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# Coefficient Estimate Of $p-$valent Bazilevič Functions with a Bounded Positive Real Part

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ABSTRACT: By considering a $p-$valent Bazilevič function in the open unit disk $\triangle$ which maps $\triangle$ onto the strip domain $w$ with $p\alpha < \Re w < p\beta$, we estimate bounds of coefficients and solve Fekete-Szegö problem for functions in this class.

Key Words: Analytic function, univalent function, $p-$ valent function, starlike function, Bazilevič function, subordination, coefficient estimate, Fekete-Szegö problem.

## Contents



## 1. Introduction

Let $\mathcal{A}_p$ denote the class of all functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic and $p-$valent in the open unit disk $\triangle = \{z : z \in \mathbb{C} : |z| < 1\}$. Note that $\mathcal{A}_1 := \mathcal{A}$the class of analytic functions further $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all univalent functions $f$ in $\triangle$. A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$ in $\triangle$ if it satisfies

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha.$$

This class is denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. The class $\mathcal{S}^*(\alpha)$ was introduced by Robertson [4]. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}$. Furthermore, let $\mathcal{M}(\beta)$ be the class of functions $f \in \mathcal{A}$ which satisfy

$$\Re \left( \frac{zf'(z)}{f(z)} \right) < \beta \qquad (z \in \triangle)$$

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for some real number $\beta$ with $\beta > 1$. The class $\mathcal{M}(\beta)$ was investigated by Uralegaddi et. al [7].

Let $P(z)$ and $Q(z)$ be analytic in $\triangle$. Then the function $P(z)$ is said to subordinate to $Q(z)$ in $\triangle$ written by

$$P(z) \prec Q(z) \qquad (z \in \triangle), \tag{1.1}$$

if there exists a function $w(z)$ which is analytic in $\triangle$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \triangle)$, and such that $P(z) = Q(w(z))$ $(z \in \triangle)$. From the definition of the subordinations, it is easy to show that the subordination (1.1) implies that

$$P(0) = Q(0) \quad \text{and} \quad P(\triangle) \subset Q(\triangle). \tag{1.2}$$

In particular, if $Q(z)$ is univalent in $\triangle$, then the subordination (1.1) is equivalent to the condition (1.2).

**Remark 1.1.** *Let $P(z)$ and $Q(z)$ be analytic in $\triangle$. Then the subordination (1.1) implies that*

$$|P'(0)| \leq |Q'(0)|, \tag{1.3}$$

*and $|P'(0)| = |Q'(0)|$ if and only if $P(z) = Q(xz)$ for some real numbers $x$ with $|x| = 1$ (cf. [1]).*

## 2. Bazilevič Functions with bounded positive real part

Motivated by the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{M}(\beta)$, we define a new class for certain $p-$valent functions.

**Definition 2.1.** *Let $\alpha$ and $\beta$ be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{S}^p_\lambda(\alpha, \beta)$ if $f$ satisfies the following inequality*

$$\alpha < \Re\left\{\left(\frac{f(z)}{z^p}\right)^{\lambda-1} \frac{f'(z)}{pz^{p-1}}\right\} < \beta \qquad (z \in \triangle, \lambda \geq 0). \tag{2.1}$$

By taking $\lambda = 0$ we further define a new class $\mathcal{S}^p_\lambda(\alpha, \beta) \equiv \mathcal{S}^p(\alpha, \beta)$.

**Definition 2.2.** *The function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{S}^p(\alpha, \beta)$ if $f$ satisfies the following inequality*

$$\alpha < \Re\left(\frac{zf'(z)}{pf(z)}\right) < \beta \qquad (z \in \triangle, p \in \mathbb{N}) \tag{2.2}$$

*for some real number $\alpha (\alpha < 1)$ and some real number $\beta (\beta > 1)$.*

**Remark 2.3.** *$f \in \mathcal{S}^p_\lambda(\alpha, \beta)$ if and only if $f$ satisfies each of the following two subordination relations:*

$$\left(\frac{f(z)}{z^p}\right)^{\lambda-1} \frac{f'(z)}{pz^{p-1}} \prec \frac{1 + (1 - 2\alpha)z}{1 - z} \qquad (z \in \triangle, \lambda \geq 0)$$

*and*

$$\left(\frac{f(z)}{z^p}\right)^{\lambda-1} \frac{f'(z)}{pz^{p-1}} \prec \frac{1 - (1 - 2\beta)z}{1 + z} \qquad (z \in \triangle, \lambda \geq 0).$$

**Remark 2.4.** *When $\lambda = 0,\ p = 1,\ \mathcal{S}_{\lambda}^{p}(\alpha, \beta)$ reduces to $\mathcal{S}(\alpha, \beta)$, studied by Kuroki and Owa [3]and further $p = 1,\ \mathcal{S}_{\lambda}^{p}(\alpha, \beta)$ reduces to $\mathcal{S}_{\lambda}(\alpha, \beta)$, the class of Bazilevic functions with bounded positive real part.*

Now, we define an analytic function $\mathcal{S}_{\alpha,\beta}(z) : \triangle \longrightarrow \mathbb{C}$ by

$$\mathcal{S}_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi}\, i \log \left( \frac{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha}} z} \right) \tag{2.3}$$

due to Kuroki and Owa [3] and they proved $\mathcal{S}_{\alpha,\beta}(z)$ maps $\triangle$ onto a convex domain $w$ with $\alpha < \Re(w) < \beta$, conformaly.

We give some example for $f(z) \in \mathcal{S}_{\lambda}^{p}(\alpha, \beta)$ as follows:

**Example 2.5.** *Let us consider the function $f(z)$ given by*

$$f(z) = \begin{cases} \left[ z^{p\lambda} + \frac{p\lambda(\beta-\alpha)}{\pi}\, i \int\limits_0^z t^{p\lambda - 1} \log \left( \frac{1-e^{\frac{i\pi(1-\alpha)}{\beta-\alpha}} t}{1-e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha}} t} \right) dt \right]^{\frac{1}{\lambda}}, & \lambda > 0 \\ z^p \exp \left[ \frac{p(\beta-\alpha)}{\pi}\, i \int\limits_0^z \frac{1}{t} \log \left( \frac{1-e^{\frac{i\pi(1-\alpha)}{\beta-\alpha}} t}{1-e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha}} t} \right) dt \right], & \lambda = 0 \end{cases} \tag{2.4}$$

*with $\alpha < 1$ and $\beta > 1$. Then we have*

$$\left( \frac{f(z)}{z^p} \right)^{\lambda - 1} \frac{f'(z)}{pz^{p-1}} = 1 + \frac{\beta - \alpha}{\pi}\, i \log \left( \frac{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha}} z} \right) = \mathcal{S}_{\alpha,\beta}(z) \quad (z \in \triangle).$$

*Hence, the function $f(z)$ given by (2.4) satisfies the inequality (2.1) which implies that $f(z) \in \mathcal{S}_{\lambda}^{p}(\alpha, \beta)$.*

**Example 2.6.** *Let us consider the function $f(z)$ given by*

$$f(z) = z^p \exp \left\{ \frac{p(\beta - \alpha)}{\pi} i \int\limits_0^z \frac{1}{t}\ \log \left( \frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} t}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} t} \right) dt \right\} \qquad (z \in \triangle) \tag{2.5}$$

*with $\alpha < 1$ and $\beta > 1$. Then we have*

$$\frac{zf'(z)}{pf(z)} = 1 + \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z} \right) = \mathcal{S}_{\alpha,\beta}(z) \qquad (z \in \triangle)$$

*It is clear that the function $f(z)$ given by (2.5) satisfies the inequality (2.2), which implies that $f(z) \in \mathcal{S}^{p}(\alpha, \beta)$.*

Applying the function $\mathcal{S}_{\alpha,\beta}(z)$ defined by (2.3), we give a necessary and sufficient condition for $f(z) \in \mathcal{A}_p$ to belong to the class $\mathcal{S}_{\lambda}^{p}(\alpha, \beta)$.

**Lemma 2.7.** *Let $f(z) \in \mathcal{A}_p$ and $0 \leq \alpha < 1 < \beta$. Then $f(z) \in \mathcal{S}^p_\lambda(\alpha, \beta)$ if and only if*

$$\left(\frac{f(z)}{z^p}\right)^{\lambda-1} \frac{f'(z)}{pz^{p-1}} \prec 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1-e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}\right) \tag{2.6}$$

*in* $\triangle$.

By taking $\lambda = 0$ we state the following Lemma.

**Lemma 2.8.** *Let $f \in \mathcal{A}_p$. Then $f(z) \in \mathcal{S}^p(\alpha, \beta)$ if and only if*

$$\frac{zf'(z)}{pf(z)} \prec 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1-e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}\right) \qquad (z \in \triangle) \tag{2.7}$$

*where $\alpha < 1$ and $\beta > 1$.*

We note that

$$\mathcal{S}_{\alpha,\beta}(z) = 1 + \frac{\beta-\alpha}{\pi}\, i \log\left(\frac{1-e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1-e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}\right) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{2(\beta-\alpha)}{n\pi} \sin\frac{n\pi(1-\alpha)}{\beta-\alpha} \qquad (n = 1, 2, 3, \ldots). \tag{2.8}$$

**Theorem 2.9.** *Let $f \in \mathcal{A}_p$, $\frac{1}{2} \leq \alpha < 1 < \beta < \frac{3}{2}$ and $\alpha < \Re\left\{\frac{zf'(z)}{pf(z)}\right\} < \beta$ in $\triangle$. Then*

$$\frac{1}{2p\lambda(1-\alpha)+1} < \Re\left\{\left(\frac{f(z)}{z^p}\right)^{\lambda}\right\} < \frac{1}{2p\lambda(1-\beta)+1} \qquad (z \in \triangle, p \in \mathbb{N}, \lambda > 0).$$

*Proof.* The proof is similar to that of Theorem 4 by Sim and Kwon [6]. □

**Remark 2.10.** *When $\lambda = 1$, Theorem 2.9 yields the following for $f \in \mathcal{S}^p(\alpha, \beta)$*

*Let $f \in \mathcal{A}_p$, $\frac{1}{2} \leq \alpha < 1 < \beta < \frac{3}{2}$ and $\alpha < \Re\left\{\frac{zf'(z)}{pf(z)}\right\} < \beta$ in $\triangle$. Then*

$$\frac{1}{2p(1-\alpha)+1} < \Re\left\{\frac{f(z)}{z^p}\right\} < \frac{1}{2p(1-\beta)+1} \qquad (z \in \triangle, p \in \mathbb{N}).$$

*When $\lambda = 1, p = 1$, Theorem 2.9 reduces to Theorem 4 of [6].*

## 3. Some coefficient problems

Using the subordination (2.6), we find sharp bounds on the second and third coefficients for $f(z) \in \mathcal{S}_\lambda^p(\alpha,\beta)$, by applying the following lemma due to Rogosinki [5].

**Lemma 3.1.** *Let* $P(z) = \sum_{n=1}^{\infty} A_n z^n$ *and* $Q(z) = \sum_{n=1}^{\infty} B_n z^n$ *be analytic in* $\triangle$. *If* $P(z) \prec Q(z)$ $(z \in \triangle)$, *then*

$$\sum_{k=1}^{m} |A_k|^2 \leq \sum_{k=1}^{m} |B_k|^2 \qquad (m = 1, 2, \ldots).$$

Applying Lemma 3.1 with

$$\begin{aligned} P(z) &= \left(\frac{f(z)}{z^p}\right)^{\lambda-1} \frac{f'(z)}{pz^{p-1}} = 1 + \frac{(1+p\lambda)}{p} a_{p+1} z \\ &\quad + \frac{(2+p\lambda)}{p}\left[a_{p+2} - \frac{(1-\lambda)}{2} a_{p+1}^2\right] z^2 + \cdots \end{aligned}$$

and

$$Q(z) = \mathcal{S}_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{3.1}$$

where $B_n$ is as in (2.8) and using Remark 1.1, we obtain the following theorem.

**Theorem 3.2.** *If the function* $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \mathcal{S}_\lambda^p(\alpha,\beta)$, *then*

$$|a_{p+1}| \leqq \frac{2p(\beta-\alpha)}{\pi(1+p\lambda)} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$

*and*

$$\begin{aligned} |a_{p+2}| \leqq \frac{2p(\beta-\alpha)}{\pi} \sin\frac{\pi(1-\alpha)}{\beta-\alpha} &\left(\frac{1}{(2+p\lambda)} \cos\frac{\pi(1-\alpha)}{\beta-\alpha}\right. \\ &\left. + \frac{|1-\lambda| p(\beta-\alpha)}{\pi(1+p\lambda)^2} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right). \end{aligned}$$

*Moreover, the equality holds in either inequality if and only if*

$$f(z) = \begin{cases} z^{p\lambda} + p\lambda \int_0^z \frac{\mathcal{S}_{\alpha,\beta}(e^{i\theta}t)-1}{t^{1-p\lambda}} dt, & \text{for } \lambda > 0 \\ z^p \exp\left[p \int_0^z \frac{\mathcal{S}_{\alpha,\beta}(e^{i\theta}t)-1}{t} dt\right], & \text{for } \lambda = 0 \end{cases}$$

*for some real number* $\theta$ $(0 \leq \theta < 2\pi)$, *where* $\mathcal{S}_{\alpha,\beta}(z)$ *is defined by* (2.3).

When $\lambda = 0$ we state the following corollary:

**Corollary 3.3.** *If the function* $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \mathcal{S}^p(\alpha, \beta)$, *then*

$$|a_{p+1}| \leqq \frac{2p(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}$$

*and*

$$|a_{p+2}| \leqq \frac{p(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \left( \cos \frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2p(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \right).$$

*Moreover, the equality holds in either inequality if and only if*

$$f(z) = z^p \exp \left\{ p \int_0^z \frac{\mathcal{S}_{\alpha,\beta}(e^{i\theta}t) - 1}{t} dt \right\}$$

*for some real number* $\theta$ $(0 \leq \theta < 2\pi)$, *where* $\mathcal{S}_{\alpha,\beta}(z)$ *is defined by* (2.3).

When $p = 1$, from Theorem 3.2, we state the following corollary:

**Corollary 3.4.** *If the function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_\lambda(\alpha, \beta)$, *then*

$$|a_2| \leq \frac{2(\beta-\alpha)}{\pi(1+\lambda)} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}$$

*and*

$$|a_3| \leq \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \left( \frac{1}{(2+\lambda)} \cos \frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{|1-\lambda|(\beta-\alpha)}{\pi(1+\lambda)^2} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \right).$$

*Moreover, the equality holds in either inequality if and only if*

$$f(z) = \begin{cases} z^\lambda + \lambda \int_0^z \frac{\mathcal{S}_{\alpha,\beta}(e^{i\theta}t)-1}{t^{1-\lambda}} dt, & \text{for } \lambda > 0 \\ z \exp \left[ \int_0^z \frac{\mathcal{S}_{\alpha,\beta}(e^{i\theta}t)-1}{t} dt \right], & \text{for } \lambda = 0 \end{cases}$$

*for some real number* $\theta$ $(0 \leq \theta < 2\pi)$, *where* $\mathcal{S}_{\alpha,\beta}(z)$ *is defined by* (2.3).

Making use of the following lemma we shall solve the Fekete-Szegö problem for $f(z) \in \mathcal{S}_\lambda^p(\alpha, \beta)$.

**Lemma 3.5.** *(Keogh and Merkers [2]) Let* $h(z) = 1 + h_1 z + h_2 z^2 + \cdots$ *be a function with positive real part in* $\triangle$. *Then for any complex number* $\nu$,

$$|h_2 - \nu h_1^2| \leqq 2 \max\{1, |1-2\nu|\}.$$

**Theorem 3.6.** *Let $0 \leq \alpha < 1 < \beta$ and let the function $f$ given by $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ be in the class $\mathcal{S}_\lambda^p(\alpha,\beta)$. Then for any complex number $\mu$,*

$$|a_{p+2} - \mu a_{p+1}^2| \leqq \frac{2p(\beta-\alpha)}{\pi(2+p\lambda)} \sin\frac{\pi(1-\alpha)}{\beta-\alpha} \times \max\left\{1, \left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{p(2+p\lambda)(1-\lambda-2\mu)(\beta-\alpha)}{\pi(1+p\lambda)^2}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right|\right\}.$$

*Proof.* Let $P(z) = \left(\frac{f(z)}{z^p}\right)^{\lambda-1}\frac{f'(z)}{pz^{p-1}}$. Then, since $f \in \mathcal{S}_\lambda^p(\alpha,\beta)$, we have $P(z) \prec Q(z)$, where $Q(z)$ is given by (3.1).

Let

$$h(z) = \frac{1+Q^{-1}(P(z))}{1-Q^{-1}(P(z))} = 1 + h_1 z + h_2 z^2 + \cdots \qquad (z \in \triangle).$$

Then $h$ is analytic and has positive real part in the open disk $\triangle$. We also have

$$P(z) = Q\left(\frac{h(z)-1}{h(z)+1}\right) \qquad (z \in \triangle). \tag{3.2}$$

We find from the equation (3.2) that

$$\begin{aligned} a_{p+1} &= \frac{pB_1h_1}{2(1+p\lambda)} \\ a_{p+2} &= \frac{p}{(2+p\lambda)}\left[\frac{B_2h_1^2}{4} - \frac{B_1h_1^2}{4} + \frac{B_1h_2}{2}\right] + \frac{(1-\lambda)}{2}\frac{p^2B_1^2h_1^2}{4(1+p\lambda)^2} \end{aligned}$$

which imply that

$$a_{p+2} - \mu a_{p+1}^2 = \frac{pB_1}{2(2+p\lambda)}(h_2 - \nu h_1^2),$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{(1-\lambda)(2+p\lambda)pB_1}{2(1+p\lambda)^2} + \frac{(2+p\lambda)\mu pB_1}{(1+p\lambda)^2}\right).$$

Applying Lemma 3.5, we obtain

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| &= \frac{p}{2(2+p\lambda)}|B_1||h_2 - \nu h_1^2| \\ &\leq \frac{p}{(2+p\lambda)}B_1 \max\{1, |1-2\nu|\}. \end{aligned} \tag{3.3}$$

Substituting $B_1 = \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$ and $B_2 = \frac{(\beta-\alpha)}{\pi}\sin\frac{2\pi(1-\alpha)}{\beta-\alpha}$ in (3.3), we can obtain the results as asserted. □

By taking $\lambda = 0$ we state the following:

**Corollary 3.7.** *Let $0 \leq \alpha < 1 < \beta$ and let the function $f$ given by $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ be in the class $\mathcal{S}^p(\alpha, \beta)$. Then for any complex number $\mu$,*

$$|a_{p+2} - \mu a_{p+1}^2| \leqq \frac{p(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \max\left\{1, \left|\cos \frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2p(1-2\mu)(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}\right|\right\}.$$

Putting $p = 1$ in Theorem 3.6, we get the following corollary.

**Corollary 3.8.** *Let $0 \leq \alpha < 1 < \beta$ and let the function $f$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_\lambda(\alpha, \beta)$. Then for any complex number $\mu$,*

$$|a_3 - \mu a_2^2| \leqq \frac{2(\beta-\alpha)}{\pi(2+\lambda)} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \times \max\left\{1, \left|\cos \frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{(2+\lambda)(1-\lambda-2\mu)(\beta-\alpha)}{\pi(1+\lambda)^2} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}\right|\right\}.$$

## 4. coefficient estimates for $f \in \mathcal{S}^p(\alpha, \beta)$

In this section by making use of the following lemma we deduced some coefficient estimates for $f(z) \in \mathcal{S}^p(\alpha, \beta)$.

**Lemma 4.1.** *[5] Let $Q(z) = \sum_{n=1}^{\infty} B_n z^n$ be analytic and univalent in $\triangle$ and suppose that $Q(z)$ maps $\triangle$ onto a convex domain. If $P(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in $\triangle$ and satisfies the following subordination*

$$P(z) \prec Q(z) \qquad (z \in \triangle),$$

*then*

$$|A_n| \leqq |B_1| \qquad (n = 1, 2, \ldots).$$

**Theorem 4.2.** *If the function $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \mathcal{S}^p(\alpha, \beta)$, then*

$$|a_{p+n}| \leqq \prod_{k=p+1}^{n+p-1} \frac{[k-(p+1)] + \frac{2p(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}}{(n-1)!} \qquad (n = 2, 3, \ldots).$$

*Proof.* According to the assertion of Lemma 4.1, the function $f(z)$ satisfies the subordination (2.7). Let us define $P(z)$ and $Q(z)$ by

$$P(z) = \frac{zf'(z)}{pf(z)} \qquad (z \in \triangle) \tag{4.1}$$

and

$$Q(z)=1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1-e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}\right)\qquad(z\in\triangle)\tag{4.2}$$

Then, the subordination (2.7) satisfies (1.1).

Note that the function $Q(z)$ defined by (4.2) is convex in $\triangle$ and has the form

$$Q(z)=1+\sum_{n=1}^{\infty}B_nz^n,$$

where

$$B_n=\frac{2(\beta-\alpha)}{n\pi}\sin\frac{n\pi(1-\alpha)}{\beta-\alpha}\qquad(n=1,2,\ldots).\tag{4.3}$$

If we let

$$P(z)=1+\sum_{n=1}^{\infty}A_nz^n,$$

then by Lemma 4.1 we see that the subordination (2.7) implies that

$$|A_n|\leqq|B_1|\qquad(n=1,2,\ldots).\tag{4.4}$$

where

$$|B_1|=\frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}.\tag{4.5}$$

Now, the equality (4.1) implies that

$$zf'(z)=p\,P(z)f(z).$$

that is,

$$pz^p+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}=p\left[1+\sum_{n=1}^{\infty}A_nz^n\right]\left[z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\right].$$

Then, the coefficient of $z^{n-1}$ in both sides lead to

$$a_{p+n-1}=\frac{p}{n-1}(A_{n-1}+A_{n-2}a_{p+1}+\cdots+A_1a_{p+n-2}).$$

A simple calculation combined with the inequality (4.4) yields that

$$\begin{aligned}|a_{p+n-1}|&=\frac{p}{n-1}|A_{n-1}+A_{n-2}a_{p+1}+\cdots+A_1a_{p+n-2}|\\&\leqq\frac{p}{n-1}(|A_{n-1}|+|A_{n-2}||a_{p+1}|+\cdots+|A_1||a_{p+n-2}|)\\&\leqq\frac{p|B_1|}{n-1}\sum_{k=p+1}^{n+p-1}|a_{k-1}|\qquad(|a_p|=1),\end{aligned}$$

where $B_1$ is given in (4.5). To prove the assertion of the theorem, we need to show that

$$|a_{p+n-1}| \leqq \frac{p|B_1|}{n-1} \sum_{k=p+1}^{n+p-1} |a_{k-1}| \leqq \prod_{k=p+1}^{n+p-1} \frac{[k-(p+1)]+p|B_1|}{(n-1)!}. \tag{4.6}$$

We now use the mathematical induction for the proof of the theorem.

Since

$$|a_{p+1}| \leq p|B_1||a_p| = p|B_1|,$$

it is clear that the assertion is holds true for $n = 2$.

We assume that the proposition is true for $n = m$. Then, some calculation gives us that

$$\begin{aligned} |a_{p+m}| &\leqq \frac{p|B_1|}{(m+1)-1} \sum_{k=p+1}^{m+p} |a_{k-1}| = \frac{p|B_1|}{m} \left( \sum_{k=p+1}^{m+p-1} |a_{k-1}| + |a_{p+m-1}| \right) \\ &\leqq \frac{p|B_1|}{m} \left(1 + \frac{p|B_1|}{m-1}\right) \sum_{k=p+1}^{m+p-1} |a_{k-1}| = \frac{m-1+p|B_1|}{m} \frac{p|B_1|}{m-1} \sum_{k=p+1}^{m+p-1} |a_{k-1}| \\ &\leqq \frac{m-1+p|B_1|}{m} \prod_{k=p+1}^{m+p-1} \frac{[k-(p+1)]+p|B_1|}{(m-1)!} = \prod_{k=p+1}^{m+p} \frac{[k-(p+1)]+p|B_1|}{((m+1)-1)!} \end{aligned}$$

which implies that the inequality (4.6) is true for $n = m + 1$. By mathematical induction, we proved that

$$|a_{p+n-1}| \leqq \prod_{k=p+1}^{n+p-1} \frac{[k-(p+1)]+p|B_1|}{(n-1)!} \qquad (n = 2, 3, \ldots),$$

where $B_1$ is given in (4.5). This completes the proof of the theorem. □

**Remark 4.3.** *By specializing the parameters $\lambda = 0, p = 1$, the results proved in this paper , leads the results obtained in [3]. Further by taking $\lambda = 0$ one can deduce the results for functions $f \in \mathcal{S}^p(\alpha, \lambda)$, hence we omit the details*

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