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Common fixed point theorems under implicit relations on ordered metric spaces and application to integral equations

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Abstract In this paper, we prove existence results for common fixed points of two or three relatively asymptotically regular mappings satisfying the orbital continuity of one of the involved maps under implicit relation on ordered orbitally complete metric spaces. We furnish suitable examples to demonstrate the validity of the hypotheses of our results. At the end of the results, an application to the study of existence and uniqueness of solutions for a class of nonlinear integral equations is presented.

Keywords Fixed point · Common fixed point · Ordered metric space · Implicit relation · Weakly increasing maps · Dominating map · Well ordered set

Mathematics Subject Classification (1991) Primary 54H25; Secondary 47H10

1 Introduction

Fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. Most of the related results are a hybrid of two fundamental principles: order iterative technique and various contractive conditions. Indeed, they deal with a monotone (either order-preserving or order-reversing) mapping $\mathcal{F}$ satisfying, with some restriction, a classical contractive condition, and are such that for some $x_0 \in \mathcal{X}$, either $x_0 \preceq \mathcal{F}x_0$ or $\mathcal{F}x_0 \preceq x_0$ holds. The first result in this direction was given by Ran and Reurings [28] who presented its applications to matrix equations. Subsequently,

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Nieto and Rodríguez-López [25] used the contractive condition

$$d(\mathcal{F}x, \mathcal{F}y) \leq kd(x, y) \quad \text{for} \quad y \preceq x. \tag{1.1}$$

where $k \in [0, 1)$ and extended this result for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Later, in [27] O'Regan and Petruşel gave some existence results for Fredholm and Volterra type integral equations. In some of the above works, the fixed point results are given for non-decreasing mappings. In [3], Agarwal el al. used the nonlinear contractive condition, that is,

$$d(\mathcal{F}x, \mathcal{F}y) \leq \psi(d(x, y)) \text{ for } y \preceq x, \tag{1.2}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a non-decreasing function with $\lim_{n\to\infty} \psi^n(t) = 0$ for $t > 0$, instead of (1.1). Also in [3], the authors proved a fixed point theorem using generalized nonlinear contractive condition, that is,

$$d(\mathcal{F}x, \mathcal{F}y) \leq \psi\left(\max\left\{d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{F}y), \tfrac{1}{2}[d(x, \mathcal{F}y) + d(y, \mathcal{F}x)]\right\}\right) \tag{1.3}$$

for $y \preceq x$, where ψ is as above.

Recently, Altun and Simsek [4] proved the fixed point results using implicit relations for one map and two maps and generalized the results given in [3,25,27,28]. Also, an application to an existence theorem for common solution of two integral equations is given. The main results are the following:

Theorem 1 *Let $(\mathcal{X}, d, \preceq)$ be a partially ordered complete metric space. Suppose $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a non-decreasing mapping such that for all comparable $x, y \in \mathcal{X}$,*

$$T(d(\mathcal{F}x, \mathcal{F}y), d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{F}y), d(x, \mathcal{F}y), d(y, \mathcal{F}x)) \leq 0,$$

where $T : \mathbb{R}_+^6 \to \mathbb{R}$ belongs to the set of functions $\mathcal{T}$ as given in [4]. Also suppose that

$$\mathcal{F} \text{ is continuous}$$

or

$$\begin{cases} \text{if } \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to x \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq x \text{ for all } n \end{cases}$$

holds. If there exists an $x_0 \in \mathcal{X}$ with $x_0 \preceq \mathcal{F}(x_0)$ then $\mathcal{F}$ has a fixed point.

Theorem 2 *Let $(\mathcal{X}, d, \preceq)$ be a partially ordered complete metric space. Suppose $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{X}$ are two weakly increasing mappings such that for all comparable $x, y \in \mathcal{X}$,*

$$T(d(\mathcal{F}x, \mathcal{G}y), d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{G}y), d(x, \mathcal{G}y), d(y, \mathcal{F}x)) \leq 0$$

where $T : \mathbb{R}_+^6 \to \mathbb{R}$ *belongs to the set of functions* $\mathfrak{T}'$ *as given in Definition 1 of Sect. 2. Also suppose that*

$$\mathcal{F} \text{ is continuous or } \mathcal{G} \text{ is continuous}$$

or

$$\begin{cases} \text{if } \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to x \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq x \text{ for all } n \end{cases}$$

holds. Then $\mathcal{F}$ *and* $\mathcal{G}$ *have a common fixed point.*

Thereafter, several authors worked in this direction and proved fixed point theorems in ordered metric spaces. For more details see [1–8,10,11,14,18,21,23,24,27,29,31–34] and the references cited therein.

The aim of this work is to generalize Theorem 1 and Theorem 2 (and, hence, some other related common fixed point results) in two directions. The first is treated in Sect. 3, where a pair of asymptotically regular mappings in an orbitally complete ordered metric space is considered. The existence and (under additional assumptions) uniqueness of their common fixed point is obtained under assumptions that these mappings are strictly weakly isotone increasing, one is orbitally continuous and they satisfy a implicit relation condition.

In Sect. 4 we consider the case of three self-mappings $\mathcal{F}, \mathcal{G}, \mathcal{R}$ where the pair $\mathcal{F}, \mathcal{G}$ is $\mathcal{R}$-relatively asymptotically regular and relatively weakly increasing, with the implicit relation.

We furnish suitable examples to demonstrate the validity of the hypotheses of our results. We conclude the paper applying the obtained results to prove an existence theorem for solutions of a system of integral equations.

2 Notation and definitions

First, we introduce some notation and definitions that will be used later.

2.1 Implicit relation and related concepts

In recent years, Popa [26] have used implicit functions rather than contraction conditions to prove fixed point theorems in metric spaces whose strength lie in their unifying power, as an implicit function can cover several contraction conditions at the same time, which include known as well as some unknown contraction conditions. This fact is evident from examples furnished in Popa [26].

In this section, in order to prove our results, we define a set of suitable implicit functions involving six real non-negative arguments that was given in [4].

Definition 1 [4] Let $\mathbb{R}_+$ denote the set of non-negative real numbers and let $\mathfrak{T}'$ be the set of all continuous functions $T : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

($\mathfrak{T}'_1$): $T(t_1, \ldots, t_6)$ is non-increasing in variables $t_2, \ldots, t_6$.

($\mathfrak{T}'_2$): There exists a right continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = 0$, $f(t) < t$ for $t > 0$, such that for $u, v \geq 0$,

$$T(u, v, u, v, 0, u + v) \leq 0$$

or

$$T(u, v, v, u, u + v, 0) \leq 0$$

or

$$T(u, v, 0, 0, v, v) \leq 0$$

implies $u \leq f(v)$.

($\mathfrak{T}'_3$): $T(u, 0, u, 0, 0, u) > 0$ and $T(u, 0, 0, u, u, 0) > 0$, $\forall u > 0$.

Example 1 $T(t_1, \ldots, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)[at_5 + bt_6]$, where $0 \leq \alpha < 1, 0 \leq a < \frac{1}{2}, 0 \leq b < \frac{1}{2}$.

Example 2 $T(t_1, \ldots, t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$, where $k \in (0, 1)$.

Example 3 $T(t_1, \ldots, t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\})$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is right continuous and $\phi(0) = 0$, $\phi(t) < t$ for $t > 0$.

Example 4 $T(t_1, \ldots, t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, where $a > 0, b, c, d \geq 0$, $a + b + c < 1$ and $a + d < 1$.

2.2 Asymptotic regularity, orbitally completeness and related concepts

Browder and Petryshyn [12] introduced the concept of asymptotic regularity in Hilbert spaces. It can be formulated for metric spaces as follows.

Definition 2 [13] A self-map $\mathcal{F}$ on a metric space $(\mathcal{X}, d)$ is said to be asymptotically regular at a point $x \in \mathcal{X}$ if $\lim_{n\to\infty} d(\mathcal{F}^n x, \mathcal{F}^{n+1} x) = 0$.

Recall that the set $\mathcal{O}(x_0; \mathcal{F}) = \{\mathcal{F}^n x_0 : n = 0, 1, 2, \ldots\}$ is called the orbit of the self-map $\mathcal{F}$ at the point $x_0 \in \mathcal{X}$.

Definition 3 [13] A metric space $(\mathcal{X}, d)$ is said to be $\mathcal{F}$-orbitally complete for some $x \in \mathcal{X}$ if every Cauchy sequence contained in $\mathcal{O}(x; \mathcal{F})$ converges in $\mathcal{X}$.

Here, it can be pointed out that every complete metric space is $\mathcal{F}$-orbitally complete for any $\mathcal{F}$, but an $\mathcal{F}$-orbitally complete metric space need not be complete.

Definition 4 [12] A self-map $\mathcal{F}$ defined on a metric space $(\mathcal{X}, d)$ is said to be orbitally continuous at a point z in $\mathcal{X}$ if for some $x \in \mathcal{X}$ and for any sequence $\{x_n\} \subset \mathcal{O}(x; \mathcal{F})$, $x_n \to z$ as $n \to \infty$ implies $\mathcal{F}x_n \to \mathcal{F}z$ as $n \to \infty$.

Clearly, every continuous self-mapping of a metric space is orbitally continuous, but not conversely.

Sastry et al. [30] extended the above concepts to two and three mappings and employed them to prove common fixed point results for commuting mappings. In what follows, we collect such definitions for three maps.

Definition 5 Let $\mathcal{G}, \mathcal{F}, \mathcal{R}$ be three self-mappings defined on a metric space $(\mathcal{X}, d)$.

(1) If for a point $x_0 \in \mathcal{X}$, there exits a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\mathcal{R}x_{2n+1} = \mathcal{G}x_{2n}$, $\mathcal{R}x_{2n+2} = \mathcal{F}x_{2n+1}$, $n = 0, 1, 2, \ldots$, then the set $\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R}) = \{\mathcal{R}x_n : n = 1, 2, \ldots\}$ is called the orbit of $(\mathcal{G}, \mathcal{F}, \mathcal{R})$ at x_0.
(2) The space $(\mathcal{X}, d)$ is said to be $(\mathcal{G}, \mathcal{F}, \mathcal{R})$-orbitally complete at x_0 if every Cauchy sequence in $\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})$ converges in $\mathcal{X}$.
(3) The map $\mathcal{R}$ is said to be orbitally continuous at x_0 if it is continuous on $\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})$.
(4) The pair $(\mathcal{G}, \mathcal{F})$ is said to be asymptotically regular with respect to $\mathcal{R}$ at x_0 if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\mathcal{R}x_{2n+1} = \mathcal{G}x_{2n}$, $\mathcal{R}x_{2n+2} = \mathcal{F}x_{2n+1}$, $n = 0, 1, 2, \ldots$, and $d(\mathcal{R}x_n, \mathcal{R}x_{n+1}) \to 0$ as $n \to \infty$.
(5) If $\mathcal{R}$ is the identity mapping on $\mathcal{X}$, we omit "$\mathcal{R}$" in respective definitions.

2.3 Partially ordered sets and related concepts

If $(\mathcal{X}, \preceq)$ is a partially ordered set then $x, y \in \mathcal{X}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. A subset $\mathcal{K}$ of $\mathcal{X}$ is said to be well ordered if every two elements of $\mathcal{K}$ are comparable. If $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is such that, for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{F}x \preceq \mathcal{F}y$, then the mapping $\mathcal{F}$ is said to be non-decreasing.

Definition 6 Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $\mathcal{G}, \mathcal{F} : \mathcal{X} \to \mathcal{X}$.

(1) [1] The mapping $\mathcal{F}$ is called dominating if $x \preceq \mathcal{F}x$ for each $x \in \mathcal{X}$.
(2) [15,16] The pair $(\mathcal{G}, \mathcal{F})$ is called weakly increasing if $\mathcal{G}x \preceq \mathcal{F}\mathcal{G}x$ and $\mathcal{F}x \preceq \mathcal{G}\mathcal{F}x$ for all $x \in \mathcal{X}$.
(3) [15,16,35] The mapping $\mathcal{G}$ is said to be $\mathcal{F}$-weakly isotone increasing if for all $x \in \mathcal{X}$ we have $\mathcal{G}x \preceq \mathcal{F}\mathcal{G}x \preceq \mathcal{G}\mathcal{F}\mathcal{G}x$.
(4) [24] The mapping $\mathcal{G}$ is said to be $\mathcal{F}$-strictly weakly isotone increasing if, for all $x \in \mathcal{X}$ such that $x \prec \mathcal{G}x$, we have $\mathcal{G}x \prec \mathcal{F}\mathcal{G}x \prec \mathcal{G}\mathcal{F}\mathcal{G}x$.
(5) [23] Let $\mathcal{R} : \mathcal{X} \to \mathcal{X}$ be such that $\mathcal{F}\mathcal{X} \subseteq \mathcal{R}\mathcal{X}$ and $\mathcal{G}\mathcal{X} \subseteq \mathcal{R}\mathcal{X}$, and denote $\mathcal{R}^{-1}(x) := \{u \in \mathcal{X} : \mathcal{R}u = x\}$, for $x \in \mathcal{X}$. We say that $\mathcal{F}$ and $\mathcal{G}$ are weakly increasing with respect to $\mathcal{R}$ if and only if for all $x \in \mathcal{X}$, we have:

$$\mathcal{F}x \preceq \mathcal{G}y, \ \forall\, y \in \mathcal{R}^{-1}(\mathcal{F}x)$$

and

$$\mathcal{G}x \preceq \mathcal{F}y, \ \forall\, y \in \mathcal{R}^{-1}(\mathcal{G}x).$$

Example 5 [1] Let $\mathcal{X} = [0, 1]$ be endowed with the usual ordering. Let $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ be defined by $\mathcal{F}x = \sqrt[n]{x}$, $n \in \mathbb{N}$. Since $x \leq \sqrt[n]{x} = \mathcal{F}x$ for all $x \in \mathcal{X}$, $\mathcal{F}$ is a dominating map.

Remark 1 (1) None of two weakly increasing mappings need be non-decreasing. There exist some examples to illustrate this fact in [4].

(2) If $\mathcal{G}, \mathcal{F} : \mathcal{X} \to \mathcal{X}$ are weakly increasing, then $\mathcal{G}$ is $\mathcal{F}$-weakly isotone increasing.
(3) $\mathcal{G}$ can be $\mathcal{F}$-strictly weakly isotone increasing, while some of these two mappings can be not strictly increasing (see the following example).
(4) If $\mathcal{R}$ is the identity mapping ($\mathcal{R}x = x$ for all $x \in \mathcal{X}$), then $\mathcal{F}$ and $\mathcal{G}$ are weakly increasing with respect to $\mathcal{R}$ if and only if they are weakly increasing mappings.

Example 6 Let $\mathcal{X} = [0, +\infty)$ be endowed with the usual ordering and define $\mathcal{G}, \mathcal{F} : \mathcal{X} \to \mathcal{X}$ as

$$\mathcal{G}x = \begin{cases} 2x, & \text{if } x \in [0, 1], \\ 3x, & \text{if } x > 1; \end{cases} \qquad \mathcal{F}x = \begin{cases} 2, & \text{if } x \in [0, 1], \\ 2x, & \text{if } x > 1. \end{cases}$$

Clearly, we have $x \prec \mathcal{G}x \prec \mathcal{F}\mathcal{G}x \prec \mathcal{G}\mathcal{F}\mathcal{G}x$ for all $x \in \mathcal{X}$, and so, $\mathcal{G}$ is $\mathcal{F}$-strictly weakly isotone increasing; $\mathcal{F}$ is not strictly increasing.

Definition 7 [19,20]. Let $(\mathcal{X}, d)$ be a metric space and $f, g : \mathcal{X} \to \mathcal{X}$.

(1) If $w = fx = gx$, for some $x \in \mathcal{X}$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. If $w = x$, then x is a common fixed point of f and g.
(2) The mappings f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Definition 8 Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preceq)$ is called an ordered metric space if

(i) $(\mathcal{X}, d)$ is a metric space,
(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

The space $(\mathcal{X}, d, \preceq)$ is called regular if the following hypothesis holds: if $\{z_n\}$ is a non-decreasing sequence in $\mathcal{X}$ with respect to $\preceq$ such that $z_n \to z \in \mathcal{X}$ as $n \to \infty$, then $z_n \preceq z$.

3 Common fixed point theorem for $\mathcal{F}$-trictly weakly isotone increasing mappings

In this section, we improve the results of Altun and Simsek [4] by considering the following:

(1) a pair of asymptotically regular mappings;
(2) orbital continuity of one of the involved maps;
(3) strictly weakly isotone increasing condition;
(4) implicit condition, and
(5) an ordered orbitally complete metric space.

The first main result of this section is as follows:

Theorem 3 *Let $(\mathcal{X}, d, \preceq)$ be an ordered metric space. Let $\mathcal{G}, \mathcal{F} : \mathcal{X} \to \mathcal{X}$ be two mappings satisfying*

$$T(d(\mathcal{F}x, \mathcal{G}y), d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{G}y), d(x, \mathcal{G}y), d(y, \mathcal{F}x)) \leq 0 \quad (3.1)$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F})}$ such that x and y are comparable, where $T \in \mathfrak{T}'$.

We assume the following hypotheses:

(i) *$(\mathcal{F}, \mathcal{G})$ is asymptotically regular at some $x_0 \in \mathcal{X}$;*
(ii) *$\mathcal{X}$ is $(\mathcal{G}, \mathcal{F})$-orbitally complete at x_0;*
(iii) *$\mathcal{G}$ is $(\mathcal{G}, \mathcal{F})$-orbitally continuous at x_0;*
(iv) *$\mathcal{G}$ is $\mathcal{F}$-strictly weakly isotone increasing;*
(v) *$x_0 \prec \mathcal{G}x_0$.*

Then $\mathcal{G}$ and $\mathcal{F}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{G}, \mathcal{F}$ in $\overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F})}$ is well ordered if and only if it is a singleton.

Proof First of all we show that, if $\mathcal{F}$ or $\mathcal{G}$ has a fixed point, then it is a common fixed point of $\mathcal{F}$ and $\mathcal{G}$. Indeed, let z be a fixed point of $\mathcal{F}$. Assume that $d(z, \mathcal{G}z) > 0$. If we use the inequality (3.1), for $x = y = z$, we have

$$T(d(z, \mathcal{G}z), 0, 0, d(z, \mathcal{G}z), d(z, \mathcal{G}z), 0) \leq 0,$$

which is a contradiction to $(\mathfrak{T}'_3)$. Thus $d(z, \mathcal{G}z) = 0$ and so z is a common fixed point of $\mathcal{F}$ and $\mathcal{G}$. Analogously, one can observe that if z is a fixed point of $\mathcal{G}$, then it is also a fixed point of $\mathcal{F}$. Now let x_0 be a point assumed in (i). If $x_0 = \mathcal{F}x_0$, the proof is finished, so assume $x_0 \neq \mathcal{F}x_0$.

Since $(\mathcal{F}, \mathcal{G})$ is asymptotically regular at x_0 in $\mathcal{X}$, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$x_{2n+1} = \mathcal{G}x_{2n} \text{ and } x_{2n+2} = \mathcal{F}x_{2n+1} \text{ for } n \in \{0, 1, \dots\} \quad (3.2)$$

and

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \quad (3.3)$$

If $x_{n_0} = \mathcal{G}x_{n_0}$ or $x_{n_0} = \mathcal{F}x_{n_0}$ for some n_0, then the proof is finished. So assume $x_n \neq x_{n+1}$ for all n.

Since $\mathcal{G}$ is $\mathcal{F}$-strictly weakly isotone increasing, we have

$$x_1 = \mathcal{G}x_0 \prec \mathcal{F}\mathcal{G}x_0 = \mathcal{F}x_1 = x_2 \prec \mathcal{G}\mathcal{F}\mathcal{G}x_0 = \mathcal{G}\mathcal{F}x_1 = \mathcal{G}x_2 = x_3,$$
$$x_3 = \mathcal{G}x_2 \prec \mathcal{F}\mathcal{G}x_2 = \mathcal{F}x_3 = x_4 \prec \mathcal{G}\mathcal{F}\mathcal{G}x_2 = \mathcal{G}\mathcal{F}x_3 = \mathcal{G}x_4 = x_5,$$

and continuing this process we get

$$x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} \prec \cdots . \quad (3.4)$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence in the metric space $\mathcal{O}(x_0; \mathcal{G}, \mathcal{F})$. We proceed by negation and suppose that $\{x_n\}$ is not a Cauchy sequence. That is, there exists $\delta > 0$ such that $d(x_n, x_m) \geq \delta$ for infinitely many values of m and n with $m < n$. This assures that there exist two sequences $\{m(k)\}$, $\{n(k)\}$ of natural numbers, with $m(k) < n(k)$, such that for each $k \in \mathbb{N}$

$$d(x_{2n(k)}, x_{2m(k)}) \geq \delta \text{ for } k \in \{1, 2, \ldots\}. \tag{3.5}$$

We may also assume

$$d(x_{2m(k)-2}, x_{2n(k)}) < \delta \tag{3.6}$$

by choosing $2m(k)$ to be smallest number exceeding $2n(k)$ for which (3.5) holds. Now (3.5) and (3.6) imply

$$\begin{aligned} 0 < \delta &\leq d(x_{2n(k)}, x_{2m(k)}) \\ &\leq d(x_{2n(k)}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}) \\ &\leq \delta + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}) \end{aligned}$$

and so

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)}) = \delta. \tag{3.7}$$

Also, by the triangular inequality,

$$\left|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})\right| \leq d(x_{2m(k)-1}, x_{2m(k)})$$

and

$$\left|d(x_{2n(k)+1}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})\right| \leq d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2n(k)}, x_{2n(k)+1}).$$

Therefore we get

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}) = \delta \tag{3.8}$$

and

$$\lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)-1}) = \delta. \tag{3.9}$$

Also we have

$$\begin{aligned} \delta &\leq d(x_{2n(k)}, x_{2m(k)}) \\ &\leq d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)}). \end{aligned} \tag{3.10}$$

On the other hand, since $x_{2n(k)}$ and $x_{2m(k)-1}$ are comparable we can use the condition (3.1) for these points. Therefore we have

$$T(d(\mathcal{F}x_{2n(k)}, \mathcal{G}x_{2m(k)-1}), d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, \mathcal{F}x_{2n(k)}),$$
$$d(x_{2m(k)-1}, \mathcal{G}x_{2m(k)-1}), d(x_{2n(k)}, \mathcal{G}x_{2m(k)-1}), d(x_{2m(k)-1}, \mathcal{F}x_{2n(k)})) \leq 0$$

and so

$$T(d(x_{2n(k)+1}, x_{2m(k)}), d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, x_{2n(k)+1}),$$
$$d(x_{2m(k)-1}, x_{2m(k)}), d(x_{2n(k)}, x_{2m(k)}), d(x_{2m(k)-1}, x_{2n(k)+1})) \leq 0.$$

Now, considering (3.7), (3.8) and (3.9), letting $k \to \infty$ in the last inequality we have, by continuity of T, that

$$T\left(\lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)})\right), \delta, 0, 0, \delta, \delta) \leq 0.$$

From $(\mathfrak{T}_2')$, we have $\lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)}) \leq f(\delta)$, i.e., $\delta \leq f(\delta)$. This is a contradiction since $f(t) < t$ for $t > 0$. Thus $\{x_{2n}\}$ is a Cauchy sequence in $\mathcal{O}(x_0; \mathcal{G}, \mathcal{F})$. Since $\mathcal{X}$ is $(\mathcal{F}, \mathcal{G})$-orbitally complete at x_0, there exists a $z \in \mathcal{X}$ with $\lim_{n\to\infty} x_n = z$.

If $\mathcal{G}$ is orbitally continuous, then clearly $z = \mathcal{G}z = \mathcal{F}z$ □

Theorem 4 *Let $(\mathcal{X}, d, \preceq)$ and $\mathcal{G}, \mathcal{F} : \mathcal{X} \to \mathcal{X}$ satisfy all the conditions of Theorem 3, except that condition (iii) is substituted by*

(iii')$\mathcal{X}$ is regular.

Then the same conclusions as in Theorem 3 hold.

Proof Following the proof of Theorem 3, we have that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$ which is $(\mathcal{G}, \mathcal{F})$-orbitally complete at x_0. Then, there exists $z \in \mathcal{X}$ such that

$$\lim_{n\to\infty} x_n = z.$$

Now suppose that $d(z, \mathcal{F}z) > 0$. From regularity of $\mathcal{X}$, we have $x_{2n-1} \preceq z$ for all $n \in \mathbb{N}$. Hence, we can apply the considered contractive condition. Then, setting $x = z$ and $y = x_{2n-1}$ in (3.1), we obtain:

$$T(d(\mathcal{F}z, \mathcal{G}x_{2n-1}), d(z, x_{2n-1}), d(z, \mathcal{F}z), d(x_{2n-1}, \mathcal{G}x_{2n-1}),$$
$$d(x, \mathcal{G}x_{2n-1}), d(x_{2n-1}, \mathcal{F}x)) \leq 0.$$

so letting $n \to \infty$ in the last inequality, we have

$$T(d(\mathcal{F}z, z), 0, d(z, \mathcal{F}z), 0, 0, d(z, \mathcal{F}z)) \leq 0$$

which is a contradiction to $(\mathfrak{T}_3')$. Thus $d(z, \mathcal{F}z) = 0$ and so $z = \mathcal{F}z = \mathcal{G}z$. Hence, z is a common fixed point of $\mathcal{F}$ and $\mathcal{G}$. □

If we take $\mathcal{G}$ = identity mapping in Theorem 3, then we have the following consequence:

Corollary 1 *Let $(\mathcal{X}, d, \preceq)$ be an ordered metric space. Let $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying*

$$T(d(\mathcal{F}x, \mathcal{F}y), d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{F}y), d(x, \mathcal{F}y), d(y, \mathcal{F}x)) \leq 0 \quad (3.11)$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{F})}$ such that x and y are comparable, where $T \in \mathfrak{T}'$.
We assume the following hypotheses:

(i) *$\mathcal{F}$ is asymptotically regular at some point $x_0 \in \mathcal{X}$;*
(ii) *$\mathcal{X}$ is $\mathcal{F}$-orbitally complete at x_0;*
(iii) *$\mathcal{F}$ is orbitally continuous at x_0 or $\mathcal{X}$ is regular.*

Also suppose that $\mathcal{F}x \prec \mathcal{F}(\mathcal{F}x)$ for all $x \in \mathcal{X}$ such that $x \prec \mathcal{F}x$, and that $x_0 \prec \mathcal{F}x_0$. Then $\mathcal{F}$ has a fixed point. Moreover, the set of fixed points of $\mathcal{F}$ in $\overline{\mathcal{O}(x_0; \mathcal{F})}$ is well ordered if and only if it is a singleton.

If we combine Theorem 3 with some examples of T, we obtain the following result.

Corollary 2 *Let $(\mathcal{X}, d, \preceq)$ and $\mathcal{G}, \mathcal{F} : \mathcal{X} \to \mathcal{X}$ satisfy all the conditions of Theorem 3 (or Theorem 4), except that condition (3.1) is substituted by*

$$d(\mathcal{F}x, \mathcal{G}y) \leq \phi(d(x, y))$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{F})}$ such that x and y are comparable, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a right continuous function such that $\phi(0) = 0, \phi(t) < t$ for $t > 0$.
Then the same conclusions as in Theorem 3 (or Theorem 4) hold.

Proof If $T(t_1, \dots t_6) = t_1 - \phi(t_2)$, then it is obvious that $T \in \mathfrak{T}'$. Therefore the proof follows from Theorem 3 (or Theorem 4). □

Corollary 3 *Let $(\mathcal{X}, d, \preceq)$ and $\mathcal{G}, \mathcal{F} : \mathcal{X} \to \mathcal{X}$ satisfy all the conditions of Theorem 3 (or Theorem 4), except that condition (3.1) is substituted by*

$$d(\mathcal{F}x, \mathcal{G}y) \leq \phi\left(\max\left\{d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{G}y), \frac{1}{2}(d(x, \mathcal{G}y) + d(y, \mathcal{F}x))\right\}\right) \quad (3.12)$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{F})}$ such that x and y are comparable, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a right continuous function such that $\phi(0) = 0, \phi(t) < t$ for $t > 0$.
Then the same conclusions as in Theorem 3 (or Theorem 4) hold.

Proof If $T(t_1, \dots t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\})$, then it is obvious that $T \in \mathfrak{T}'$. Therefore the proof follows from Theorem 3 (or Theorem 4). □

We illustrate Corollary 3 by an example which is obtained by modifying the one from [17].

Example 7 Let the set $\mathcal{X} = [0, +\infty)$ be equipped with the usual metric d and the order defined by

$$x \preceq y \iff x \geq y.$$

Consider the following self-mappings on $\mathcal{X}$:

$$\mathcal{F}x = \begin{cases} \frac{1}{2}x, \ 0 \leq x \leq \frac{1}{2}, \\ 2x, \ x > \frac{1}{2}, \end{cases} \qquad \mathcal{G}x = \begin{cases} \frac{1}{3}x, \ 0 \leq x \leq \frac{1}{3}, \\ 3x, \ x > \frac{1}{3}. \end{cases}$$

Take $x_0 = \frac{1}{3}$. Then it is easy to show that

$$\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}) \subset \left\{ \frac{1}{2^k \cdot 3^l} : k, l \in \mathbb{N} \right\}$$

and all the conditions (i)–(v) of Corollary 3 are fulfilled (condition (iv) on $\mathcal{O}(x_0; \mathcal{G}, \mathcal{F})$). Take $\phi(t) = \frac{5}{6}t$ for $t > 0$. Then the contractive condition (3.12) takes the form

$$\left| \frac{1}{2}x - \frac{1}{3}y \right| \leq \frac{5}{6} \max \left\{ |x-y|, \frac{1}{2}x, \frac{2}{3}y, \frac{1}{2}\left[\left| x - \frac{1}{3}y \right| + \left| y - \frac{1}{2}x \right| \right] \right\},$$

for $x, y \in \mathcal{O}(x_0; \mathcal{G}, \mathcal{F})$. Using substitution $y = tx$, $t > 0$, the last inequality reduces to

$$|3 - 2t| \leq 5 \max \left\{ |1-t|, \frac{1}{2}, \frac{2}{3}t, \frac{1}{2}\left[|1 - \frac{1}{3}t| + |t - \frac{1}{2}| \right] \right\},$$

and can be checked by discussion on possible values for $t > 0$. Hence, all the conditions of Corollary 3 are satisfied and $\mathcal{G}, \mathcal{F}$ have a common fixed point (which is 0). Note that $\mathcal{G}$ and $\mathcal{F}$ do not satisfy the contractive condition for arbitrary $x, y \in \mathcal{X}$.

4 Common fixed points for relatively weakly increasing mappings

In this section, we improve and generalize the results of Altun and Simsek [4] by taking into account the following for three maps $\mathcal{R}, \mathcal{G}, \mathcal{F}$:

(1) $(\mathcal{G}, \mathcal{F})$ is a pair of asymptotically regular mappings with respect to $\mathcal{R}$;
(1) orbital continuity of one of the involved maps;
(2) $(\mathcal{G}, \mathcal{F})$ is a pair of weakly increasing mappings with respect to $\mathcal{R}$;
(3) $(\mathcal{G}, \mathcal{F})$ is a pair of dominating maps;
(4) $(\mathcal{G}, \mathcal{F})$ is a pair of compatible maps, and
(5) the basic space is an ordered orbitally complete metric space.

The first result of this section is the following.

Theorem 5 *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{R}$ be self-maps on $\mathcal{X}$ satisfying*

$$T(d(\mathcal{F}x, \mathcal{G}y), d(\mathcal{R}x, \mathcal{R}y), d(\mathcal{R}x, \mathcal{F}x), d(\mathcal{R}y, \mathcal{G}y), d(\mathcal{R}x, \mathcal{G}y), d(\mathcal{R}y, \mathcal{F}x)) \leq 0 \tag{4.1}$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})}$ such that $\mathcal{R}x$ and $\mathcal{R}y$ are comparable, where $T \in \mathfrak{T}'$.

We assume the following hypotheses:

(i) *$(\mathcal{G}, \mathcal{F})$ is asymptotically regular with respect to $\mathcal{R}$ at some $x_0 \in \mathcal{X}$;*
(ii) *$\mathcal{X}$ is $(\mathcal{G}, \mathcal{F}, \mathcal{R})$-orbitally complete at x_0;*
(iii) *$\mathcal{F}$ and $\mathcal{G}$ are weakly increasing with respect to $\mathcal{R}$;*
(iv) *$\mathcal{F}$ and $\mathcal{G}$ are dominating maps;*
(v) *$\mathcal{R}$ is monotone and orbitally continuous at x_0.*

Assume either

(a) $\mathcal{G}$ and $\mathcal{R}$ are compatible; or
(b) $\mathcal{F}$ and $\mathcal{R}$ are compatible.

Then $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{R}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})}$ is well ordered if and only if it is a singleton.

Proof Since $(\mathcal{G}, \mathcal{F})$ is asymptotically regular with respect to $\mathcal{R}$ at x_0 in $\mathcal{X}$, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$\mathcal{R}x_{2n+1} = \mathcal{G}x_{2n}, \qquad \mathcal{R}x_{2n+2} = \mathcal{F}x_{2n+1}, \qquad \forall\, n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \tag{4.2}$$

and

$$\lim_{n\to\infty} d(\mathcal{R}x_n, \mathcal{R}x_{n+1}) = 0 \tag{4.3}$$

holds. We claim that

$$\mathcal{R}x_n \preceq \mathcal{R}x_{n+1}, \ \forall\, n \in \mathbb{N}_0. \tag{4.4}$$

To this aim, we will use the increasing property with respect to $\mathcal{R}$ satisfied by the mappings $\mathcal{F}$ and $\mathcal{G}$. From (4.2), we have

$$\mathcal{R}x_1 = \mathcal{G}x_0 \preceq \mathcal{F}y, \ \forall\, y \in \mathcal{R}^{-1}(\mathcal{G}x_0).$$

Since $\mathcal{R}x_1 = \mathcal{G}x_0$, then $x_1 \in \mathcal{R}^{-1}(\mathcal{G}x_0)$, and we get

$$\mathcal{R}x_1 = \mathcal{G}x_0 \preceq \mathcal{F}x_1 = \mathcal{R}x_2.$$

Again,

$$\mathcal{R}x_2 = \mathcal{F}x_1 \preceq \mathcal{G}y, \ \forall\, y \in \mathcal{R}^{-1}(\mathcal{F}x_1).$$

Since $x_2 \in \mathcal{R}^{-1}(\mathcal{F}x_1)$, we get

$$\mathcal{R}x_2 = \mathcal{F}x_1 \preceq \mathcal{G}x_2 = \mathcal{R}x_3.$$

Hence, by induction, (4.4) holds. Therefore, we can apply (4.1) for $x = x_p$ and $y = x_q$ for all p and q.

Now, we assert that $\{\mathcal{R}x_n\}$ is a Cauchy sequence in the metric space $\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})$. We proceed by negation and suppose that $\{\mathcal{R}x_{2n}\}$ is not Cauchy. Then, there exists $\delta > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)}) \geq \delta \text{ for } k \in \{1, 2, \ldots\}. \tag{4.5}$$

We may also assume

$$d(\mathcal{R}x_{2m(k)-2}, \mathcal{R}x_{2n(k)}) < \delta \tag{4.6}$$

by choosing $2m(k)$ to be the smallest number exceeding $2n(k)$ for which (4.5) holds. Now (4.5) and (4.6) imply

$$\begin{aligned} 0 < \delta &\leq d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)}) \\ &\leq d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)-2}) + d(\mathcal{R}x_{2m(k)-2}, \mathcal{R}x_{2m(k)-1}) + d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2m(k)}) \\ &\leq \delta + d(\mathcal{R}x_{2m(k)-2}, \mathcal{R}x_{2m(k)-1}) + d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2m(k)}) \end{aligned}$$

and so

$$\lim_{k\to\infty} d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)}) = \delta. \tag{4.7}$$

Also, by the triangular inequality,

$$\left|d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)-1}) - d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)})\right| \leq d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2m(k)})$$

and

$$\begin{aligned} \left|d(\mathcal{R}x_{2n(k)+1}, \mathcal{R}x_{2m(k)-1}) - d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)})\right| &\leq d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2m(k)}) \\ &\quad + d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}). \end{aligned}$$

Therefore we get

$$\lim_{k\to\infty} d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)-1}) = \delta \tag{4.8}$$

and

$$\lim_{k\to\infty} d(\mathcal{R}x_{2n(k)+1}, \mathcal{R}x_{2m(k)-1}) = \delta. \tag{4.9}$$

Also we have

$$\begin{aligned}\delta &\leq d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)})\\ &\leq d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}) + d(\mathcal{R}x_{2n(k)+1}, \mathcal{R}x_{2m(k)}).\end{aligned} \tag{4.10}$$

On the other hand, since $\mathcal{R}x_{2n(k)}$ and $\mathcal{R}x_{2m(k)-1}$ are comparable we can use the condition (4.1) for these points. Therefore we have

$$T(d(\mathcal{F}x_{2n(k)}, \mathcal{G}x_{2m(k)-1}), d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)-1}), d(\mathcal{R}x_{2n(k)}, \mathcal{F}x_{2n(k)}),\\ d(\mathcal{R}x_{2m(k)-1}, \mathcal{G}x_{2m(k)-1}), d(\mathcal{R}x_{2n(k)}, \mathcal{G}x_{2m(k)-1}), d(\mathcal{R}x_{2m(k)-1}, \mathcal{F}x_{2n(k)})) \leq 0$$

and so

$$T(d(\mathcal{R}x_{2n(k)+1}, \mathcal{R}x_{2m(k)}), d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)-1}), d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}),\\ d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2m(k)}), d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)}), d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2n(k)+1})) \leq 0.$$

Now, considering (4.7), (4.8) and (4.9), letting $k \to \infty$ in the last inequality we have, by continuity of T, that

$$T\left(\lim_{k\to\infty} d(\mathcal{R}x_{2n(k)+1}, \mathcal{R}x_{2m(k)})\right), \delta, 0, 0, \delta, \delta) \leq 0.$$

From ($\mathfrak{T}_2'$), we have $\lim_{k\to\infty} d(\mathcal{R}x_{2n(k)+1}, \mathcal{R}x_{2m(k)}) \leq f(\delta)$, i.e., $\delta \leq f(\delta)$. This is a contradiction since $f(t) < t$ for $t > 0$.

Hence, we deduce that $\{\mathcal{R}x_n\}$ is a Cauchy sequence in $\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})$. Since $\mathcal{X}$ is $(\mathcal{G}, \mathcal{F}, \mathcal{R})$-orbitally complete at x_0, there exists some $z \in \mathcal{X}$ such that

$$\mathcal{R}x_n \to z \text{ as } n \to \infty. \tag{4.11}$$

We will prove that z is a common fixed point of the three mappings $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{R}$.

Suppose, to the contrary, that $d(z, \mathcal{R}z) > 0$. We have

$$\mathcal{G}x_{2n} = \mathcal{R}x_{2n+1} \to z \text{ as } n \to \infty \tag{4.12}$$

and

$$\mathcal{F}x_{2n+1} = \mathcal{R}x_{2n+2} \to z \text{ as } n \to \infty. \tag{4.13}$$

Suppose that (a) holds, i.e., $\mathcal{G}$ and $\mathcal{R}$ are compatible. Then, using condition (v),

$$\lim_{n\to\infty} \mathcal{G}\mathcal{R}x_{2n+2} = \lim_{n\to\infty} \mathcal{R}\mathcal{G}x_{2n+2} = \mathcal{R}z. \tag{4.14}$$

From (4.11) and the orbital continuity of $\mathcal{R}$, we have also

$$\mathcal{R}(\mathcal{R}x_n) \to \mathcal{R}z \text{ as } n \to \infty. \tag{4.15}$$

Now, using (iv), $x_{2n+1} \preceq \mathcal{F}x_{2n+1} = \mathcal{R}x_{2n+2}$ and since $\mathcal{R}$ is monotone, $\mathcal{R}x_{2n+1}$ and $\mathcal{R}\mathcal{R}x_{2n+2}$ are comparable. Thus, we can apply (4.1) to obtain

$$T(d(\mathcal{F}x_{2n+1}, \mathcal{G}\mathcal{R}x_{2n+2}), d(\mathcal{R}x_{2n+1}, \mathcal{R}\mathcal{R}x_{2n+2}), d(\mathcal{R}x_{2n+1}, \mathcal{F}x_{2n+1}), d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{G}\mathcal{R}x_{2n+2}), d(\mathcal{R}x_{2n+1}, \mathcal{G}\mathcal{R}x_{2n+2}), d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{F}x_{2n+1})) \leq 0 \quad (4.16)$$

Passing to the limit as $n \to \infty$ in (4.16), using (4.11)–(4.15), we obtain

$$T(d(z, \mathcal{R}z), d(z, \mathcal{R}z), 0, 0, d(z, \mathcal{R}z), d(\mathcal{R}z, z) \leq 0$$

which is a contradiction to $(\mathfrak{T}'_2)$. Thus $d(z, \mathcal{R}z) = 0$ and

$$\mathcal{R}z = z. \quad (4.17)$$

Now, $x_{2n+1} \preceq \mathcal{F}x_{2n+1}$ and $\mathcal{F}x_{2n+1} \to z$ as $n \to \infty$, so by the assumption we have $x_{2n+1} \preceq z$ and $\mathcal{R}x_{2n+1}$ and $\mathcal{R}z$ are comparable. Hence (4.1) gives

$$T(d(\mathcal{F}x_{2n+1}, \mathcal{G}z), d(\mathcal{R}x_{2n+1}, \mathcal{R}z), d(\mathcal{R}x_{2n+1}, \mathcal{F}x_{2n+1}), d(\mathcal{R}z, \mathcal{G}z), d(\mathcal{R}x_{2n+1}, \mathcal{G}z), d(\mathcal{R}z, \mathcal{F}x_{2n+1})) \leq 0.$$

Passing to the limit as $n \to \infty$ in the above inequality and using (4.17), it follows that

$$T(d(z, \mathcal{G}z), 0, 0, d(z, \mathcal{G}z), d(z, \mathcal{G}z), 0) \leq 0.$$

which is a contradiction to $(\mathfrak{T}'_3)$. Thus

$$\mathcal{G}z = z. \quad (4.18)$$

Similarly, $x_{2n} \preceq \mathcal{G}x_{2n}$ and $\mathcal{G}x_{2n} \to z$ as $n \to \infty$, implies that $x_{2n} \preceq z$, hence $\mathcal{R}x_{2n}$ and $\mathcal{R}z$ are comparable. From (4.1) we get

$$T(d(\mathcal{F}z, \mathcal{G}x_{2n}), d(\mathcal{R}z, \mathcal{R}x_{2n}), d(\mathcal{R}z, \mathcal{F}z), d(\mathcal{R}x_{2n}, \mathcal{G}x_{2n}), d(\mathcal{R}z, \mathcal{G}x_{2n}), d(\mathcal{R}x_{2n}, \mathcal{F}z)) \leq 0$$

Passing to the limit as $n \to \infty$, we have

$$T(d(\mathcal{F}z, z), 0, d(z, \mathcal{F}z), 0, 0, d(z, \mathcal{F}z)) \leq 0$$

which is a contradiction to $(\mathfrak{T}'_3)$. Thus

$$z = \mathcal{F}z. \quad (4.19)$$

Therefore, $\mathcal{G}z = \mathcal{F}z = \mathcal{R}z = z$, hence z is a common fixed point of $\mathcal{R}$, $\mathcal{F}$ and $\mathcal{G}$.

Similarly, the result follows when condition (b) holds.

Now, suppose that the set of common fixed points of $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})}$ is well ordered. We claim that there is a unique common fixed point of $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})}$. Assume to the contrary that $\mathcal{G}u = \mathcal{F}u = \mathcal{R}u = u$ and $\mathcal{G}v = \mathcal{F}v = \mathcal{R}v = v$ but $u \neq v$. By supposition, we can replace x by u and y by v in (4.1) to obtain

$$T(d(\mathcal{F}u, \mathcal{G}v), d(\mathcal{R}u, \mathcal{R}v), d(\mathcal{R}u, \mathcal{F}u), d(\mathcal{R}v, \mathcal{G}v), d(\mathcal{R}u, \mathcal{G}v), d(\mathcal{R}v, \mathcal{F}u)) \leq 0$$

or

$$T(d(u, v), d(u, v), 0, 0, d(u, v), d(v, u)) \leq 0$$

which is a contradiction to $(\mathfrak{T}'_2)$. Hence, $u = v$. The converse is trivial. □

We obtain the following corollaries of Theorem 5 which improve Theorem 4.5 [4] by considering orbital continuity of maps and orbitally complete space instead of continuity of maps on complete space.

Corollary 4 *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let $\mathcal{F}$ and $\mathcal{G}$ be self-maps on $\mathcal{X}$ satisfying*

$$T(d(\mathcal{F}x, \mathcal{G}y), d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{G}y), d(x, \mathcal{G}y), d(y, \mathcal{F}x)) \leq 0 \quad (4.20)$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F})}$ such that x and y are comparable, where $T \in \mathfrak{T}'$.

We assume the following hypotheses:

(i) *$(\mathcal{G}, \mathcal{F})$ is asymptotically regular at some point $x_0 \in \mathcal{X}$;*
(ii) *$\mathcal{X}$ is $(\mathcal{G}, \mathcal{F})$-orbitally complete at x_0;*
(iii) *$\mathcal{F}$ and $\mathcal{G}$ are weakly increasing;*
(iv) *$\mathcal{F}$ and $\mathcal{G}$ are dominating maps.*

Then $\mathcal{F}$ and $\mathcal{G}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{F}$ and $\mathcal{G}$ in $\overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F})}$ is well ordered if and only if it is a singleton.

Corollary 5 *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let $\mathcal{F}$ and $\mathcal{R}$ be self-maps on $\mathcal{X}$ satisfying*

$$T(d(\mathcal{F}x, \mathcal{F}y), d(\mathcal{R}x, \mathcal{R}y), d(\mathcal{R}x, \mathcal{F}x), d(\mathcal{R}y, \mathcal{F}y), d(\mathcal{R}x, \mathcal{F}y), d(\mathcal{R}y, \mathcal{F}x)) \leq 0$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{T}, \mathcal{T}, \mathcal{R})}$ such that $\mathcal{R}x$ and $\mathcal{R}y$ are comparable, where $T \in \mathfrak{T}'$.

We assume the following hypotheses:

(i) *$\mathcal{F}$ is asymptotically regular with respect to $\mathcal{R}$ at some point $x_0 \in \mathcal{X}$;*
(ii) *$\mathcal{X}$ is $(\mathcal{F}, \mathcal{R})$-orbitally complete at x_0;*
(iii) *$\mathcal{F}$ is weakly increasing with respect to $\mathcal{R}$;*
(iv) *$\mathcal{F}$ is a dominating map;*
(v) *$\mathcal{R}$ is monotone and orbitally continuous at x_0.*

Then $\mathcal{F}$ and $\mathcal{R}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{F}$ and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{F}, \mathcal{F}, \mathcal{R})}$ is well ordered if and only if it is a singleton.

Corollary 6 *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let $\mathcal{F}$ be a self-map on $\mathcal{X}$ satisfying for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{F})}$ such that x and y are comparable,*

$$T(d(\mathcal{F}x, \mathcal{F}y), d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{F}y), d(x, \mathcal{F}y), d(y, \mathcal{F}x)) \leq 0$$

where $T \in \mathfrak{T}'$.

We assume the following hypotheses:

(i) *$\mathcal{F}$ is asymptotically regular at some point x_0 of $\mathcal{X}$;*
(ii) *$\mathcal{X}$ is $\mathcal{F}$-orbitally complete at x_0;*
(iii) *$\mathcal{F}x \preceq \mathcal{F}(\mathcal{F}x)$ for all $x \in \mathcal{X}$;*
(iv) *$\mathcal{F}$ is a dominating map.*

Then $\mathcal{F}$ has a fixed point. Moreover, the set of fixed points of $\mathcal{F}$ in $\overline{\mathcal{O}(x_0; \mathcal{F})}$ is well ordered if and only if it is a singleton.

We illustrate Theorem 5 by an example which is obtained by modifying the one from [17].

Example 8 Let the set $\mathcal{X} = [0, +\infty)$ be equipped with the usual metric d and the order defined by

$$x \preceq y \iff x \geq y.$$

Consider the following self-mappings on $\mathcal{X}$:

$$\mathcal{R}x = 6x, \qquad \mathcal{F}x = \begin{cases} \frac{1}{2}x, & 0 \leq x \leq \frac{1}{2}, \\ x, & x > \frac{1}{2}, \end{cases} \qquad \mathcal{G}x = \begin{cases} \frac{1}{3}x, & 0 \leq x \leq \frac{1}{3}, \\ x, & x > \frac{1}{3}. \end{cases}$$

Take $x_0 = \frac{1}{2}$. Then it is easy to show that

$$\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R}) \subset \left\{ \frac{1}{2^k \cdot 3^l} : k, l \in \mathbb{N} \right\} \quad \text{and} \quad \overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})} = \mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R}) \cup \{0\}$$

and all the conditions (i)–(v) and (a)-(b) of Theorem 5 are fulfilled (condition (iii) on $\overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})}$). Take $\phi(t) = \frac{1}{6}t$ for $t > 0$. Then contractive condition (4.1) takes the form

$$\left| \frac{1}{2}x - \frac{1}{3}y \right| \leq \frac{1}{6} \max \left\{ |6x - 6y|, \frac{11}{2}x, \frac{17}{3}y, \frac{1}{2}\left[\left| 6x - \frac{1}{3}y \right| + \left| 6y - \frac{1}{2}x \right| \right] \right\},$$

for $x, y \in \overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})}$. Using substitution $y = tx$, $t \geq 0$, the last inequality reduces to

$$|3 - 2t| \leq \max\{6|1 - t|, \tfrac{11}{2}, \tfrac{17}{3}t, \tfrac{1}{2}[|6 - \tfrac{1}{3}t| + |6t - \tfrac{1}{2}|]\},$$

and can be checked by discussion on possible values for $t \geq 0$. If we suppose $T(t_1, \ldots, t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\})$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ right continuous and $\phi(0) = 0$, $\phi(t) < t$ for $t > 0$, then all the conditions of Theorem 5 are satisfied and $\mathcal{G}, \mathcal{F}, \mathcal{R}$ have a unique common fixed point in $\overline{\mathcal{O}(x_0; \mathcal{G}, \mathcal{F}, \mathcal{R})}$ (which is 0).

5 An application to integral equations

In this section, we apply the result given by Corollary 4 to study the existence and uniqueness of solutions to a class of nonlinear integral equations.

We consider the nonlinear integral equations

$$x(t) = \int_a^b K_1(t, s, x(s))\,ds + g(t), \; t \in [a, b], \tag{5.1}$$

and

$$x(t) = \int_a^b K_2(t, s, x(s))\,ds + g(t), \; t \in [a, b]. \tag{5.2}$$

The purpose of this section is to give an existence theorem for common solution of (5.1) and (5.2) using Corollary 4.

Let $\ll$ be a partial order relation on $\mathbb{R}$. Let $\mathcal{X} := C([a, b], \mathbb{R})$ with the usual maximum norm, i.e., $\| x\| = \max_{t\in[a,b]} |x(t)|$, for $x \in \mathcal{X}$. Consider on $\mathcal{X}$ the partial order defined by

$$x \preceq y \text{ if and only if } x(t) \ll y(t) \text{ for every } t \in [a, b].$$

Then $(\mathcal{X}, \preceq)$ is a partially ordered set. Also $(\mathcal{X}, \|\cdot\|)$ is a complete metric space. Moreover for every increasing sequence $\{x_n\}$ in $\mathcal{X}$ converging to $x^* \in \mathcal{X}$, we have $x_n(t) \ll x^*(t)$ for every $t \in [a, b]$. Also for every $x, y \in \mathcal{X}$ there exists $c(x, y) \in \mathcal{X}$ which is comparable with x and y [25].

Define $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{X}$, by

$$\mathcal{F}x(t) = \int_a^b K_1(t, s, x(s))\,ds + g(t), \quad x \in \mathcal{X}, \quad t \in [a, b],$$

and

$$\mathcal{G}x(t) = \int_a^b K_2(t, s, x(s))\,ds + g(t), \quad x \in \mathcal{X}, \quad t \in [a, b].$$

Theorem 6 *Consider the integral equations* (5.1) *and* (5.2)*. Assume*

(i) $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous,*
(ii) *for each* $t, s \in [a, b]$,

$$K_1(t,s,x(s)) \ll K_2\left(t,s,\int_a^b K_1(s,\tau,x(\tau))\,d\tau + g(s)\right)$$

and

$$K_2(t,s,x(s)) \ll K_1\left(t,s,\int_a^b K_2(s,\tau,x(\tau))\,d\tau + g(s)\right),$$

(iii) *there exist a continuous function* $p : [a,b]\times[a,b] \to \mathbb{R}_+$ *and a right continuous and non-decreasing function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\phi(0) = 0$ *and* $\phi(t) < t$ *for* $t > 0$*, such that*

$$\begin{aligned}&|K_1(t,s,x(s)) - K_2(t,s,y(s))|\\&\quad \le p(t,s)\phi\left(\max\left\{|x(s)-y(s)|, |x(s)-\mathcal{F}x(s)|, |y(s)-\mathcal{G}y(s)|,\right.\right.\\&\qquad \left.\left.\tfrac{1}{2}(|x(s)-\mathcal{G}y(s)| + |y(s)-\mathcal{F}x(s)|)\right\}\right)\end{aligned}$$

for each $t,s \in [a,b]$ *and comparable* $x,y \in \mathcal{X}$*,*

(iv) $\sup_{t\in[a,b]} \int_a^b p(t,s)\,ds \le 1$.

Then the integral equations (5.1) *and* (5.2) *have a unique common solution* x^* *in* $C([a,b],\mathbb{R})$.

Proof From (ii), we have, for all $t \in [a,b]$,

$$\begin{aligned}\mathcal{F}x(t) &= \int_a^b K_1(t,s,x(s))\,ds + g(t)\\&\ll \int_a^b K_2(t,s,\int_a^b K_1(s,\tau,x(\tau))\,d\tau + g(s))\,ds + g(t)\\&= \int_a^b K_2(t,s,\mathcal{F}x(s))\,ds + g(t)\\&= \mathcal{G}\mathcal{F}x(t)\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}x(t) &= \int_a^b K_2(t,s,x(s))\,ds + g(t)\\ &\ll \int_a^b K_1\Big(t,s,\int_a^b K_2(s,\tau,x(\tau))\,d\tau + g(s)\Big)\,ds + g(t)\\ &= \int_a^b K_1(t,s,\mathcal{G}x(s))\,ds + g(t)\\ &= \mathcal{F}\mathcal{G}x(t).\end{aligned}$$

Thus, we have $\mathcal{F}x \preceq \mathcal{G}\mathcal{F}x$ and $\mathcal{G}x \preceq \mathcal{F}\mathcal{G}x$ for all $x \in \mathcal{X}$. This shows that $\mathcal{F}$ and $\mathcal{G}$ are weakly increasing. Also for each comparable $x, y \in \mathcal{X}$, we have

$$\begin{aligned}&|\mathcal{F}x(t) - \mathcal{G}y(t)|\\ &= \left|\int_a^b K_1(t,s,x(s))\,ds - \int_a^b K_2(t,s,y(s))\,ds\right|\\ &\le \int_a^b |K_1(t,s,x(s)) - K_2(t,s,y(s))|\,ds\\ &\le \int_a^b p(t,s)\phi(\max\{|x(s)-y(s)|, |x(s)-\mathcal{F}x(s)|, |y(s)-\mathcal{F}y(s)|,\\ &\qquad \tfrac{1}{2}(|x(s)-\mathcal{F}y(s)| + |y(s)-\mathcal{F}x(s)|)\})\,ds\\ &\le \phi\left(\max\left\{\|x-y\|, \|x-\mathcal{F}x\|, \|y-\mathcal{F}y\|, \tfrac{1}{2}(\|x-\mathcal{F}y\|+\|y-\mathcal{F}x\|)\right\}\right)\\ &\quad\times \int_a^b p(t,s)\,ds\\ &\le \phi\left(\max\left\{\|x-y\|, \|x-\mathcal{F}x\|, \|y-\mathcal{F}y\|, \tfrac{1}{2}(\|x-\mathcal{F}y\|+\|y-\mathcal{F}x\|)\right\}\right),\end{aligned}$$

for every $t \in [a,b]$. Hence

$$\|\mathcal{F}x - \mathcal{G}y\| \le \phi\left(\max\left\{\|x-y\|, \|x-\mathcal{F}x\|, \|y-\mathcal{F}y\|, \tfrac{1}{2}(\|x-\mathcal{F}y\|+\|y-\mathcal{F}x\|)\right\}\right)$$

for each comparable $x, y \in \mathcal{X}$.

If we suppose $T(t_1,\ldots,t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, \frac{1}{2}(t_5+t_6)\})$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is right continuous and $\phi(0) = 0$, $\phi(t) < t$ for $t > 0$, then we get (4.20). Therefore all conditions of Corollary 4 are satisfied. This completes the proof of the theorem. □

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