Available online at www.sciencedirect.com

ScienceDirect

Procedia Computer Science 57 (2015) 1096 – 1103

Procedia Computer Science

3rd International Conference on Recent Trends in Computing 2015 (ICRTC-2015)

Combinatorial Properties of Root-fault Hypertrees

R. Sundara Rajan[a], R. Jayagopal[a,*], Indra Rajasingh[a,**], T.M. Rajalaxmi[b], N. Parthiban[c]

[a]*School of Advanced Sciences, VIT University, Chennai, India*
[b]*Department of Mathematics, SSN College of Engineering, Chennai, India*
[c]*School of Computing Science and Engineering, VIT University, Chennai, India*

Abstract

Combinatorial properties have become more and more important recently in the study of reliability, fault tolerance, randomized routing, and transmission delay in interconnection networks. In this paper, we prove that hypertrees are planar. We also discuss certain combinatorial properties of root-fault hypertrees.

Peer-review under responsibility of organizing committee of the 3rd International Conference on Recent Trends in Computing 2015 (ICRTC-2015)

Keywords: Wide diameter; fault wide diameter; Rabin number; hypertree

1. Introduction

Combinatorial properties have become more and more important recently in the study of reliability, fault tolerance, randomized routing, and transmission delay in interconnection networks [1]. Reliability and efficiency are important criteria in the design of interconnection networks. Connectivity is a widely used measure for network fault-tolerance capacities, while diameter determines routing efficiency along individual paths. In practice, we are interested in high-connectivity, small-diameter networks. Recently, the w-wide diameter, $(w-1)$-fault diameter and the w-Rabin number have to measure network reliability and efficiency [2].

The distance $d_G(x,y)$ form from a vertex x to another vertex y in a network G is the minimum number of edges of a path from x to y. The diameter $d(G)$ of a network G is the maximum distance from one vertex to another. The connectivity $k(G)$ of a network G is the minimum number of vertices whose removal results in a disconnected or

*Corresponding author E-mail address: jgopal89@gmail.com.

**This work is supported by Project no. SR/S4/MS: 846/13, Department of Science and Technology, SERC, Government of India.

Peer-review under responsibility of organizing committee of the 3rd International Conference on Recent Trends in Computing 2015 (ICRTC-2015)
doi:10.1016/j.procs.2015.07.393

trivial network. Accoding to Menger's theorem, there are at least k (internally) vertex-disjoint paths from a vertex x to another vertex y in a network of connectivity k [3].

The classical approach to study routing in interconnection networks is to find the shortest path between the sending station and the receiving station. Whenever some stations are faulty on the path between the sending station and the receiving station, the management protocol has to find a way to bypass those faulty stations and set up a new path between them. Similarly, if this new path is disconnected again, a third path needs to be set up, if it is possible [4]. In this context, diameter is the measurement for maximum transmission delay and connectivity is a good parameter to study the tolerance of the network on occasions when nodes fail. Fault tolerant interconnection networks can be found in [3].

For a graph (network) G with connectivity $k(G)$, the parameters w-wide diameter $d_w(G)$, $(w-1)$ –fault diameter $D_w(G)$ and the Rabin number $r_w(G)$ for any $w \leq k(G)$ arise from the study of parallel routing, fault-tolerant systems and randomized routing respectively [5, 6, 7, 8]. Due to the widespread use of reliable, efficient and fault-tolerant networks, these three parameters have been the subject of extensive study over the past decade [5].

In 1994, Chen et al. determined the wide diameter of the cycle prefix network [9]. In 1998, Liaw et al. found fault-tolerant routing in circulant directed graphs and cycle prefix networks [10]. The line connectivity and the fault diameters in pyramid networks were studied by Cao et al. in 1999 [4]. In the same year Liaw et al. determined the Rabin number and wide diameter of butterfly networks [2, 7]. In 2005, Liaw et al. found the wide diameters and Rabin numbers of generalized folded hypercube networks [11]. In 2009, Jia and Zhang found the wide diameter of Cayley graphs of Z_m, the cyclic group of residue classes modulo m and they proved that the k-wide diameter of the Cayley graph Cay(Z_m,A) [12]. In 2011, Rajasingh et al. determined the reliability measures in circulant network [13].

2. Basic concepts

In this section we give the basic definitions and preliminaries that are required for the study.

Definition 2.1. [4] *A container $C(x,y)$ between two distinct nodes x and y in a network G is a set of node-disjoint paths between x and y. The number of paths in $C(x,y)$ is called the width of $C(x,y)$. A $C(x,y)$ container with width w is denoted by $C_w(x,y)$. The length of $C_w(x,y)$, written as $l(C_w(x,y))$, is the length of a longest path in $C_w(x,y)$.*

Definition 2.2. [9] *For $w \leq k(G)$, the w-wide distance from x to y in a network G is defined as*

$$d_w(x,y) = min\,\{l(C_w(x,y)) : C_w(x,y)\} \text{ is a container with width } w \text{ between } x \text{ and } y\}.$$

The w-wide diameter of G is defined as $d_w(G) = max_{x,y \epsilon V(G)}\{d_w(x,y)\}$.

In other words, for $w \leq k(G)$, the ω-wide diameter $d_w(G)$ of a network G is the minimum l such that for any two distinct vertices x and y there exist ω vertex-disjoint paths of length at most l from x to y.

The notion of w-wide diameter was introduced by Hsu [5] to unify the concepts of diameter and connectivity. It is desirable that an ideal interconnection network G should be one with con- nectivity $k(G)$ as large as possible and diameter $d(G)$ as small as possible. The wide-diameter $d_w(G)$ combines connectivity $k(G)$ and diameter $d(G)$, where $1 \leq w \leq k(G)$. Hence $d_w(G)$ is a more suitable parameter than $d(G)$ to measure fault-tolerance and efficiency of parallel processing computer networks. Thus, determining the value of $d_w(G)$ is of significance for a given graph G and an integer w. Hsu proved that this problem is NP-complete [5].

Remark 2.3. *If there exists a container $C_w^*(x,y)$ such that each of the w paths in $C_w^*(x,y)$ is a shortest path between x and y in G, then $d_w(x,y) = l(C_w^*(x,y))$.*

Definition 2.4. [2] *For $w \leq k(G)$, the $(w-1)$-fault distance frtom x to y in a network G is*

$$D_w(x,y) = max\,\{(d_{G-|S|}(x,y) : S \subseteq V \text{ with } |S| = w-1 \text{ and } x,\ y \text{ are not in } S\}$$

where $d_{G-|S|}(x,y)$ *denotes the shortest distance between* x *and* y *in* $G-|S|$.

The notion of $D_w(x,y)$ was defined by Hsu [5] and the special case in which $w = k(G)$ was studied by Krishnamoorthy et al. [6].

Definition 2.5. [13] *For* $w \le k(G)$, *the* $(w-1)$*-fault wide distance from* x *and* y *in a network* G *is*

$$\rho_w(x,y) = max\,\{d_{k(G)-|S|}(x,y) : S \subseteq V \; with \; |S| = w-1 \; and \; x,y \; are \; not \; in \; S\}.$$

The $(w-1)$*-fault wide diameter of* G *is*

$$\rho_w(G) = max\,\{\rho_w(x,y) : x \; and \; y \; are \; nodes \; in \; G\}.$$

Definition 2.6. [2] *The* w*-Rabin number* $r_w(G)$ *of a network* G *is the minimum* l *such that, for any* $w+1$ *distinct vertices* $x, \; y_1, \dots, y_w$ *there exists* w *vertex-disjoint paths of length at most* l *form* x *to* y_i, $1 \le i \le w$.

This concept was first defined by Hsu [5]. It is clear that when $w = 1, d_1(G) = D_1(G) = \rho_w(G) = r_1(G) = d(G)$ for any network G.
The following are basic properties and relationships among $d_w(G), D_w(G), \rho_w(G)$ and $r_w(G)$.

Lemma 2.7. [2] *The following statements hold for any network* G *of connectivity* k:
1. $D_1(G) \le D_2(G) \le \dots \le D_k(G)$
2. $d_1(G) \le d_2(G) \le \dots \le d_k(G)$
3. $r_1(G) \le r_2(G) \le \dots \le r_k(G)$
4. $D_w(G) \le d_w(G)$ and $D_w(G) \le r_w(G)$ for $1 \le w \le k$

Lemma 2.8. [13] *The following statements hold for any network* G *of connectivity* k:
1. $\rho_1(G) \le \rho_2(G) \le \dots \le \rho_w(G)$
2. $D_w(G) \le d_w(G)$ and $D_w(G) \le r_w(G)$ for $1 \le w \le k$

3. Main Results

A tree is a connected graph that contains no cycles. The most common type of tree is the binary tree. It is so named because each node can have at most two descendents. A binary tree is said to be a complete binary tree if each internal node has exactly two descendents. These descendents are described as left and right children of the parent node. Binary trees are widely used in data structures because they are easily stored, easily manipulated, and easily retrieved. Also, many operations such as searching and storing can be easily performed on tree data structures. Furthermore, binary trees appear in communication pattern of divided-and-conquer type algorithms, functional and logic programming, and graph algorithms [3].

For any non-negative integer r, the complete binary tree of height $r-1$, denoted by T_r, is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. Clearly, a complete binary tree T_r has $r-1$ levels and level i, $0 \le i \le r-1$, contains 2^i vertices. Thus T_r has exactly $2^r - 1$ vertices. The rooted complete binary tree RT_r is obtained from a complete binary tree T_{r-1} by attaching to its root a pendant edge. The new vertex is called the root of RT_r and is considered to be at level 0 and level i in T_{r-1} becomes $i+1$ in RT_r, where $0 \le i \le r-1$. See Figure 1.

Definition 3.1. *Let* T_r *be a complete binary tree,* $r \ge 1$. *A graph which is obtained from two copies of complete binary tree* T_r, *say* T_r^1, T_r^2 *by joining each vertex of* T_r^1 *and the corresponding vertex of* T_r^2 *by an edge is called a extended theta mesh and is denoted by* $ETM(r)$. *See Figure* 2.

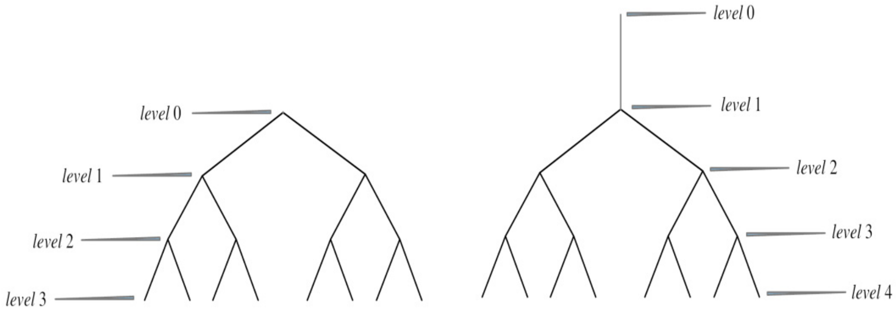

Figure 1: Complete binary tree T_4 and Rooted complete binary tree RT_5

Remark 3.2. *$ETM(r)$ has $2^{r+1} - 2$ vertices and $3.2^r - 5$ edges. The diameter $d(ETM(r)) = 2r - 1$ and it is 2-connected planar biregular graph, where $r \geq 1$.*

Definition 3.3. *Let RT_r be a rooted complete binary tree, $r \geq 1$. A graph which is obtained from two copies of rooted complete binary tree RT_r, say RT_r^1, RT_r^2 by joining each vertex of RT_r^1 and the corresponding vertex of RT_r^2, by an edge except level 0 is called an extended rooted theta mesh and is denoted by $ETM(r)$. See Figure 2.*

Remark 3.4. *$ETM(r)$ has 2^r vertices and $3.2^{r-1} - 3$ edges, where $r \geq 1$.*

Definition 3.5. *A graph which is obtained from $ETM(r)$ by identifying the pendant vertices is known as identified extended rooted theta mesh. For brevity, we call this graph as identified theta mesh and denote it by $ITM(r), r \geq 2$. The identified vertex is called the root of $ITM(r)$.*

Remark 3.6. *$ITM(r)$ has $2^r - 1$ vertices and $3.2^{r-1} - 3$ edges, $r \geq 2$.*

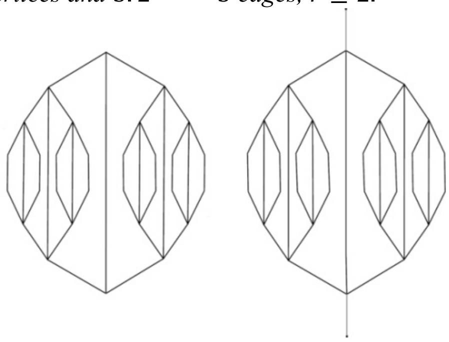

Figure 2: Extended theta mesh $ETM(4)$ and extended rooted theta mesh $ERTM(5)$

A hypergraph is a generalization of a graph in which an edge can connect any number of vertices and are called hyperedges. Hypergraphs arise naturally in important practical problems, including circuit layout, boolean satisfiability and numerical linear algebra [14]. Hypergraphs are also considered a useful tool for modeling system architectures and data structures and to represent a partition, covering and clustering in the area of circuit design [15]. A transversal of a hypergraph H is a set of vertices that contains at least one vertex of each hyperedge [16]. Computing the transversal hypergraph has applications in combinatorial optimization [17], in game theory, and in several fields of computer science such as machine learning [18], indexing of databases, the satisfiability problem, data mining [19], and computer program optimization [20].

A hypertree is a hypergraph H if there is a tree T such that the hyperedges of H induce subtrees in T [21]. In the literature, hypertree is also called a subtree hypergraph or arboreal hypergraph [16, 21].

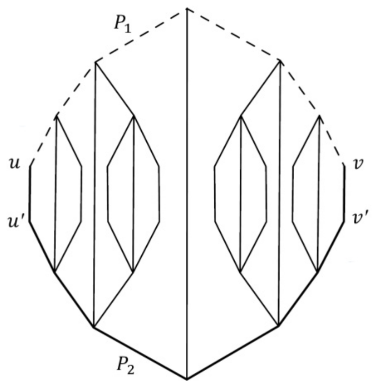

Figure 3: 2-wide diameter of $ETM(4)$

The basic skeleton of a hypertree is a complete binary tree T_r. Here the nodes of the tree are numbered as follows: The root node has label 1. The root is said to be at level 1. Labels of left and right children are formed by appending a 0 and 1, respectively, to the label of the parent node. See Figure $4(a)$. The decimal labels of the hypertree in Figure $4(a)$ is depicted in Figure $4(b)$. Here the children of the node x are labeled as $2x$ and $2x+1$. Additional links in a hypertree are horizontal and two nodes are joined in the same level i of the tree if their label difference is 2^{i-2}. We denote an r-dimensional(level) hypertree as $HT(r)$. It has $2^r - 1$ vertices and $3.2^{r-1} - 3$ edges [22].

A hypertree is an interconnection topology for incrementally expansible multicomputer systems, which combines the easy expansibility of tree structures with the compactness of the hypercube; that is, it combines the best features of the binary tree and the hypercube. These two properties make this topology particulary attractive for implementation of multiprocessor networks of the future, where a complete computer with a substantial amount of memory can fit on a single VLSI chip [22] .

The crossing number [3] of interconnection networks is an important property in VLSI Layout. The highlight of this paper is the fact that $ITM(r)$ is isomorphic to $HT(r)$, thereby proving that $HT(r)$ is planar.

Isomorphic Algorithm

Input : The r -dimensional identified theta mesh $ITM(r)$ and the r-dimensional hypertree $HT(r)$, $r \geq 2$.
Algorithm : Removal of root vertex and the edges joining T_r^1 and T_r^2 of $ITM(r)$ leaves T_r^1 and T_r^2. Label the vertices of T_r^1 from 0 to $2^{r-1} - 2$ and the vertices of T_r^2 from 2^{r-1} to $2^r - 2$ using inorder labeling [23] and the label the root vertex as $2^{r-1} - 1$. Removal the horizontal edges in hypertree $HT(r)$ leaves a complete binary tree T_r and label the vertices of T_r using inorder labeling [23]. See Figure 5.
Output : $ITM(r)$ is isomorphic to $HT(r)$, $r \geq 2$. See Figure 5.

Theorem 3.7. *The identified theta mesh* $ITM(r)$ *is isomorphic to the hypertree* $HT(r)$ *, where* $r \geq 2$.
Proof. Label the vertices of $ITM(r)$ and $HT(r)$, using Isomorphic Algorithm. Let u be any vertex in $ITM(r)$ with label x. We define a function g from $V(ITM(r))$ to $V(HT(r))$ as follows:

$$g(x) = x.$$

The function g is obviously bijective. Let u and v be two distinct vertices in $ITM(r)$ with label x and y respectively. It follows that $g(x)$ and $g(y)$ are the labels of two distinct vertices in $HT(r)$ given as follows:

$$g(x) = x, g(y) = y.$$

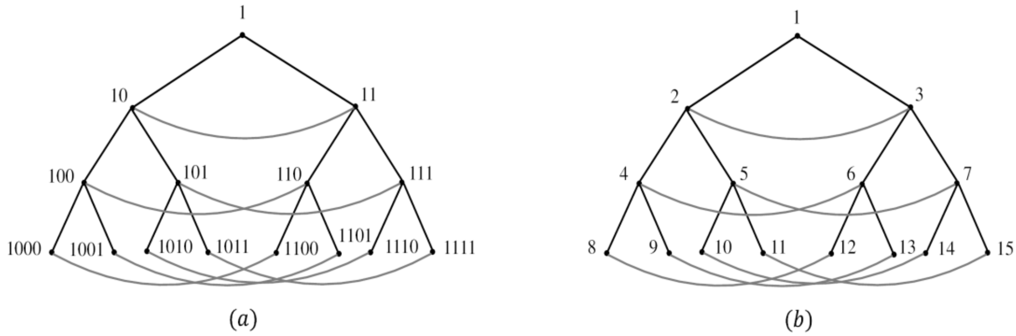

Figure 4: (a) $HT(4)$ with binary labels (b) $HT(4)$ with decimal labels

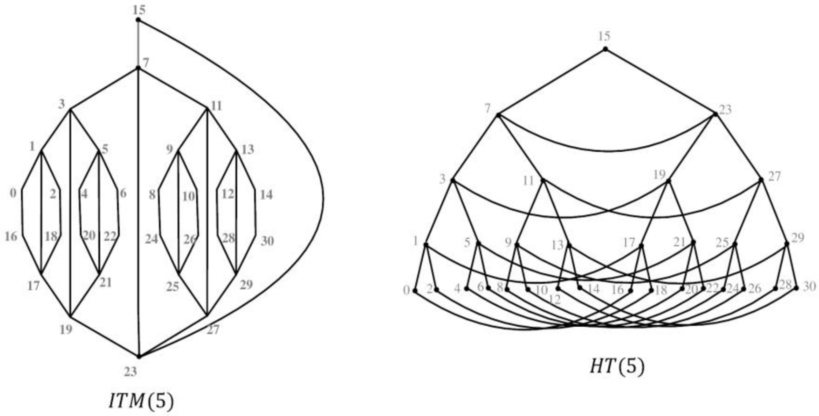

Figure 5: $ITM(5)$ is isomorphic to $HT(5)$

Let the labels x and y be adjacent in $ITM(r)$. Then, we have the following three cases.

Case 1 $(x, y \in T_r^1$ or $T_r^2)$

By definition of complete binary tree, $g(x)$ and $g(y)$ are adjacent in $HT(r)$.

Case 2 $(x \in T_r^1$ and $y \in T_r^2)$

$\Rightarrow |y-x| = 2^{r-1}$ [by inorder labeling of $IHT(r)$]

$\Rightarrow |y-x| = 2^{r-1}$ [by inorder labeling of $HT(r)$]

$\Rightarrow g(x)$ and $g(y)$ are adjacent in $HT(r)$.

Case 3 (x is the root and $y \in T_r^1$ or T_r^2)

$\Rightarrow |y-x| = 2^{r-2}$.

As in Case 2, $g(x)$ and $g(y)$ are adjacent in $HT(r)$.

Similarly, we prove the converse. □

Corollary 3.8. *The network $ITM(r)\backslash u$ is isomorphic to $HT(r)\backslash v$, where u and v are the root vertices of $ITM(r)$ and $HT(r)$ respectively. We call the graph $HT(r)\backslash v$ as the root-fault hypertree and denote it by $HT^*(r)$, $r \geq 2$.*

Now we discuss certain combinatorial parameters of the r-dimensional root-fault hypertree $HT^*(r)$.

Theorem 3.9. *Let G be the r-dimensional root-fault hypertree $HT^*(r)$. Then $d_2(G) = 2r$, $r \geq 2$.*

Proof. Let u, v be the left most and right most vertices of degree 2 in the same level of G. Then length of P_1 is $d(u,v) = 2r-2$ and length of P_2 is $d(u,v) = d(u,u') + d(u',v') + d(v',v) = 2r$ or vice-versa, where u' and v' are the vertices adjacent to u and v respectively. See Figure 3.

For any other pair of vertices $i, j \in G, d(i, j) < 2r$. Hence $d_2(G) = 2r, r \geq 2$. □

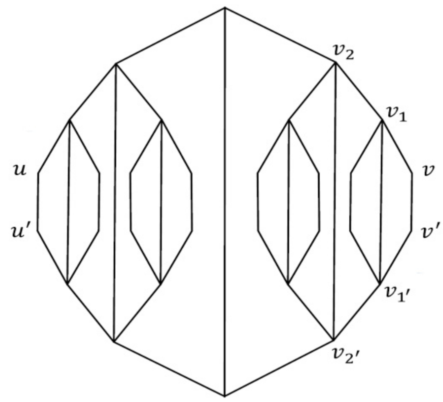

Figure 6: $ETM(4)$ with identifying vertices

Theorem 3.10. *Let G be the r-dimensional root-fault hypertree $HT^*(r), r \geq 2$. Then $D_2(G) = \rho_2(G) = 2r$.*
Proof. *G is isomorphic to $ETM(r)$. Let u, v be the left most and right most vertices of degree 2 in the $(r-1)^{th}$ level of T_r^1 in $ETM(r)$. Then $d(u, v) = 2r - 2$.*

Let $G' = G \backslash \{v_1\}$, where v_1 is the faulty vertex in the $(r-2)^{th}$ level of T_r^1 in $ETM(r)$, which is adjacent to v. Then $d_G(u, v) = d(u, v_2) + d(v_2, v_2') + d(v_2', v_1') + d(v_1', v') + d(v', v)$

$= 2r - 4 + 1 + 1 + 1 + 1$

$= 2r$

Where v_2 is a vertex in the $(r-3)^{th}$ level of T_r^1 and adjacent to v_1. Also, by Theorem 3.9, $d_G(i, j) \leq 2r$, where $i, j \in ETM(r)$. See Figure 6.

For any other faulty vertex in $ETM(r)$, $d_G(x, y) \leq 2r$, where x, y in $ETM(r)$. Hence $D_2(G) = 2r$. Proceeding in the same way, we prove $\rho_2(G) = 2r$. □

Theorem 3.11. *Let G be the r-dimensional root-fault hypertree$HT^*(r), r \geq 2$. Then $r_2(G) = 2r$.*
Proof. G is isomorphic to $ETM(r)$. Let u, v, v_1 be the three vertices in $ETM(r)$ as shown in the Figure 6. Then by Theorem 3.9, $d_2(u, v) = 2r$. Again by Remark 3.2, $d_2(u, v_1) = 2r - 1$. For any other vertices $i, j, k \in ETM(r)$, $d_2(i, j) \leq 2r$ and $d_2(i, k) \leq 2r$. Hence the proof. □

4. Concluding Remark

In this paper, we prove that $d_2(G) = D_2(G) = \rho_2(G) = r_2(G) = 2r$, $r \geq 2$, when G is a root-fault hypertree. It is very interesting to note that this is one of the important networks since various diameters discussed in this paper are equal.

References

1. D.R. Duh, G.H. Chen and D.F. Hsu, *Combinatorial properties of generalized hypercube graphs*, Information Processing Letters, Vol. 57, 41 - 45, 1996.
2. S.C. Liaw and G.J. Chang, *Rabin number of Butterfly Networks*, Discrete Math., Vol. 196, no. 1-3, 219 - 227, 1999.
3. J.M. Xu, *Topological Structure and Analysis of Interconnection Networks*, Kluwer Academic Publishers, 2001.
4. F. Cao, D. Du, D.F. Hsu and S. Teng, *Fault Tolerance Properties of Pyramid Networks*, IEEE Transactions on Computers, Vol. 48, no. 1, 88 - 93, 1999.
5. D.F. Hsu, *On Container Width and Length in Graphs, Groups and Networks*, IEICE Trans. Fundamentals of

Electronics, Comm., and Computer Sciences, Vol. E-77A, 668 - 680, 1994.
6. M.S. Krishnamoorthy and B. Krishnamurthy, *Fault diameter of interconnection networks*, Comput. Math. Appl., Vol. 13, no. 5-6, 577 - 582, 1987.
7. S.C. Liaw and G.J. Chang, *Wide Diameters of Butterfly Networks*, Taiwanese Journal of Mathematics, Vol. 3, no. 1, 83 - 88, 1999.
8. I. Rajasingh, B. Rajan, R.S. Rajan and P. Manuel, *Topological properties of Fat trees*, Journal of Combinatorial Mathematics and Combinatorial Computing, Vol. 79, 139 - 146, 2011.
9. W.Y.C. Chen, V. Faber and E. Knill, *Restricted Routing and Wide Diameter of the Cycle Prefix Network*, DIMACS Series in Discrete Mathematics and Theoretial Computer Science, Vol. 21, 31 - 46, 1995.
10. S.C. Liaw, G.J. Chang, F. Cao and D.F. Hsu, *Fault-tolerant Routing in Circulant Networks and Cycle Prefix Networks*, Annals of Combinatorics, Vol. 2, no. 2, 165 - 172, 1998.
11. S.C. Liaw and P.S. Lan, Wide Diameters and Rabin Numbers of Generalized Folded Hypercube Networks, PhD Thesis, Taiwan, Republic of China, June 2005.
12. K.E. Jia and A.Q. Zhang, *On Wide diameter of Cayley graphs*, Journal of Interconnection Networks, Vol. 10, no. 3, 219 - 231, 2009.
13. I. Rajasingh, B. Rajan and R.S. Rajan, *Combinatorial properties of circulant networks*, IAENG International Journal of Applied Mathematics, Vol. 41, no. 4, 352 - 356, 2011.
14. D.A. Papa and I.L. Markov, *Hypergraph Partitioning and Clustering: In Approximation Algorithms and Metaheuristics*, CRC Press, 2007.
15. M. Akram and W.A. Dudek, *Intuitionistic fuzzy hypergraphs with applications*, Information Sciences, Vol. 218, 182–193, 2013.
16. J.V.D. Heuvel and M. Johnson, *Transversals of subtree hypergraphs and the source location problem in digraphs*, Networks, Vol. 51, no. 2, 113–119, 2008.
17. A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, Springer-Verlag, Berlin Hiedelberg, 2003.
18. T. Mitchell, *Machine Learning*, McGraw Hill, 1997.
19. M. Kantardzic, *Data Mining: Concepts, Models, Methods, and Algorithms*, John Wiley & Sons , 2003.
20. V.I. Voloshin, *Introduction to Graph and Hypergraph Theory*, Nova Science Publishers, Inc., 2009.
21. A. Brandstadt, V.D. Chepoi and F.F. Dragan, *The algorithmic use of hypertree structure and maximum neighbourhood orderings*, Discrete Applied Mathematics, Vol. 82, 43–77, 1998.
22. J.R. Goodman and C.H. Sequin, *A multiprocessor interconnection topology*, IEEE Transactions on Computers, Vol. c-30, no. 12, 923–933, 1981.
23. T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein, *Introduction to Algorithms*, MIT Press and McGraw-Hill, New York, 2001.