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Certain Results on Prime and Prime Distance Labeling of Graphs

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Abstract.

Let G be a graph on n vertices. A bijective function $f : V(G) \rightarrow \{1, 2, \dots, n\}$ is said to be a prime labeling if for every $e = xy$, $\text{GCD}\{f(x), f(y)\} = 1$. A graph which permits a prime labeling is a "prime graph". On the other hand, a graph G is a prime distance graph if there is an injective function $g : V(G) \rightarrow Z$ (the set of all integers) so that for any two vertices s & t which are adjacent, the integer $|g(s) - g(t)|$ is a prime number and g is called a prime distance labeling of G . A graph G is a prime distance graph (PDL) iff there exists a "prime distance labeling" (PDL) of G . In this paper, we obtain the prime labeling and prime distance labeling of certain classes of graphs.

1. Introduction

Only simple, finite, connected, undirected, and non-trivial graphs are considered throughout this paper. Let $G = (V, E)$ be a graph. While $|V|$ denotes the "number of vertices" G , $|E|$ denotes the "number of edges" in G . For various graph theoretic terminology and notations we follow [17] whereas for number theory we refer to [5]. We recall a few definitions which are more useful and relevant for the present work. If the vertices of G are given (assigned) values (integers) subject to some constraints, it is known as a vertex labeling of G . For the latest survey on graph labeling one can refer to [2]. A prime labeling of G is a bijective function $f : V(G) \rightarrow \{1, 2, \dots, |V|\}$ and for any pair of vertices u & v which are adjacent, $\text{GCD}(f(u), f(v)) = 1$ where GCD denotes the greatest common divisor. A graph which permits a prime labeling is said to be a prime graph (or simply, prime). "The idea of prime labeling was originated by Entringer and discussed in detail by Tout et al. [16]". For more results on prime labeling of graphs one can see [2].

The concept of distance graph was introduced by Eggleton et al. [1]. Let Z be the set of all integers and D be a subset of the set of all positive integers. The integer distance graph $G(Z, D)$ is the graph with Z as its vertex set and two vertices s and t are joined iff $|s - t| \in D$. Then the prime distance graph $G(Z, P)$ is the distance graph with $D = P$. In 2013, Laison et al. [6] considered the finite subgraphs of $G(Z, P)$. They defined that a graph H is a PDG (prime

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distance graph) if there exists an injective function $g : V(H) \rightarrow Z$ such that for any two vertices s and t which are adjacent, the integer $|g(s) - g(t)|$ is a prime number and g is called a PDL (prime distance labeling) of H . So H is a PDG if and only if there exists a PDL of H . Note that in a PDL, the labels on vertices of H are distinct, but the edge labels are not so. For more results on PDL of graphs one can see [8–13]. In this paper, we establish the prime labeling and PDL of certain graphs. By WLG, we mean without loss of generality.

Note: The prime labeling and PDL of a G are not unique.

2. Main Results

In this section we derive the prime labeling and prime distance labeling of certain classes of graphs.

2.1. Prime Labeling of Certain Graphs

This section is devoted to obtain the prime labeling of some graphs such as lotus inside a circle, line graph of sunlet graph, and the generalized butterfly graph. First we recall the definitions of these graphs for the sake of completeness.

Definition 1. [17] A vertex of degree one is known as a pendant vertex. An edge of a graph is said to be pendant if one of its end vertices is a pendant vertex.

Definition 2. [17] A cycle C_n is a "closed path" whose terminal and initial vertices are the same.

Definition 3. [14] A lotus inside a circle, denoted LC_n , is obtained from $C_n : v_1, v_2, \dots, v_n, v_1$ and $K_{1,n}$ (with the central vertex v_0 & the pendant vertices u_1, u_2, \dots, u_n) by joining each u_i to v_i and $v_{i+1} \pmod n$.

Theorem 1. The louts inside a circle graph LC_n admits a prime labeling for $n \geq 3$.

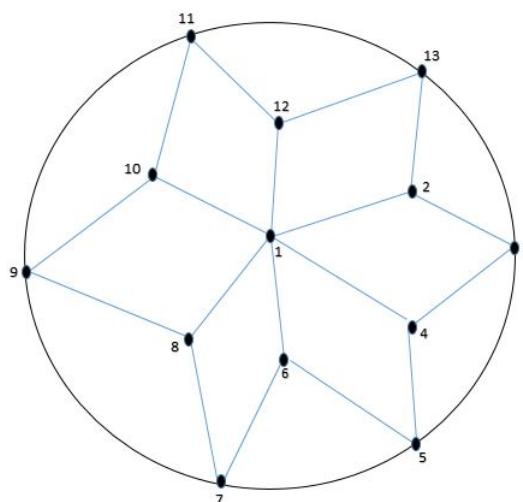


Figure 1. A prime labeling of LC_6

Proof. Let $LC_n, n \geq 3$ be the given louts inside a circle graph. Clearly $|LC_n| = 2n + 1$. We call v_0 the central vertex. Label the pendant vertices of a star as v_1, v_2, \dots, v_n and label the vertices

of a cycle as u_1, u_2, \dots, u_n . Define a bijective function $f : V(LC_n) \rightarrow \{1, 2, \dots, 2n + 1\}$ as follows: WLG, let $f(v_0) = 1, f(v_1) = 2$, and $f(u_1) = 3$. Then $f(v_i) = f(v_{i-1}) + 2$ and $f(u_i) = f(u_{i-1}) + 2$ for $2 \leq i \leq n$. An easy check shows that f is the required prime labeling of LC_n . \square

Definition 4. [15] "A sunlet graph, S_n , on $2n$ vertices is obtained by attaching n -pendant edges to C_n ".

Definition 5. [15] "The line graph of $G, L(G)$, is a graph whose vertices are the edges of G , and if $uv \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G ".

Theorem 2. The line graph of a sunlet graph $L(S_n)$ permits a prime labeling for $n \geq 3$.

Proof. Let $S_n, n \geq 3$ be the sunlet graph on $2n$ vertices. Obtain the line graph of the sunlet graph $L(S_n)$ whose vertex set $V(L(S_n))$ is defined as follows: label the vertices on the cycle as v_1, v_2, \dots, v_n and the outer vertices as u_1, u_2, \dots, u_n . Now define an one-to-one and on-to function $f : V(L(S_n)) \rightarrow \{1, 2, \dots, 2n\}$ as follows: WLG, let $f(v_1) = 1$ and $f(u_1) = 2$. Then $f(v_i) = f(v_{i-1}) + 2$ & $f(u_i) = f(u_{i-1}) + 2$ for $2 \leq i \leq n$. One can see that f is the desired prime labeling of $L(S_n)$. \square

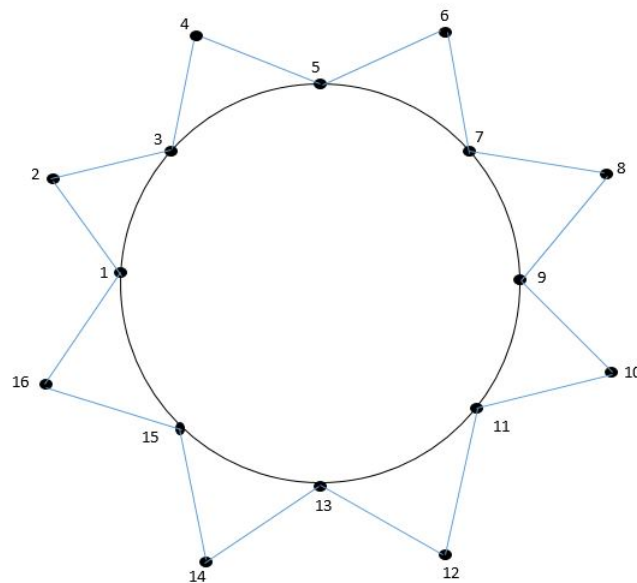


Figure 2. A prime labeling of $L(S_8)$

Definition 6. [3] The generalized butterfly graph, denoted BF_n , is a graph obtained by "inserting vertices to every wing with the assumption that sum of inserting vertices to every wing are same".

Theorem 3. The generalized butterfly graph BF_n admits a prime labeling for $n \geq 2$.

Proof. Let BF_n be the generalized butterfly graph. We take $n \geq 3$. One can note that $|V(BF_n)| = 2n + 1$ and $|E(BF_n)| = 4n - 2$. Let v_0 be the apex (central vertex). Define the vertex set $V(BF_n)$ as $\{v_i : i = 1, 2, \dots, 2n\}$ and the edge $E(BF_n)$ as $\{(v_i, v_{i+1}) : i = 1, 2, \dots, n - 1, n + 1, \dots, 2n - 1\} \cup \{(v_0, v_i) : i = 1, 2, \dots, 2n\}$. We label the vertices on right wing as $\{v_1, v_2, v_3, \dots, v_{n-1}, v_n\}$ and the vertices on left wing as $\{v_{n+1}, v_{n+2}, v_{n+3}, \dots, v_{2n-1}, v_{2n}\}$. Now

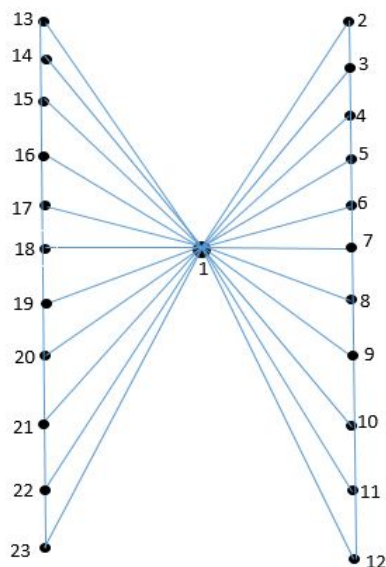


Figure 3. A prime labeling of BF_{11}

define a bijective function $f : V(BF_n) \rightarrow \{1, 2, \dots, 2n + 1\}$ as follows: WLG, let $f(v_0) = 1$. Then $f(v_i) = f(v_{i-1}) + 1$. An easy check clearly shows that f is the required prime labeling of BF_n . \square

2.2. Prime Distance Labeling of Certain Graphs

In this section, the PDL of some graphs such as butterfly graph, pyramid grid graph are established. We also recall a few relevant definitions and results needed for the present investigation. Let N be the set of natural numbers.

Definition 7. [17] A graph G is a bipartite graph if it does not contain any cycle of odd length.

Conjecture 1. [5] Any even numberer than or equal to 4 is a addition of two primes.

Theorem 4. [6] Every bipartite graph is a PDG.

Definition 8. [7] A graph $B_{n,m}$ (where n and m are any arbitrary positive integers) is said to be a butterfly graph if "two cycles of the same order sharing a common vertex with an arbitrary number of pendant edges attached at the common vertex".

Theorem 5. A butterfly graph $B_{n,m}$ admits a PDL for $n \geq 3$ and any $m \in N$ if the Goldbach's conjecture is true.

Proof. Let $B_{n,m}$ be the given butterfly graph with $n \geq 3$ and $m \in N$. Let the vertex set of $B_{n,m}$ be $V(B_{n,m}) = \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_n\} \cup \{w_1, w_2, \dots, w_m\}$. Clearly $|V(B_{n,m})| = 2n - 1 + m$ as $u_1 = v_1$. Now two cases arise.

Case 1: C_n , when n is even

The proof is direct from Theorem 4.

Case 2: C_n , when n is odd

Define an one-to-one labeling $f : V(B_{n,m}) \rightarrow Z$ as follows: Without loss of generality, let $f(v_i) = 2(i - 1)$ for $1 \leq i \leq n - 1$. By Conjecture 1, $f(v_{n-1}) = p_1 + p_2$ then $f(v_n) = p_1$ or p_2 whichever is unused. Next $f(u_i) = -f(v_i)$ for $2 \leq i \leq n$ and $f(w_i) = p*_i : 1 \leq i \leq m$ where $p*_i$

are sufficiently larger primes than $f(v_{n-1})$. An easy verification shows that the vertex labels are distinct and f is the desired prime distance labeling of $B_{n,m}$. \square

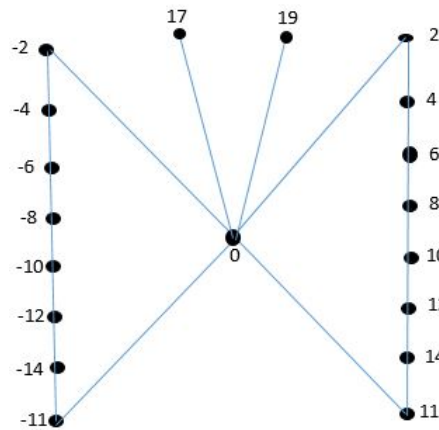


Figure 4. A prime distance labeling of $B_{9,2}$

Definition 9. [4] The Mobius ladder, M_n , is "graph with even number n of vertices, formed from a cycle C_n by adding edges connecting opposite pairs of vertices in C_n ".

Lemma 1. [4] The Mobius ladder M_n is bipartite when $n \equiv 2(\text{mod } 4)$ and not bipartite when $n \equiv 0(\text{mod } 4)$.

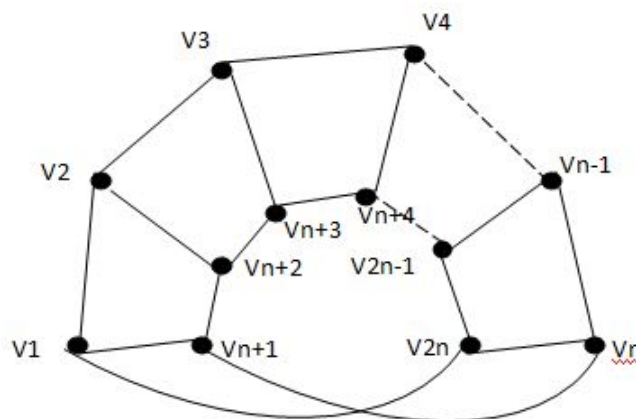


Figure 5. A Mobius ladder M_n on $2n$ vertices

Conjecture 2. The Mobius ladder M_n admits a prime distance labeling for $n \geq 3$.

Remark 1. There are two cases in proving Conjecture 2.

Case 1: When $n \equiv 2(\text{mod } 4)$

The proof is direct from Lemma 1 and Theorem 4.

Case 2: When $n \equiv 0(\text{mod } 4)$

Define an one-one function $f : V(M_n) \rightarrow Z$ as follows: without loss of generality, let

$f(v_i) = 2(i - 1)$ for $1 \leq i \leq n$. Then let $f(v_{n+1}) = f(v_n) + p$, where p is a sufficiently large unused prime. Then there arise two subcases:

Subcase 1: When $f(v_{2n})$ is a prime

We are through.

Subcase 2: When $f(v_{2n})$ is not a prime

The proof is complete if one can able to give a general pattern of prime distance labeling of M_n .

CONCLUSION

The prime labeling and prime distance labeling of certain graphs have been investigated. Establishing the prime and prime distance labeling for other families of graphs is still open and this is for future work. We also believe that the concepts of prime labeling and prime distance labeling may find applications in graph-based cryptography.

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