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Coefficient bounds for a subclass of Sakaguchi type functions using Chebyshev Polynomial

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ABSTRACT

In this work, considering a general subclass of bi-univalent Sakaguchi type functions and using Chebyshev polynomials, we obtain coefficient expansions for functions in this class.

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1. Introduction

Let D be the unit disk $\{z : z \in C \text{ and } |z| < 1\}$, A be the class of all functions analytic in D, satisfying the conditions

$$f(0) = 0 \text{ and } f'(0) = 1.$$

Then each function f in A has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Further, by S we shall denote the class of all functions in A which are univalent in D. The koebe one-quarter theorem [7] states that the image of D under every function f from S contains a disk of radius $\frac{1}{4}$, Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in D)$$

and

$$f(f^{-1}(w)) = w, \qquad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots$$

A function $f(z) \in A$ is said to be bi-univalent in D if both f(z) and $f^{-1}(z)$ are univalent in D. If the functions f and g are analytic in D, then f is said to be subordinate to g, written as

$$f(z) \prec g(z), \quad (z \in D)$$

if there exists a Schwarz function w(z), analytic in D, with

$w(0) = 0$ and $|w(z)| < 1$ $(z \in D)$

such that $f(z) = g(w(z))$ $(z \in D)$.

Let Σ denote the class of bi-univalent functions defined in the unit disk D. Lewin [11] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficients $|a_2|$. Netanyahu [14] showed that $\max|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Brannan and Taha [3] introduced certain subclasses of the bi-univalent function class Σ similar to find familiar subclasses. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1], [6], [12], [15], [17]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2], [5], [9], [10]). The coefficient estimate problem for each of $|a_n|$ $(n \in N / \{1, 2\};\ N = \{1, 2, 3, ...\})$ is still an open problem.

Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. The majority of books and research papers dealing with specific orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$ and their numerous uses in different applications, see for example, Doha [8] and Mason [13].

The Chebyshev polynomials of the first and second kinds are well known. In the case of a real variable x on [−1, 1], they are defined by

$$T_n(x) = \cos n\theta,$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where the subscript n denotes the polynomial degree and where x = cos θ.

Definition 1

For $0 \le \lambda \le 1$, $l \in (\frac{1}{2}, 1]$, $|t| \le 1$ and $t \ne 1$ (t is real), a function $f(z) \in \Sigma$ is said to be in the class $C_\Sigma(\lambda, l, t)$, if the following subordination hold

$$\left\{\frac{(1-t)[zf'(z)+(2\lambda^2-\lambda)z^2f''(z)]}{\begin{array}{r}[4(\lambda-\lambda^2)(z-tz)+(2\lambda^2-\lambda)z(f'(z)-tf'(tz))\\ +(2\lambda^2-3\lambda+1)(f(z)-f(tz))]\end{array}}\right\} \prec H(z,l)=\frac{1}{1-2lz+z^2} \quad (z\in D) \tag{1.2}$$

and

$$\left\{\frac{(1-t)[wg'(w)+(2\lambda^2-\lambda)w^2g''(w)]}{\begin{array}{r}[4(\lambda-\lambda^2)(w-tw)+(2\lambda^2-\lambda)w(g'(w)-tg'(tw))\\ +(2\lambda^2-3\lambda+1)(g(w)-g(tw))]\end{array}}\right\} \prec H(w,l)=\frac{1}{1-2lw+w^2} \quad (w\in D) \tag{1.3}$$

where $g(w) = f^{-1}(w)$.

We note that if $l = \cos\alpha$, $\alpha \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, then

$$H(z,l)=\frac{1}{1-2lz+z^2}=1+\sum_{n=1}^{\infty}\frac{\sin(n+1)\alpha}{\sin\alpha}z^n \quad (z\in D).$$

Thus

$$H(z,l)=1+2\cos\alpha z+(3\cos^2\alpha-\sin^2\alpha)z^2+\cdots \quad (z\in D).$$

Following see, we write

$$H(z,l)=1+U_1(l)z+U_2(l)z^2+\cdots \quad (z\in D,\ l\in(-1,1)),$$

where

$$U_{n-1}=\frac{\sin(n\arccos l)}{\sqrt{1-l^2}}$$

($n \in N$) are the chebyshev polynomials of the second kind. Also it is known that

$$U_n(l) = 2lU_{n-1}(l) - U_{n-2}(l),$$

and

$$U_1(l) = 2l, \quad U_2(l) = 4l^2 - 1, \quad U_3(l) = 8l^3 - 4l, \ldots \tag{1.4}$$

The Chebyshev polynomials $T_n(l)$, $l \in [-1, 1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(l) z^n = \frac{1 - lz}{1 - 2lz + z^2} \quad (z \in D).$$

However, the Chebyshev polynomials of the first kind $T_n(l)$ and the second kind $U_n(l)$ are well connected by the following relationships

$$\frac{dT_n(l)}{dl} = nU_{n-1}(l),$$
$$T_n(l) = U_n(l) - lU_{n-1}(l),$$
$$2T_n(l) = U_n(l) - U_{n-2}(l).$$

In this paper, motivated by the earlier work of Srutha keerthi [16].

2. Coefficient bounds for the function class $C_\Sigma(\lambda, l, t)$

Theorem 1

Let the function f(z) given by (1.1) be in the class $C_\Sigma(\lambda, l, t)$. Then

$$|a_2| \le \frac{2l\sqrt{2l}(2\lambda^2 t - \lambda t + 1)}{\sqrt{|A|}},$$

$$|a_3| \le \frac{4l^2(2\lambda^2 t - \lambda t + 1)^2}{[2 - u_2((2\lambda^2 - 3\lambda + 1) + 2t(2\lambda^2 - \lambda))]^2} + \frac{2l(2\lambda^2 t - \lambda t + 1)}{B},$$

$$A = 4l^2 B(2\lambda^2 t - \lambda t + 1) - \left\{ \begin{matrix} (2 - u_2[(2\lambda^2 - 3\lambda + 1) + 2t(2\lambda^2 - \lambda)]) \\ [4l^2(4\lambda^2 - 2\lambda + 2) + (2 - u_2[(2\lambda^2 - 3\lambda + 1) + 2t(2\lambda^2 - \lambda)])] \end{matrix} \right\},$$

$$B = [(6\lambda^2 - 3\lambda + 3) - u_3[(2\lambda^2 - 3\lambda + 1) + 3t(2\lambda^2 - \lambda)]].$$

and $u_n = \dfrac{1-t^n}{1-t}$, $n \in N$.

Proof.

Let $f \in C_\Sigma(\lambda, l, t)$. From (1.2) and (1.3), we have

$$\left\{ \frac{(1-t)[zf'(z)+(2\lambda^2-\lambda)z^2f''(z)]}{[4(\lambda-\lambda^2)(z-tz)+(2\lambda^2-\lambda)(f'(z)-tf'(tz)) +(2\lambda^2-3\lambda+1)(f(z)-f(tz))]} \right\} = 1+U_1(l)w(z)+U_2(l)w^2(z)+\cdots, \tag{2.1}$$

and

$$\left\{ \frac{(1-t)[wg'(w)+(2\lambda^2-\lambda)w^2g''(w)]}{[4(\lambda-\lambda^2)(w-tw)+(2\lambda^2-\lambda)w(g'(w)-tg'(tw)) +(2\lambda^2-3\lambda+1)(g(w)-g(tw))]} \right\} = 1+U_1(l)v(w)+U_2(l)v^2(w)+\cdots, \tag{2.2}$$

for some analytic functions w, v such that $w(0) = v(0) = 0$ and $|w(z)| < 1$, $|v(z)| < 1$ for all $z \in D$. From the equalities (2.1) and (2.2), we obtain that

$$\left\{ \frac{(1-t)[zf'(z)+(2\lambda^2-\lambda)z^2f''(z)]}{[4(\lambda-\lambda^2)(z-tz)+(2\lambda^2-\lambda)(f'(z)-tf'(tz)) +(2\lambda^2-3\lambda+1)(f(z)-f(tz))]} \right\} = 1+U_1(l)c_1z+[U_1(l)c_2+U_2(l)c_1^2]z^2+\cdots, \tag{2.3}$$

$$\left\{\frac{(1-t)[wg'(w)+(2\lambda^2-\lambda)w^2g''(w)]}{[4(\lambda-\lambda^2)(w-tw)+(2\lambda^2-\lambda)w(g'(w)-tg'(tw))+(2\lambda^2-3\lambda+1)(g(w)-g(tw))]}\right\}$$

$$=1+U_1(l)d_1w+[U_1(l)d_2+U_2(l)d_1^2]w^2+\cdots. \tag{2.4}$$

It is fairly well-known that if $|w(z)| = |c_1z + c_2z^2 + c_3z^3 + ...| < 1$ and $|v(w)| = |d_1w + d_2 w^2 + d_3w^3 + ...| < 1$, $z, w \in D$, then

$|c_j| \leq 1, \quad \forall\, j \in N.$

It follows from (2.3) and (2.4) that

$$\frac{[2-u_2((2\lambda^2-3\lambda+1)+2t(2\lambda^2-\lambda))]}{(2\lambda^2t-\lambda t+1)}a_2=U_1(l)c_1, \tag{2.5}$$

$$\frac{Ba_3}{(2\lambda^2t-\lambda t+1)}-\frac{Ca_2^2}{(2\lambda^2t-\lambda t+1)^2}=U_1(l)c_2+U_2(l)c_1^2, \tag{2.6}$$

$$-\frac{[2-u_2((2\lambda^2-3\lambda+1)+2t(2\lambda^2-\lambda))]}{(2\lambda^2t-\lambda t+1)}a_2=U_1(l)d_1, \tag{2.7}$$

$$\frac{B(2a_2^2-a_3)}{(2\lambda^2t-\lambda t+1)}-\frac{Ca_2^2}{(2\lambda^2t-\lambda t+1)^2}=U_1(l)d_2+U_2(l)d_1^2, \tag{2.8}$$

Where

$$C=\left\{(2-u_2[(2\lambda^2-3\lambda+1)+2t(2\lambda^2-\lambda)])[2(2\lambda^2-\lambda)[1+tu_2]+u_2(2\lambda^2-3\lambda+1)]\right\}.$$

From (2.5) and (2.7) we obtain

$$c_1=-d_1 \tag{2.9}$$

and

$$\frac{2[2-u_2((2\lambda^2-3\lambda+1)+2t(2\lambda^2-\lambda))]^2}{(2\lambda^2t-\lambda t+1)^2}a_2^2=U_1^2(l)(c_1^2+d_1^2). \tag{2.10}$$

By adding (2.6) and (2.8),we get

$$\frac{2Ba_2^2}{(2\lambda^2 t-\lambda t+1)}-\frac{2Ca_2^2}{(2\lambda^2 t-\lambda t+1)^2}=U_1(l)(c_2+d_2)+U_2(l)(c_1^2+d_1^2), \qquad (2.11)$$

By using (2.10) inequality (2.11), we have

$$\frac{2[B(2\lambda^2-\lambda t+1)\ -C]a_2^2}{(2\lambda^2 t-\lambda t+1)^2}-\frac{2U_2(l)[2-u_2((2\lambda^2-3\lambda+1)+2t(2\lambda^2-\lambda))]^2a_2^2}{U_1^2(l)(2\lambda^2 t-\lambda t+1)^2}=U_1(l)(c_2+d_2).$$

(2.12)

From (1.4) and (2.12) we get

$$|a_2|\le\frac{2l\sqrt{2l}(2\lambda^2 t-\lambda t+1)}{\sqrt{|A|}},$$

Next, in order find the bound on $|a_3|$, by subtracting (2.8) from (2.6), we obtain

$$\frac{2B(a_3-a_2^2)}{(2\lambda^2 t-\lambda t+1)}=U_1(l)(c_2-d_2)+U_2(l)(c_1^2-d_1^2).$$

(2.13)

Then, in view of (2.9) and (2.10), we have from (2.13)

$$a_3=\frac{U_1^2(l)(c_1^2+d_1^2)(2\lambda^2 t-\lambda t+1)^2}{2[2-u_2((2\lambda^2-3\lambda+1)+2t(2\lambda^2-\lambda))]^2}+\frac{U_1(l)(c_2-d_2)(2\lambda^2 t-\lambda t+1)}{2B}.$$

Notice that (1.4), we get

$$|a_3|\le\frac{4l^2(2\lambda^2 t-\lambda t+1)^2}{[2-u_2((2\lambda^2-3\lambda+1)+2t(2\lambda^2-\lambda))]^2}+\frac{2l(2\lambda^2 t-\lambda t+1)}{B}.$$

□

3. Fekete-Szegö inequalities for the function class $C_\Sigma(\lambda, l, t)$

Theorem 2

Let f given by (1.1) be in the class $C_\Sigma(\lambda, l, t)$ and $\mu \in R$. Then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{2l(2\lambda^2 t-\lambda t+1)}{B}; & \text{for } \left|\mu-1\right|\le\dfrac{A}{4l^2B}\\ \dfrac{8l^3\left|1-\mu\right|(2\lambda^2 t-\lambda t+1)}{A}; & \text{for } \left|\mu-1\right|\ge\dfrac{A}{4l^2B}.\end{cases}$$

Proof.

From (2.12) and (2.13)

$$a_3-\mu a_2^2=(1-\mu)\frac{U_1^3(l)(2\lambda^2 t-\lambda t+1)^2(c_2+d_2)}{2U_1^2(l)[B(2\lambda^2-\lambda t+1)\ -C]-2U_2(l)[2-u_2((2\lambda^2-3\lambda+1)+2t(2\lambda^2-\lambda))]^2}$$
$$+\frac{U_1(l)(c_2-d_2)(2\lambda^2-\lambda t+1)}{2B}$$
$$=U_1(l)(2\lambda^2 t-\lambda t+1)\left[\left(h(\mu)+\frac{1}{2B}\right)c_2+\left(h(\mu)-\frac{1}{2B}\right)d_2\right]$$

where

$$h(\mu)=\frac{(1-\mu)U_1^2(l)(2\lambda^2 t-\lambda t+1)}{2U_1^2(l)[B(2\lambda^2-\lambda t+1)\ -C]-2U_2(l)[2-u_2((2\lambda^2-3\lambda+1)+2t(2\lambda^2-\lambda))]^2}.$$

Then , in view of (1.4), we conclude that

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{2l(2\lambda^2 t-\lambda t+1)}{B} & 0\le\left|h(\mu)\right|\le\dfrac{1}{2B}\\ 4l\left|h(\mu)\right| & \left|h(\mu)\right|\ge\dfrac{1}{2B}.\end{cases}$$

4. Conclusions

A modest attempt has been made in this report to study certain class of analytic functions and Sakaguchi type functions on the open unit disk . Introduced new Subclasses of analytic univalent functions in the class $C_\Sigma(\lambda, l, t)$ and by selecting the values($l\in(\frac{1}{2}, 1]$, $0\le\lambda\le 1$ and $|t|\le 1$, $t\ne 1$), we used the Chebyshev polynomial expansions to provided estimates for the initial coefficients of bi-univalent Sakaguchi type functions in $C_\Sigma(\lambda, l, t)$.

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