs. [3-5] show the variation of the frequency of the limit cycle with parameters and the corresponding phase plots are presented in Figs. [6-8]. To develop the numerical solutions we have used the values of ω and a from the homotopy solutions after minimizing the square residual error. Furthermore, to check the convergence of obtained solution we plot h-curves [1] in Fig. [9]. The flat region of Fig. [9] confirms the convergence of the obtained solution. We have done it for one set of parameter values, but it can be extanded for another set of parameter values also.

Increasing the values of the damping parameter μ and the external forcing g decreases the frequency of the limit cycle as shown in Figs. [3] and [5]. Therefore, the corresponding period increases. This is physically reasonable because once the amplitude of the external forcing increases it is likely that the period of the oscillator increases. We find the reverse effect when we vary the nonlinear term β in Fig. [4]. So, the corresponding period decreases with an increase of the nonlinear term β.

Numerical investigations of Eq. (1) have been carried out by Venkatesan and Lakshmanan [17]. They have classified regions for different dynamics in terms of the parameter space. Basically the parameter space is defined in terms of the external forcing and the external frequency. We note that at $\alpha = -0.5, \beta = 0.5, \mu = 0.1$ and an appropriate choice of the external frequency like $g = 0.20$ and $g = 0.15$, our solutions are in the region of periodic solutions given in [17].

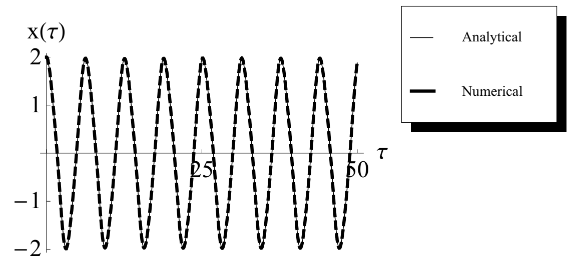

FIGURE 1. Time trajectory $x(\tau)$ vs. τ, at $\mu = 0.5, \alpha = 1, \beta = 1, g = 1$.

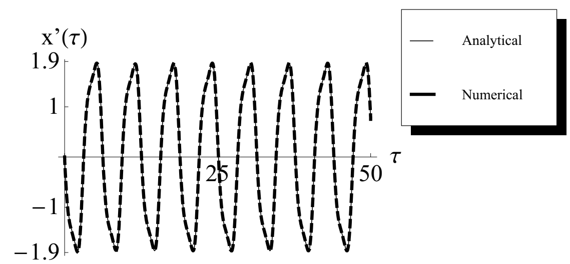

FIGURE 2. Time trajectory of the first derivative: $x'(\tau)$ vs. τ, at $\mu = 0.5, \alpha = 1, \beta = 1, g = 1$.

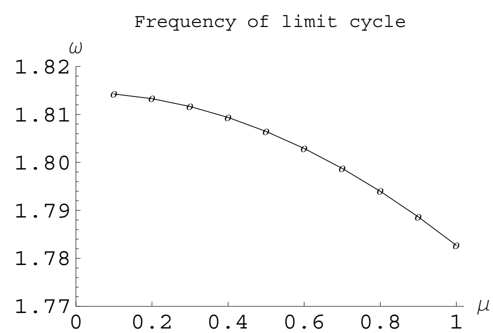

FIGURE 3. Variation of the frequency of the limit cycle ω vs. μ, at $\alpha = 1, \beta = 1, g = 1$.

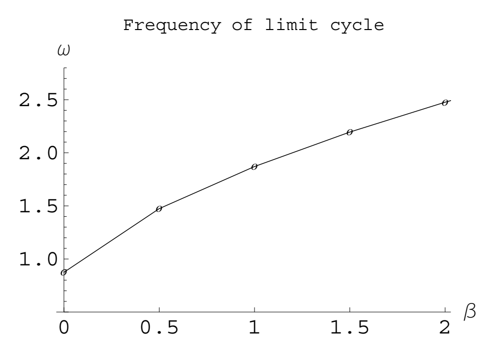

FIGURE 4. Variation of the frequency of the limit cycle ω vs. β, at $\alpha = 1, \mu = 0.1, g = 0.5$.

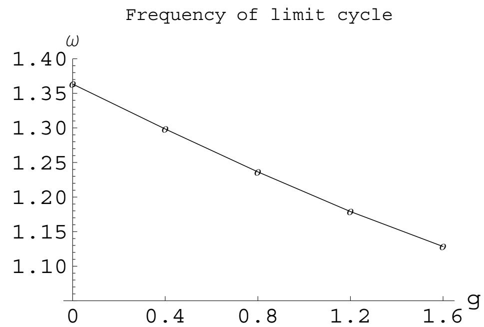

FIGURE 5. Variation of the frequency of the limit cycle ω vs. g, at $\alpha = 0.5, \mu = 0.1, \beta = 0.5$.

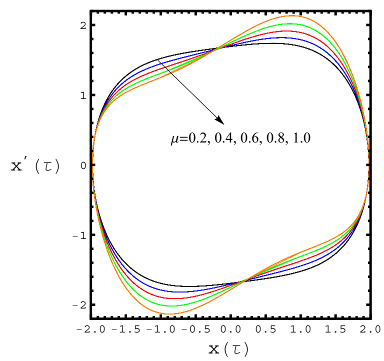

FIGURE 6. 10^{th} order HAM phase plane of $x^{'}(\tau)$ Vs. $x(\tau)$ at $\alpha = 1, \beta = 1$ and $g = 1$ for different values of μ.

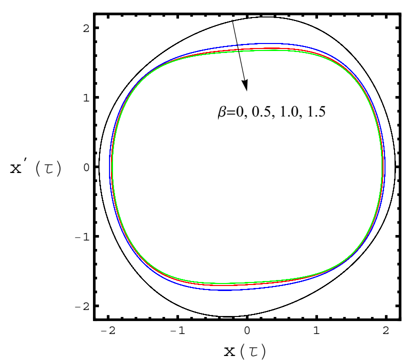

FIGURE 7. 10^{th} order HAM phase plane of $x^{'}(\tau)$ Vs. $x(\tau)$ at $\alpha = 1, g = 0.5$ and $\mu = 0.1$ for different values of β.

CONCLUSION

We have developed limit cycle solutions for the forced Van der Pol Duffing oscillator under the condition that the external frequency is equal to the the resultant frequency of the forced nonlinear oscillator. Instead of choosing a proper value of the convergence control parameter h, we used the optimal value of h by minimizing the square residual error of the problem. On the basis of this analysis and the analysis proposed by Chen and Liu [5], we conclude that the frequency of the limit cycle (in case of forcing or without forcing) decreases on increasing the value of the damping parameter.

We demonstrate that in the case without forcing, we obtain limit cycle solutions as already shown in the literature

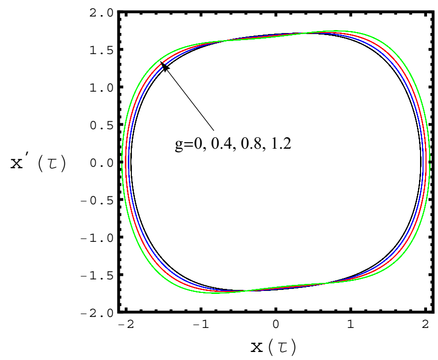

FIGURE 8. 10^{th} order HAM phase plane of $x^{'}(\tau)$ Vs. $x(\tau)$ at $\alpha = 0.5, \beta = 0.5$ and $\mu = 0.1$ for different values of g.

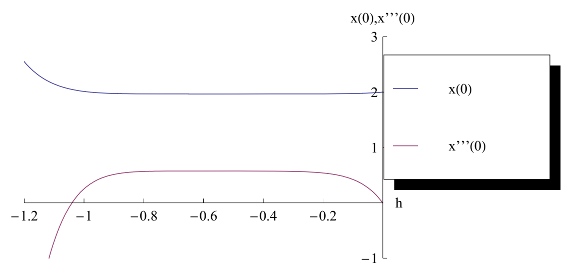

FIGURE 9. h-curves, $x(0)$ and $x'''(0)$ vs. h at $\alpha = 0.5, \mu = 0.1, \beta = 0.5, g = 0.4$.

TABLE 1. Variation of the frequency of the limit cycle at different values of μ with the square residual error for $\alpha = 1, \beta = 1$ and $g = 1$.

μ	ω	Δ_m	**optimal** h
0.1	1.81426	6.65×10^{-9}	-0.30
0.2	1.81328	8.91×10^{-9}	-0.30
0.3	1.81164	1.81×10^{-8}	-0.29
0.4	1.80937	3.20×10^{-8}	-0.29
0.5	1.80645	7.15×10^{-8}	-0.29
0.6	1.80291	2.05×10^{-7}	-0.29
0.7	1.79874	4.52×10^{-7}	-0.28
0.8	1.79398	1.11×10^{-6}	-0.28
0.9	1.78862	3.24×10^{-6}	-0.28
1.0	1.78270	9.25×10^{-6}	-0.27

by Chen and Liu [5], if we use the proper value of h. As the square residual error is minimum and the comparison between numerical and analytical solutions is quite good, we can conclude that the optimal homotopy analysis method is a strong tool for solving such nonlinear problems with high accuracy. In the case with forcing we note that the proposed regions of periodic solutions are similar to what have been demonstrated numerically in the literature by Venkatesan and Lakshmanan [17].

TABLE 2. Variation of the frequency of the limit cycle at different values of β with the square residual error for $\alpha = 1, \mu = 0.1$ and $g = 0.5$.

β	ω	Δ_m	**optimal** h
0	0.87412	3.84×10^{-13}	-1.32
0.5	1.47345	1.57×10^{-11}	-0.46
1.0	1.87083	1.56×10^{-8}	-0.26
1.5	2.19495	2.13×10^{-7}	-0.19
2.0	2.47625	8.16×10^{-7}	-0.14
2.5	2.72837	2.84×10^{-6}	-0.12

TABLE 3. Variation of the frequency of the limit cycle at different values of g with the square residual error for $\alpha = 0.5, \beta = 0.5$ and $\mu = 0.1$.

g	ω	Δ_m	**optimal** h
0	1.36352	4.80×10^{-9}	-0.47
0.4	1.29852	2.17×10^{-9}	-0.57
0.8	1.23639	4.33×10^{-9}	-0.65
1.2	1.17912	1.07×10^{-7}	-0.66
1.6	1.12854	2.19×10^{-6}	-0.60

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