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Convergence of modified S -iteration process for two asymptotically nonexpansive mappings in the intermediate sense in $CAT(0)$ spaces

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Abstract

The purpose of this paper is to study modified S -iteration process and investigate the existence and convergence theorems in the setting of $CAT(0)$ spaces for a class of mappings which is wider than that of asymptotically nonexpansive mappings. Our results generalize, unify and extend several comparable results in the existing literature.

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1 Introduction

A metric space X is a $CAT(0)$ space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $CAT(0)$ space. Other examples include pre-Hilbert spaces (see [1]), \mathbb{R} -trees (see [2]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [4]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [1].

Fixed point theory in $CAT(0)$ spaces has been first studied by Kirk (see [5, 6]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. It is worth mentioning that the results in $CAT(0)$ spaces can be applied to any $CAT(k)$ space with $k \leq 0$ since any $CAT(k)$ space is a $CAT(k')$ space for every $k' \geq k$ (see, e.g., [1]).

The Mann iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 1, \end{cases} \quad (1)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Further, the Ishikawa iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \quad (2)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0, 1)$. This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal, O'Regan and Sahu [7] introduced the S -iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)T x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \quad (3)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0, 1)$. Note that (3) is independent of (2) (and hence (1)). They showed that their process is independent of those of Mann and Ishikawa and converges faster than both of these (see [7, Proposition 3.1]).

In 1991, Schu [8] considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \end{cases} \quad (4)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

In 1994, Tan and Xu [9] studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1, \end{cases} \quad (5)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0, 1)$. This iteration process reduces to the modified Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

Recently, Agarwal, O'Regan and Sahu [7] introduced the modified S -iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1, \end{cases} \quad (6)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0, 1)$. Note that (6) is independent of (5) (and hence of (4)). Also (6) reduces to (3) when $T^n = T$ for all $n \geq 1$.

Very recently, Şahin and Başarir [10] modified iteration process (6) in a CAT(0) space as follows.

Let K be a nonempty closed convex subset of a complete CAT(0) space X , and let $T: K \rightarrow K$ be an asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{cases} \quad (7)$$

where and throughout the paper $\{\alpha_n\}, \{\beta_n\}$ are the sequences such that $0 \leq \alpha_n, \beta_n \leq 1$ for all $n \geq 1$. They studied modified S -iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space and established some strong convergence results under some suitable conditions which generalize some results of Khan and Abbas [11].

Inspired and motivated by [10] and some others, we modify iteration scheme (7) for two mappings in a CAT(0) space as follows.

Let K be a nonempty closed convex subset of a complete CAT(0) space X , and let $S, T: K \rightarrow K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F(S, T) = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)T^n x_n \oplus \alpha_n S^n y_n, \\ y_n = (1 - \beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{cases} \quad (8)$$

where and throughout the paper $\{\alpha_n\}, \{\beta_n\}$ are the sequences such that $0 \leq \alpha_n, \beta_n \leq 1$ for all $n \geq 1$.

In this paper, we study the newly defined modified S -iteration process (8) involving two asymptotically nonexpansive mappings in the intermediate sense and investigate the existence and convergence theorems for the above mentioned mappings and iteration scheme in the setting of CAT(0) spaces. Our results generalize, unify and extend several comparable results in the existing literature.

2 Preliminaries and lemmas

In order to prove the main results of this paper, we need the following definitions, concepts and lemmas.

Let (X, d) be a metric space and K be its subset. Let $T: K \rightarrow K$ be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$. We will also denote by $F(S, T)$ the set of common fixed points of S and T , that is, $F(S, T) = \{x \in K : Sx = Tx = x\}$.

The concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [12] in 1972. The iterative approximation problem for asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings was studied by many authors in a Banach space and a CAT(0) space (see, e.g., [9, 13–19]).

Definition 2.1 Let (X, d) be a metric space and K be its nonempty subset. Then $T: K \rightarrow K$ is said to be

- (1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;
- (2) asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ such that $d(T^n x, T^n y) \leq (1 + r_n)d(x, y)$ for all $x, y \in K$ and $n \geq 1$;

- (3) uniformly L -Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq Ld(x, y)$ for all $x, y \in K$ and $n \geq 1$;
- (4) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in K$.

In 1993, Bruck, Kuczumow and Reich [20] introduced a notion of asymptotically non-expansive mapping in the intermediate sense. A mapping $T: K \rightarrow K$ is said to be asymptotically nonexpansive in the intermediate sense provided that T is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in K} \{d(T^n x, T^n y) - d(x, y)\} \leq 0.$$

From the above definitions, it follows that an asymptotically nonexpansive mapping must be asymptotically nonexpansive mapping in the intermediate sense. But the converse does not hold as the following example.

Example 2.1 (See [21]) Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|\lambda| < 1$. For each $x \in K$, define

$$T(x) = \begin{cases} \lambda x \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is an asymptotically nonexpansive mapping in the intermediate sense but it is not an asymptotically nonexpansive mapping.

Remark 2.1 It is clear that the class of asymptotically nonexpansive mappings includes nonexpansive mappings, whereas the class of asymptotically nonexpansive mappings in the intermediate sense is larger than that of asymptotically nonexpansive mappings.

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y . We say that X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining x to y .

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in \mathbb{R}^2 such that $d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [1]).

CAT(0) space: A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom.

Let Δ be a geodesic triangle in X , and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ . Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{9}$$

Complete CAT(0) spaces are often called *Hadamard spaces* (see [21]). If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \quad (10)$$

Inequality (9) is the (CN) inequality of Bruhat and Tits [22]. The above inequality was extended in [23] as

$$d^2(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d^2(z, x) + (1 - \alpha)d^2(z, y) - \alpha(1 - \alpha)d^2(x, y) \quad (11)$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [1, p.163]). Moreover, if X is a CAT(0) metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1 - \alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(z, x) + (1 - \alpha)d(z, y) \quad (12)$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1 - \alpha)y : \alpha \in [0, 1]\}$.

A subset K of a CAT(0) space X is convex if, for any $x, y \in K$, we have $[x, y] \subset K$.

For the development of our main results, we recall some definitions, and some key results are listed in the form of lemmas.

Lemma 2.1 (See [17]) *Let X be a CAT(0) space.*

(i) *For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y). \quad (A)$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (A).

(ii) *For $x, y \in X$ and $t \in [0, 1]$, we have*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).$$

Let $\{x_n\}$ be a bounded sequence in a closed convex subset K of a CAT(0) space X . For $x \in X$, set

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point [24, Proposition 7].

We now recall the definition of Δ -convergence and weak convergence (\rightharpoonup) in a CAT(0) space.

Definition 2.2 (See [25]) A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

In this case we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ -limit of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In the Banach space it is known that every bounded sequence has a regular subsequence [26, Lemma 15.2].

Since in a CAT(0) space every regular sequence Δ -converges, we see that every bounded sequence in X has a Δ -convergent subsequence, also it is noticed that [25, p.3690].

Lemma 2.2 (See [27]) *Given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ -converges to x and given $y \in X$ with $y \neq x$, then*

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y).$$

In a Banach space, the above condition is known as the Opial property.

Now, recall the definition of weak convergence in a CAT(0) space.

Definition 2.3 (See [28]) Let K be a closed convex subset of a CAT(0) space X . A bounded sequence $\{x_n\}$ in K is said to converge weakly to $q \in K$ if and only if $\Phi(q) = \inf_{x \in K} \Phi(x)$, where $\Phi(x) = \limsup_{n \rightarrow \infty} d(x_n, x)$.

Note that $\{x_n\} \rightharpoonup q$ if and only if $A_K\{x_n\} = \{q\}$.

Nanjaras and Panyanak [29] established the following relation between Δ -convergence and weak convergence in a CAT(0) space.

Lemma 2.3 (See [29], Proposition 3.12) *Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X , and let K be a closed convex subset of X which contains $\{x_n\}$. Then*

- (i) $\Delta\text{-}\lim_{x_n} = x$ implies $x_n \rightharpoonup x$.
- (ii) The converse of (i) is true if $\{x_n\}$ is regular.

Lemma 2.4 (See [23, Lemma 2.8]) *If $\{x_n\}$ is a bounded sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.5 (See [30, Proposition 2.1]) *If K is a closed convex subset of a CAT(0) space X and if $\{x_n\}$ is a bounded sequence in K , then the asymptotic center of $\{x_n\}$ is in K .*

Lemma 2.6 (See [31]) *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n \rightarrow \infty} a_n$ exists.*

Lemma 2.7 (See [27, Theorem 3.1]) *Let X be a complete CAT(0) space, K be a nonempty closed convex subset of X . If $T: K \rightarrow K$ is an asymptotically nonexpansive mapping in the intermediate sense, then T has a fixed point.*

Lemma 2.8 (See [27, Theorem 3.2]) *Let X be a complete CAT(0) space, K be a nonempty closed convex subset of X . If $T: K \rightarrow K$ is an asymptotically nonexpansive mapping in the intermediate sense, then $\text{Fix}(T)$ is closed and convex.*

Lemma 2.9 (Demiclosed principle) (See [27, Proposition 3.3]) *Let K be a closed convex subset of a complete CAT(0) space X and $T: K \rightarrow K$ be an asymptotically nonexpansive mapping in the intermediate sense. If $\{x_n\}$ is a bounded sequence in K such that $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $\{x_n\} \rightharpoonup w$, then $Tw = w$.*

Lemma 2.10 (See [27, Corollary 3.4]) *Let K be a closed convex subset of a complete CAT(0) space X and $T: K \rightarrow K$ be an asymptotically nonexpansive mapping in the intermediate sense. If $\{x_n\}$ is a bounded sequence in K Δ -converging to x and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, then $x \in K$ and $Tx = x$.*

3 Main results

Now, we prove the following lemmas using modified S -iteration scheme (8) needed in the sequel.

Lemma 3.1 *Let K be a nonempty closed convex subset of a complete CAT(0) space X , and let $S, T: K \rightarrow K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F(S, T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (8). Put*

$$R_n = \max \left\{ 0, \sup_{x,y \in K, n \geq 1} (d(S^n x, S^n y) - d(x, y)), \sup_{x,y \in K, n \geq 1} (d(T^n x, T^n y) - d(x, y)) \right\} \quad (13)$$

such that $\sum_{n=1}^{\infty} R_n < \infty$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b]$ for some $a, b \in (0, 1)$. Then

- (i) $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in F(S, T)$.
- (ii) $\lim_{n \rightarrow \infty} d(x_n, F(S, T))$ exists.

Proof Let $p \in F(S, T)$. From (8), (13) and Lemma 2.1(ii), we have

$$\begin{aligned} d(y_n, p) &= d((1 - \beta_n)x_n \oplus \beta_n T^n x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(T^n x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n [d(x_n, p) + R_n] \\ &\leq d(x_n, p) + R_n. \end{aligned} \quad (14)$$

Again using (8), (13), (14) and Lemma 2.1(ii), we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1 - \alpha_n)T^n x_n \oplus \alpha_n S^n y_n, p) \\ &\leq (1 - \alpha_n)d(T^n x_n, p) + \alpha_n d(S^n y_n, p) \end{aligned}$$

$$\begin{aligned}
 &\leq (1 - \alpha_n)[d(x_n, p) + R_n] + \alpha_n[d(y_n, p) + R_n] \\
 &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(y_n, p) + R_n \\
 &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n [d(x_n, p) + R_n] + R_n \\
 &\leq d(x_n, p) + 2R_n.
 \end{aligned} \tag{15}$$

Taking infimum over all $p \in F(S, T)$, we have

$$d(x_{n+1}, p) \leq d(x_n, F(S, T)) + 2R_n. \tag{16}$$

Since by the hypothesis of the theorem $\sum_{n=1}^{\infty} R_n < \infty$, it follows from Lemma 2.6, (15) and (16) that $\lim_{n \rightarrow \infty} d(x_n, p)$ and $\lim_{n \rightarrow \infty} d(x_n, F(S, T))$ exist. \square

Lemma 3.2 *Let K be a nonempty closed convex subset of a complete CAT(0) space X , and let $S, T: K \rightarrow K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F(S, T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (8) and R_n is taken as in Lemma 3.1. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b]$ for some $a, b \in (0, 1)$. Then $\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$.*

Proof Using (8) and (11), we have

$$\begin{aligned}
 d^2(y_n, p) &= d^2((1 - \beta_n)x_n \oplus \beta_n T^n x_n, p) \\
 &\leq \beta_n d^2(T^n x_n, p) + (1 - \beta_n) d^2(x_n, p) \\
 &\quad - \beta_n(1 - \beta_n) d^2(T^n x_n, x_n) \\
 &\leq \beta_n [d(x_n, p) + R_n]^2 + (1 - \beta_n) d^2(x_n, p) \\
 &\quad - \beta_n(1 - \beta_n) d^2(T^n x_n, x_n) \\
 &\leq d^2(x_n, p) + A_n - \beta_n(1 - \beta_n) d^2(T^n x_n, x_n),
 \end{aligned} \tag{17}$$

where $A_n = R_n^2 + 2R_n d(x_n, p)$, since by the hypothesis $\sum_{n=1}^{\infty} R_n < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$. Again using (8), (11) and (17), we have

$$\begin{aligned}
 d^2(x_{n+1}, p) &= d^2((1 - \alpha_n)T^n x_n \oplus \alpha_n S^n y_n, p) \\
 &\leq \alpha_n d^2(S^n y_n, p) + (1 - \alpha_n) d^2(T^n x_n, p) \\
 &\quad - \alpha_n(1 - \alpha_n) d^2(S^n y_n, T^n x_n) \\
 &\leq \alpha_n [d(y_n, p) + R_n]^2 + (1 - \alpha_n) [d(x_n, p) + R_n]^2 \\
 &\quad - \alpha_n(1 - \alpha_n) d^2(S^n y_n, T^n x_n) \\
 &\leq \alpha_n [d^2(y_n, p) + B_n] + (1 - \alpha_n) [d^2(x_n, p) + A_n] \\
 &\quad - \alpha_n(1 - \alpha_n) d^2(S^n y_n, T^n x_n) \\
 &\leq \alpha_n d^2(y_n, p) + (1 - \alpha_n) d^2(x_n, p) + \alpha_n B_n + (1 - \alpha_n) A_n \\
 &\quad - \alpha_n(1 - \alpha_n) d^2(S^n y_n, T^n x_n)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha_n [d^2(x_n, p) + A_n - \beta_n(1 - \beta_n)d^2(T^n x_n, x_n)] \\
 &\quad + (1 - \alpha_n)d^2(x_n, p) + \alpha_n B_n + (1 - \alpha_n)A_n \\
 &\quad - \alpha_n(1 - \alpha_n)d^2(S^n y_n, T^n x_n) \\
 &\leq d^2(x_n, p) + A_n + \alpha_n B_n - \alpha_n \beta_n(1 - \beta_n)d^2(T^n x_n, x_n) \\
 &\quad - \alpha_n(1 - \alpha_n)d^2(S^n y_n, T^n x_n), \tag{18}
 \end{aligned}$$

where $A_n = R_n^2 + 2R_n d(x_n, p)$ and $B_n = R_n^2 + 2R_n d(y_n, p)$, since by the hypothesis $\sum_{n=1}^\infty R_n < \infty$, it follows that $\sum_{n=1}^\infty A_n < \infty$ and $\sum_{n=1}^\infty B_n < \infty$. This implies that

$$\begin{aligned}
 d^2(T^n x_n, x_n) &\leq \frac{1}{\alpha_n \beta_n(1 - \beta_n)} [d^2(x_n, p) - d^2(x_{n+1}, p)] \\
 &\quad + \frac{A_n}{\alpha_n \beta_n(1 - \beta_n)} + \frac{B_n}{\beta_n(1 - \beta_n)} \\
 &\leq \frac{1}{a^2(1 - b)} [d^2(x_n, p) - d^2(x_{n+1}, p)] \\
 &\quad + \frac{A_n}{a^2(1 - b)} + \frac{B_n}{a(1 - b)} \tag{19}
 \end{aligned}$$

and

$$\begin{aligned}
 d^2(S^n y_n, T^n x_n) &\leq \frac{1}{\alpha_n(1 - \alpha_n)} [d^2(x_n, p) - d^2(x_{n+1}, p)] \\
 &\quad + \frac{A_n}{\alpha_n(1 - \alpha_n)} + \frac{B_n}{(1 - \alpha_n)} \\
 &\leq \frac{1}{a(1 - b)} [d^2(x_n, p) - d^2(x_{n+1}, p)] \\
 &\quad + \frac{A_n}{a(1 - b)} + \frac{B_n}{(1 - b)}. \tag{20}
 \end{aligned}$$

Since $A_n \rightarrow 0, B_n \rightarrow 0$ as $n \rightarrow \infty$ and $d(x_n, p)$ is convergent, therefore on taking limit as $n \rightarrow \infty$ in (19) and (20), we get

$$\lim_{n \rightarrow \infty} d(T^n x_n, x_n) = 0 \tag{21}$$

and

$$\lim_{n \rightarrow \infty} d(S^n y_n, T^n x_n) = 0. \tag{22}$$

Now using (8) and (21), we get

$$d(x_n, y_n) \leq (1 - \beta_n)d(x_n, x_n) + \beta_n d(x_n, T^n x_n)$$

implies

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0. \tag{23}$$

Again using (21) and (22), we get

$$d(S^n y_n, x_n) \leq d(S^n y_n, T^n x_n) + d(T^n x_n, x_n)$$

implies

$$\lim_{n \rightarrow \infty} d(S^n y_n, x_n) = 0. \tag{24}$$

Further using (23) and (24), we get

$$\begin{aligned} d(S^n x_n, x_n) &\leq d(S^n x_n, S^n y_n) + d(S^n y_n, x_n) \\ &\leq d(x_n, y_n) + R_n + d(S^n y_n, x_n) \end{aligned}$$

implies

$$\lim_{n \rightarrow \infty} d(S^n x_n, x_n) = 0. \tag{25}$$

Now using (8), (21) and (24), we get

$$d(x_n, x_{n+1}) \leq (1 - \alpha_n)d(x_n, T^n x_n) + \alpha_n d(x_n, S^n y_n)$$

implies

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \tag{26}$$

Let $\mu_n = d(T^n x_n, x_n)$, by (21), we have $\mu_n \rightarrow 0$ as $n \rightarrow \infty$. Now, we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(x_{n+1}, x_n) + R_{n+1} + d(T^{n+1}x_n, Tx_n) \\ &\leq \mu_{n+1} + 2d(x_n, x_{n+1}) + R_{n+1} + d(T^{n+1}x_n, Tx_n) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned} \tag{27}$$

by (21), (26), $R_{n+1} \rightarrow 0$ and the uniform continuity of T . Similarly, we can prove that

$$\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0. \tag{28}$$

This completes the proof. □

Now we prove the Δ -convergence and strong convergence results.

Theorem 3.1 *Let K be a nonempty closed convex subset of a complete CAT(0) space X , and let $S, T: K \rightarrow K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F(S, T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (8) and R_n be taken as in Lemma 3.1. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b]$ for some $a, b \in (0, 1)$. Then the sequence $\{x_n\}$ is Δ -convergent to a point of $F(S, T)$.*

Proof We first show that $w_w(\{x_n\}) \subseteq F(S, T)$. Let $u \in w_w(\{x_n\})$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in K$. By Lemma 2.10, $v \in F(T)$ and $v \in F(S)$ and so $v \in F(S, T)$. By Lemma 3.1 $\lim_{n \rightarrow \infty} d(x_n, F(S, T))$ exists, so by Lemma 2.4, we have $u = v$, i.e., $w_w(\{x_n\}) \subseteq F(S, T)$.

To show that $\{x_n\}$ Δ -converges to a point in $F(S, T)$, it is sufficient to show that $w_w(\{x_n\})$ consists of exactly one point.

Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and let $A(\{x_n\}) = \{x\}$ for some $u \in w_w(\{x_n\}) \subseteq F(S, T)$ and $\{d(x_n, v)\}$ converge. By Lemma 2.4, we have $x = v \in F(S, T)$. Thus $w_w(\{x_n\}) = \{x\}$. This shows that $\{x_n\}$ is Δ -convergent to a point of $F(S, T)$. This completes the proof. \square

Theorem 3.2 *Let K be a nonempty closed convex subset of a complete CAT(0) space X , and let $S, T: K \rightarrow K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F(S, T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (8) and R_n is taken as in Lemma 3.1. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b]$ for some $a, b \in (0, 1)$. If $\liminf_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$ or $\limsup_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$, where $d(x, F(S, T)) = \inf_{p \in F(S, T)} d(x, p)$, then the sequence $\{x_n\}$ converges strongly to a point in $F(S, T)$.*

Proof From (16) of Lemma 3.1, we have

$$d(x_{n+1}, p) \leq d(x_n, F(S, T)) + 2R_n,$$

where $p \in F(S, T)$. Since by the hypothesis of the theorem $\sum_{n=1}^{\infty} R_n < \infty$, by Lemma 2.6 and $\liminf_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$ or $\limsup_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$ gives that

$$\lim_{n \rightarrow \infty} d(x_n, F(S, T)) = 0. \tag{29}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in K .

From (15), we have

$$\begin{aligned} d(x_{n+m}, p) &\leq d(x_{n+m-1}, p) + 2R_{n+m-1} \\ &\leq d(x_{n+m-2}, p) + 2[R_{n+m-2} + R_{n+m-1}] \\ &\leq \dots \\ &\leq d(x_n, p) + 2 \sum_{k=n}^{n+m-1} R_k \end{aligned} \tag{30}$$

for the natural numbers m, n and $p \in F(S, T)$. Since $\lim_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$, therefore for any $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, F(S, T)) < \varepsilon/8$ and $\sum_{k=n}^{n+m-1} R_k < \varepsilon/4$ for all $n \geq n_0$. So, we can find $p^* \in F(S, T)$ such that $d(x_{n_0}, p^*) < \varepsilon/4$. Hence, for all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq d(x_{n_0}, p^*) + \sum_{k=n_0}^{n+m-1} R_k \end{aligned}$$

$$\begin{aligned}
 &+ d(x_{n_0}, p^*) + \sum_{k=n_0}^{n+m-1} R_k \\
 &= 2d(x_{n_0}, p^*) + 2 \sum_{k=n_0}^{n+m-1} R_k \\
 &< 2\left(\frac{\varepsilon}{4}\right) + 2\left(\frac{\varepsilon}{4}\right) = \varepsilon.
 \end{aligned} \tag{31}$$

This proves that $\{x_n\}$ is a Cauchy sequence in K . Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n \rightarrow \infty} x_n = q$. Since K is closed, therefore $q \in K$. Next, we show that $q \in F(S, T)$. Since $\lim_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$, we get $d(q, F(S, T)) = 0$, closedness of $F(S, T)$ gives that $q \in F(S, T)$. This completes the proof. \square

Theorem 3.3 *Let K be a nonempty closed convex subset of a complete CAT(0) space X , and let $S, T: K \rightarrow K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F(S, T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (8) and R_n is taken as in Lemma 3.1. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b]$ for some $a, b \in (0, 1)$. If S and T satisfy the following conditions:*

- (i) $\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$;
- (ii) *If the sequence $\{z_n\}$ in K satisfies $\lim_{n \rightarrow \infty} d(z_n, Sz_n) = 0$ and $\lim_{n \rightarrow \infty} d(z_n, Tz_n) = 0$, then $\liminf_{n \rightarrow \infty} d(z_n, F(S, T)) = 0$ or $\limsup_{n \rightarrow \infty} d(z_n, F(S, T)) = 0$.*

Then the sequence $\{x_n\}$ converges strongly to a point of $F(S, T)$.

Proof It follows from the hypothesis that $\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. From (ii), $\liminf_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$ or $\limsup_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$. Therefore, the sequence $\{x_n\}$ must converge strongly to a point in $F(S, T)$ by Theorem 3.2. This completes the proof. \square

Theorem 3.4 *Let K be a nonempty closed convex subset of a complete CAT(0) space X , and let $S, T: K \rightarrow K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F(S, T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (8) and R_n is taken as in Lemma 3.1. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b]$ for some $a, b \in (0, 1)$. If either S or T is semi-compact, then the sequence $\{x_n\}$ converges strongly to a point of $F(S, T)$.*

Proof Suppose that T is semi-compact. By Lemma 3.2, we have $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow p \in K$. Now Lemma 3.2 guarantees that $\lim_{n_j \rightarrow \infty} d(x_{n_j}, Tx_{n_j}) = 0$ and so $d(p, Tp) = 0$. Similarly, we can show that $d(p, Sp) = 0$. Thus $p \in F(S, T)$. By (16), we have

$$d(x_{n+1}, p) \leq d(x_n, p) + 2R_n.$$

Since by the hypothesis $\sum_{n=1}^{\infty} R_n < \infty$, by Lemma 2.6, $\lim_{n \rightarrow \infty} d(x_n, p)$ exists and $x_{n_j} \rightarrow p \in F(S, T)$ gives that $x_n \rightarrow p \in F(S, T)$. This shows that $\{x_n\}$ converges strongly to a point of $F(S, T)$. This completes the proof. \square

We recall the following definition.

A mapping $T: K \rightarrow K$, where K is a subset of a normed linear space E , is said to satisfy Condition (A) [32] if there exists a nondecreasing function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T) \neq \emptyset\}$.

We modify this definition for two mappings.

Two mappings $S, T: K \rightarrow K$, where K is a subset of a normed linear space E , are said to satisfy Condition (B) if there exists a nondecreasing function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $a_1\|x - Sx\| + a_2\|x - Tx\| \geq f(d(x, F(S, T)))$ for all $x \in K$, where $d(x, F(S, T)) = \inf\{\|x - p\| : p \in F(S, T) \neq \emptyset\}$ and a_1 and a_2 are two nonnegative real numbers such that $a_1 + a_2 = 1$. It is to be noted that Condition (B) is weaker than the compactness of the domain K .

Remark 3.1 Condition (B) reduces to Condition (A) when $S = T$.

As an application of Theorem 3.2, we establish another strong convergence result employing Condition (B) as follows.

Theorem 3.5 *Let K be a nonempty closed convex subset of a complete CAT(0) space X , and let $S, T: K \rightarrow K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F(S, T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (8) and R_n is taken as in Lemma 3.1. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b]$ for some $a, b \in (0, 1)$. If S and T satisfy Condition (B), then the sequence $\{x_n\}$ converges strongly to a point of $F(S, T)$.*

Proof By Lemma 3.2, we know that

$$\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0. \tag{32}$$

From Condition (B) and (32), we get

$$\lim_{n \rightarrow \infty} f(d(x_n, F(S, T))) \leq a_1 \cdot \lim_{n \rightarrow \infty} d(x_n, Sx_n) + a_2 \cdot \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0,$$

i.e., $\lim_{n \rightarrow \infty} f(d(x_n, F(S, T))) = 0$. Since $f: [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(t) > 0$ for all $t \in (0, \infty)$, therefore we have

$$\lim_{n \rightarrow \infty} d(x_n, F(S, T)) = 0.$$

Now all the conditions of Theorem 3.2 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of $F(S, T)$. This completes the proof. \square

Remark 3.2 Our results generalize, unify and extend several comparable results in the existing literature.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally and significantly in this research work. All authors read and approved the final manuscript.

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