

Dark energy models from a parametrization of H : A comprehensive analysis and observational constraints

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The presented paper is a comprehensive analysis of two dark energy (DE) cosmological models wherein exact solutions of the Einstein field equations (EFEs) are obtained in a model-independent way (or by cosmological parametrization). A simple parametrization of Hubble parameter (H) is considered for the purpose in the flat Friedmann-Lemaitre-Robertson-Walker (FLRW) background. The parametrization of H covers some known models for some specific values of the model parameters involved. Two models are of special interest which show the behavior of cosmological phase transition from deceleration in the past to acceleration at late-times. The model parameters are constrained with 57 points of Hubble datasets together with the 580 points of Union 2.1 compilation supernovae datasets and baryonic acoustic oscillation (BAO) datasets. With the constrained values of the model parameters, both the models are analyzed and compared with the standard Λ CDM model and showing nice fit to the datasets. Two different candidates of DE is considered, cosmological constant Λ and a general scalar field ϕ and their dynamics are discussed on the geometrical base built. The geometrical and physical interpretations of the two models in consideration are discussed in details and the evolution of various cosmological parameters are shown graphically. The age of the Universe in both models are also calculated. Various cosmological parametrization schemes used in the past few decades to find exact solutions of the EFEs are also summarized at the end which can serve as a unified reference for the readers.

I. INTRODUCTION

Late-time cosmic acceleration is an essential constituent of precision cosmology at present. The idea of cosmic acceleration was first evidenced by the observations of high redshift supernova of type Ia [1, 2]. The idea of cosmic acceleration was later accepted quickly by the scientific community largely because of the independent observations with different methodology adopted by the supernova search teams lead by Perlmutter and Riess and also the CMB and the large scale structure data were providing substantial evidence for a cosmological constant, indirectly [3–5]. Later on some robust analysis and precise observations strengthen the idea of cosmic acceleration and a flat Universe consistent with $\Omega_\Lambda = 1 - \Omega_m = 0.75$ [6–9]. What causing the accelerating expansion is still a mystery and we are mostly in dark in this context. However, the theoretical predictions and advanced surveys in observational point of view indicating the presence of a weird form of energy in the Universe with high negative pressure with increasing density. The mysterious energy is named as *dark energy* [10] as it's nature, characteristics is speculative only without any laboratory tests. Also, the candidature of dark energy is a debatable topic at present cosmological studies. Moreover, the age crisis in the standard model need cosmic acceleration [11]. Although, the modification of gravity theory at infra red scale attracted attention to explain the late-time acceleration without invoking any extra source term [12, 13], but the theory of dark energy became quite popular [14, 15].

Very recently, gravitational wave detection and the picture of black hole shadow strengthen the Einstein's general theory of relativity and any modifications in the Einstein's theory (specifically to the geometry part) is not worth appreciated. However, Einstein himself was not convinced with the matter distribution in the Universe i.e. the right hand side of his field equations (representing matter sector) is considered to made up of low grade wood while the geometry part is of solid marble (representing the space-time). Any extra source term such as Einstein's cosmological constant (representing energy density of vacuum) could be added into the energy momentum tensor and serve as a candidate of dark energy. The most favoured candidate of dark energy is the well known cosmological constant Λ . Also, Λ CDM models have the best fit with many observational datasets. However, with this significant Λ , due to its non dynamical and the long standing fine tuning problem, researchers thought beyond it for a better candidate of dark energy. So, scalar field models were discussed after the cosmological constant for which Λ could also be generated from particle creation effect [16]. The dynamically evolving scalar field models have been utilized for the purpose

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are quintessence [17–19], K-essence [20–22], phantom [23] and tachyonic field [24–26]. The exotic fluid is also serve the purpose to explain the cosmic acceleration phenomenology that considered an equation of state producing large negative pressure e.g. Chaplygin gas equation of state [27], Polytropic gas equation of state [28] etc.

Soon after the formulation of EFEs, theoreticians worked on finding exact solutions. The first exact solution of the EFEs is the Schwarzschild exterior solution [29] wherein the perfect fluid equation of state was considered as a supplementary condition. Despite of the high non linearity of the EFEs, various exact solutions are obtained for static and spherically symmetric metrics. Einstein’s static solution [30], de-Sitter solution [31], Tolman’s solutions [32], Adler’s solutions [33], Buchdahl’s solution [34], Vaidya and Tikekar solution [35], Durgapal’s solutions [36], Knutsen’s solutions [37] and many more well-known solutions of EFEs are obtained which are summarized in the literature [38] and also discussed in [39]. Milne’s model [40], steady state model [41] are some different models proposed. All those phenomenological cosmological models explain the Universe theoretically very well. However, observations play a major role in modern cosmology which validate or discard a model. Now, numerical computations are also playing big role in modern cosmology and estimating cosmological parameters and also parameters of a model. In this study, an important discussion is given on a technique of finding exact solution of EFEs known as model independent way study. Moreover, two models are discussed and analyzed comprehensively with current trendz in theoretical cosmology.

The paper is organized as follows. The first section is an introduction to present cosmological scenario. The second section describes the Einstein’s field equations in general relativity in presence of dark energy. The third section is a motivation to the idea of model independent way or the cosmological parametrization study to obtain exact solutions to EFEs. A simple parametrization of Hubble parameter is considered in the light of cosmographical study in the fourth section. In the fifth section, observational constraints have been found for the model parameters involved in the functional form of H for the two models obtained. The sixth section is devoted to the geometrical dynamics and analysis of some important cosmological parameters describing the geometrical behavior of the Universe for both the models. In the seventh section, two candidates of dark energy is explored, cosmological constant and a general scalar field and the physical parameters such as energy density, density parameter, potential of scalar field and equation of state parameter are discussed for both the models under considerations. In the eighth section, the age of the Universe for the obtained models are calculated. The final section summarizes the physical insights of the results obtained and concluded. A brief summary of the various parametrization schemes used in the past few decades are given in the appendix.

II. EFES IN PRESENCE OF DARK ENERGY

The nature of dark energy and its candidature is a mystery and it is a matter of speculation to express it as a source term into the Einstein field equations. However, DE is speculated to be homogeneous permeating all over the space for which the energy momentum tensor can be represented in the form of a perfect fluid as

$$T_{ij}^{DE} = (\rho_{DE} + p_{DE})U_i U_j + p_{DE} g_{ij}, \quad (1)$$

with its equation of state in the form $p_{DE} = \omega_{DE}\rho_{DE}$, where ω_{DE} is the equation of state (EoS) parameter and is a function of time in general satisfying the inequality $\omega_{DE} < 0$. There is hot debate going on for a suitable value of ω_{DE} and the analysis of some observational data shows that its value lies in the range $-1.61 < \omega_{DE} < -0.78$ [7, 42, 43]. But recent analysis of datasets provide more tighter constraints on ω_{DE} [44, 45]. The different values of ω_{DE} in certain ranges gives rise to different candidates and can broadly be be classified as follows. For

- ▶ $\omega_{DE} = -1$, the case is for the cosmological constant;
- ▶ $\omega_{DE} = \text{constant} \neq -1$, the case is for cosmic strings, domain walls, etc.;
- ▶ $\omega_{DE} \neq \text{constant}$, the cases for scalar fields (quintessence, k-essence etc.), braneworlds, Dirac-Born-Infeld(DBI) action, Chaplygin gas etc.;
- ▶ $\omega_{DE} < -1$, the case is for phantom models.

For a broader list of dark energy models see [14, 15] (and refs. therein). There are interesting cases in each of them with some problems though. For example, Cosmological constant Λ is the most consistent model for dark energy explaining observations but is plagued with fine tuning problem. Similarly, the phantom models are interesting where the weak energy condition ($\rho > 0$, $\rho + p > 0$) is violated with the feature of finite time singularity [23].

In general relativity, dark energy can be introduced by supplementing the energy momentum tensor T_{ij}^{DE} into the Einstein field equations $G_{ij} = -8\pi G T_{ij}$ together with the matter source T_{ij}^M as a perfect fluid,

$$T_{ij}^{tot} = T_{ij}^m + T_{ij}^{de} = (\rho_{tot} + p_{tot})U_i U_j + p_{tot} g_{ij}, \quad (2)$$

with $\rho_{tot} = \sum \rho + \rho_{de}$ and $p_{tot} = \sum p + p_{de}$ denoting the total energy densities and total pressure due to all types of matter (baryonic matter, dark matter and radiation) and dark energy respectively. U_i is the usual four velocity vector and g_{ij} is the metric tensor. Now, the modified Einstein Field Equations for a flat FLRW metric

$$ds^2 = -dt^2 + a^2(t) [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)], \quad (3)$$

where $a(t)$ is the scale factor of the Universe, can be written as

$$M_{pl}^{-2} \rho_{tot} = 3 \left(\frac{\dot{a}}{a} \right)^2 = 3H^2, \quad (4)$$

$$M_{pl}^{-2} p_{tot} = -2 \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 = (2q - 1)H^2. \quad (5)$$

The conservation of energy-momentum (or from Eqs. (4) and (5) yields

$$\dot{\rho}_{tot} + 3(p_{tot} + \rho_{tot}) \frac{\dot{a}}{a} = 0. \quad (6)$$

The continuity equation (6) play significant role in the evolution as it deals with the matter and its interaction. In current cosmology, two kinds of dark energy models generally discussed; interacting models of dark energy (considering the interaction between cold dark matter and dark energy) [46–48] and non-interacting models of dark energy where all the matters allowed to evolve separately [49–52]. Up to date, there are no known interaction other than gravity between the matter and dark energy. The present study refers to non interacting models only. The system of equations are non linear ordinary differential equations and is difficult to find exact solutions. There are tremendous efforts to find both the exact and numerical solutions to EFEs in the past. In the next section, the solution techniques of the above system of equations will be discussed elaborately.

III. COSMOLOGICAL PARAMETRIZATION

The above system of equations (4), (5) and (6) possesses only two independent equations with five unknowns a , ρ , p , ρ_{de} , p_{de} (or ω_{de}). Due to the homogeneous distribution of matter in the Universe at large scale, it is customary to consider the barotropic equation of state $p = \omega\rho$, $\omega \in [0, 1]$. The equation of state describes different types of matter source in the Universe depending upon the discrete or dynamical values of equation of state parameter ω that includes baryonic matter ($\omega = 0$), dark matter ($\omega = 0$), radiation ($\omega = 1/3$), stiff matter ($\omega = 1$), etc. This additional equation provides the third constraint equation. Another constraint equation can be the consideration of equation of state of dark energy ($\omega_{de} = \text{constant}$ or a function of time t or function of scale factor a or function of redshift z) - best known as parametrization of dark energy equation of state. These four equations can explain the cosmological dynamics of the Universe where all the geometrical parameters (Hubble parameter H , deceleration parameter q , jerk parameter j , etc.) or physical parameters (densities ρ , ρ_{de} , pressures p , p_{de} , EoS parameter ω_{de} , density parameter Ω_i , etc.) are expressed as functions of either scale factor a or the redshift z ($= \frac{a_0}{a} - 1$, a_0 being the present value of scale factor generally normalized to $a_0 = 1$). But, there is still one more equation short to close the system for the complete determination of the system; the time evolution of scale factor a is yet to be determined. In literature, there are various schemes of parametrization of the scale factor and it's higher order derivatives (H , q , j etc.) providing the complete solution of the EFEs i.e. the explicit forms of cosmological parameters as a function of cosmic time t .

In fact, a critical analysis of the solution techniques of EFEs in general relativity theory or in modified theories has two aspects; one is the parametrization of geometrical parameters a , H , q , j giving the time dependent functions of all the cosmological parameters; another is the parametrization of the physical parameters ρ , p , ρ_{de} , p_{de} (or ω_{de}) giving the scale factor dependence or redshift dependence of all the cosmological parameters. See the appendix for a broad list of different schemes of parametrization of geometrical parameters and physical parameters and also some phenomenological ansatzs used in the past few decades to find the exact solutions of Einstein field equations. If we, observe carefully, we can say that the first kind of parametrization schemes (of geometrical parameters) are considered to find exact solutions that discusses the expansion dynamics of the Universe and provides the time evolution of the physical parameters ρ , p , ρ_{de} , p_{de} (or ω_{de}). This method is generally known as *model independent way* study of cosmological models or the cosmological parametrization [53–55]. The method do not affect the background theory anyway and provide solutions to the EFEs explicitly and also has an advantage of reconstructing the cosmic history of the Universe explaining some phenomena of the Universe. Also, this method provides the simplest way to resolve

some of the problems of standard model e.g. the initial singularity problem, cosmological constant problem, etc. and also explain the late-time acceleration conundrum, theoretically. While the second kind of parametrization (of physical parameters) are generally considered to discuss all the physical aspects (thermodynamics, structure formation, nucleosynthesis etc.) of the Universe. However, both the schemes of parametrization are adhoc choices or some phenomenological ansatzs (e.g. Λ -varying cosmologies). All parametrization schemes (see appendix-1) contain some arbitrary constants, referred to as *model parameters* which are constrained through any observational datasets.

The purpose here is to obtain an exact solution of the Einstein field equations in standard general relativity theory with a simple parametrization of the Hubble parameter H and discuss the reconstructed cosmic evolution.

IV. PARAMETRIZATION OF H & THE MODELS

The cosmographic analysis provide clues to study the evolution of the observable Universe in a model independent way in terms of the kinematic variables [56]. Moreover, analysis of cosmographic parameters helps in studying the dark energy without any assumption of any particular cosmological model except only the cosmological principle. In the standard approximation the scale factor can be expanded in Taylors series around the present time t_0 (which is the current age of the Universe also) and is the simple strategy adopted in cosmographical analysis. Here and afterwards a suffix 0 denotes the value of the parameter at present time t_0 . The Taylor's series expansion can be written as:

$$a^{(n)} = 1 + H_0(t - t_0) - \frac{1}{2!}q_0H_0^2(t - t_0)^2 + \frac{1}{3!}j_0H_0^3(t - t_0)^3 + \frac{1}{4!}s_0H_0^4(t - t_0)^4 + \frac{1}{5!}l_0H_0^5(t - t_0)^5 + \dots \quad (7)$$

where $H(t) = \frac{1}{a} \frac{da}{dt}$ is the Hubble parameter measuring velocity, $q(t) = -\frac{1}{a} \frac{d^2a}{dt^2} \left[\frac{1}{a} \frac{da}{dt} \right]^{-2}$ is the deceleration parameter measuring acceleration, $j(t) = \frac{1}{a} \frac{d^3a}{dt^3} \left[\frac{1}{a} \frac{da}{dt} \right]^{-3}$ jerk parameter measuring jerk, $s(t) = \frac{1}{a} \frac{d^4a}{dt^4} \left[\frac{1}{a} \frac{da}{dt} \right]^{-4}$ is the snap parameter and $l(t) = \frac{1}{a} \frac{d^5a}{dt^5} \left[\frac{1}{a} \frac{da}{dt} \right]^{-5}$ is the lerk parameter. All of these parameters play significant roles in the cosmographic analysis of the Universe (specifically the H , q and j) and distinguish various dark energy models.

Motivated by the above discussion, in this paper, a simple parametrization of the Hubble parameter (H) is considered as an explicit function of cosmic time ' t ' in the form [55]

$$H(t) = \frac{k_2 t^m}{(t^n + k_1)^p} \quad (8)$$

where $k_1, k_2 \neq 0, m, n, p$ are real constants (or model parameters). k_1 and k_2 both have the dimensions of time. Some specific values of the parameters m, n, p suggest some distinguished models which are elaborated by Pacif et al [55]. It is easy to see that, the single parametrization (8) generalizes several known models e.g. Λ CDM model, power law model, hybrid expansion model, bouncing model, linearly varying deceleration parameter model and some more. Out of the twelve models deduced for some integral or non integral values of m, n, p in the functional form of HP in (8), two models (with $m = -1, p = 1, n = 1$ and with $m = -1, p = 1, n = 2$) show the possibility of describing the phenomena of cosmological phase transition for negative k_1 & k_2 and is described as in the following Table-1.

Table-1: The models			
Models	$H(t)$	$a(t)$	$q(t)$
M1	$\frac{k_2}{t(k_1 - t)}$	$\beta \left(\frac{t}{k_1 - t} \right)^{\frac{k_2}{k_1}}$	$-1 + \frac{k_1}{k_2} - \frac{2}{k_2} t$
M2	$\frac{k_2}{t(k_1 - t^2)}$	$\beta \left(\frac{t^2}{k_1 - t^2} \right)^{\frac{k_2}{2k_1}}$	$-1 + \frac{k_1}{k_2} - \frac{3}{k_2} t^2$

where, β is an integrating constant which also play an important role in the evolution. Pacif et al. obtained solutions for these two models in a scalar field background and also found the observational constraints on model parameters with 28 points of $H(z)$ datasets. The present paper is an extension of the same study for these two models M1 and M2 wherein much deeper analysis have been done.

One can see, for both the models M1 and M2, the Hubble parameter and scale factor both diverge in finite time and show a big rip singularity in near future at $t = t_s = k_1$ for model M1 and at $t = t_s = \sqrt{k_1}$ for model M2. The phase transition occurs at time $t_{tr} = \frac{k_1 - k_2}{2}$ for model M1 and at time $t_{tr} = \sqrt{\frac{k_1 - k_2}{3}}$ for model M2 and suggest that k_1 must be greater than k_2 . With some suitable choice of model parameters k_1, k_2 and β , rough sketches for the

time-evolution of scale factor (SF) and the Hubble parameter (HP) are made and are shown graphically in the figures FIG. 1 and FIG. 2 respectively showing that $a(t)$ diverges in finite time and the $H(t)$ becomes asymptotic showing big rip in near future.

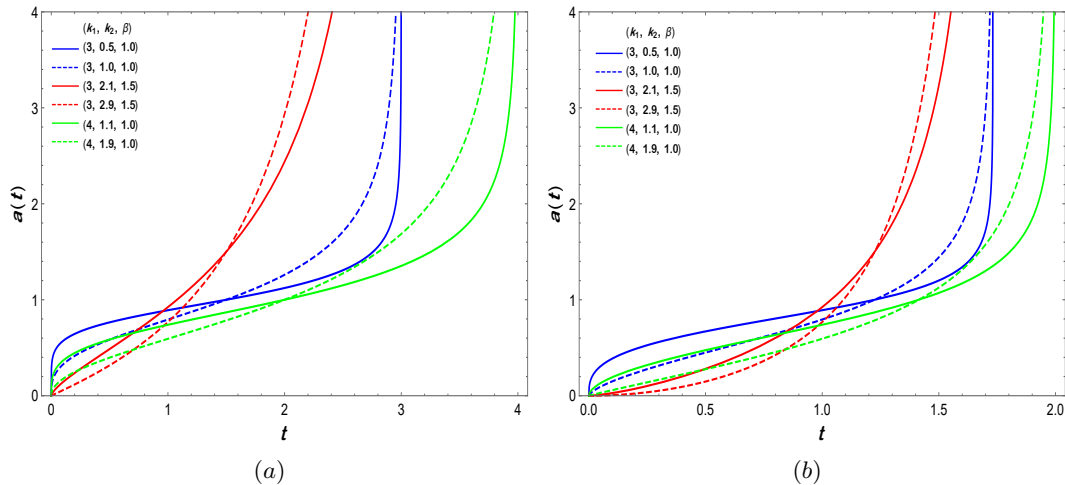


FIG. 1: Figures (a) and (b) respectively show rough sketches of the evolution of the scale factor w.r.t. cosmic time ‘ t ’ for both models M1 and M2 with some arbitrary values of the model parameters k_1, k_2, β .

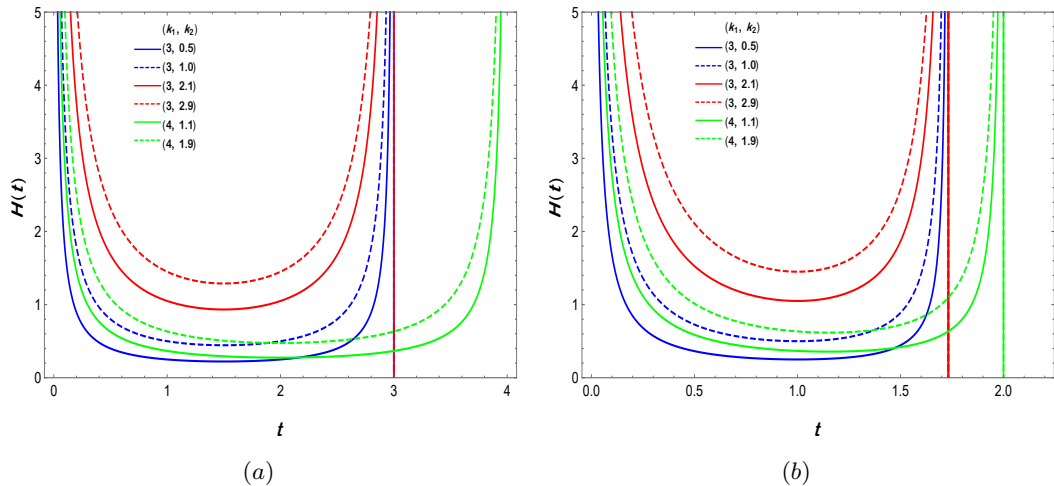


FIG. 2: Figures (a) and (b) respectively show rough sketches of the evolution of the Hubble parameter w.r.t. cosmic time ‘ t ’ for both models M1 and M2 with some arbitrary values of the model parameters k_1, k_2, β .

In order to check the consistency of the theoretical models obtained here with the observations, some available datasets are used in the next section. The model parameters are constrained through these datasets.

V. OBSERVATIONAL CONSTRAINTS

Three datasets are considered here for our analysis namely Hubble datasets (H_z), Type Ia supernovae datasets (SN) and Baryon Acoustic Oscillations datasets (BAO). The detailed datasets and the method used are explained below.

In the study of late-time Universe and the observational studies, it is convenient to express all the cosmological parameters as functions of redshift z . As the cosmological parameters here are functions of cosmic time t , the time-

redshift relationship must be established. The t - z relations are obtained as:

$$t(z) = k_1 \left[1 + \{\beta(1+z)\}^{\frac{k_1}{k_2}} \right]^{-1}, \quad (9)$$

for model M1 and

$$t(z) = \sqrt{k_1} \left[1 + \{\beta(1+z)\}^{2\frac{k_1}{k_2}} \right]^{-\frac{1}{2}} \quad (10)$$

for model M2. The above expressions (9) and (10) contain three parameters β , k_1 and k_2 but actually two model parameters are sufficient to describe these models by taking $\frac{k_1}{k_2} = \alpha$ which is also beneficial for further analysis and numerical computations for which the expressions for the Hubble parameter for both the models M1 and M2 are written in terms of redshift z as follows:

$$H(z) = H_0 (1 + \beta^\alpha)^{-2} (1+z)^{-\alpha} [1 + \{\beta(1+z)\}^\alpha]^2, \quad (11)$$

for model M1 and

$$H(z) = H_0 (1 + \beta^{2\alpha})^{-\frac{3}{2}} (1+z)^{-2\alpha} [1 + \{\beta(1+z)\}^{2\alpha}]^{\frac{3}{2}} \quad (12)$$

for model M2. The different datasets are described below.

A. $H(z)$ datasets

It is well known that the Hubble parameter ($H = \frac{\dot{a}}{a}$) directly probes the expansion history of the Universe where \dot{a} is the rate of change of the scale factor a of the Universe. Hubble parameter is also related to the differential redshift as, $H(z) = -\frac{1}{1+z} \frac{dz}{dt}$, where dz is obtained from the spectroscopic surveys and so a measurement of dt provides the Hubble parameter which will be independent of the model. In fact, two methods are generally used to measure the Hubble parameter values $H(z)$ at certain redshift and are extraction of $H(z)$ from line-of-sight BAO data and differential age method [57]-[75] estimating $H(z)$. Here, in this paper, an updated list of 57 data points are used as listed in Table-2 out of which 31 data points measured with DA method and 26 data points are obtained with BAO and other methods in the redshift range $0.07 \leq z \leq 2.42$ [76]. Moreover, the value of H_0 is taken as prior for our analysis as $H_0 = 67.8 \text{ Km/s/Mpc}$ [77]. The chi square function to determine the mean values of the model parameters α & β (which is equivalent to the maximum likelihood analysis) is,

$$\chi_H^2(\alpha, \beta) = \sum_{i=1}^{28} \frac{[H_{th}(z_i, \alpha, \beta) - H_{obs}(z_i)]^2}{\sigma_{H(z_i)}^2}, \quad (13)$$

where, H_{th} is the theoretical, H_{obs} is the observed value and $\sigma_{H(z_i)}$ is the standard error in the observed value of the Hubble parameter H . The 57 points of Hubble parameter values $H(z)$ with errors σ_H from differential age (31 points) method and BAO and other (26 points) methods are tabulated in Table-2 with references.

31 points from DA method								26 points from BAO & other method							
z	$H(z)$	σ_H	Ref.	z	$H(z)$	σ_H	Ref.	z	$H(z)$	σ_H	Ref.	z	$H(z)$	σ_H	Ref.
0.070	69	19.6	[57]	0.4783	80	99	[61]	0.24	79.69	2.99	[64]	0.52	94.35	2.64	[66]
0.90	69	12	[58]	0.480	97	62	[57]	0.30	81.7	6.22	[65]	0.56	93.34	2.3	[66]
0.120	68.6	26.2	[57]	0.593	104	13	[59]	0.31	78.18	4.74	[66]	0.57	87.6	7.8	[70]
0.170	83	8	[58]	0.6797	92	8	[59]	0.34	83.8	3.66	[64]	0.57	96.8	3.4	[71]
0.1791	75	4	[59]	0.7812	105	12	[59]	0.35	82.7	9.1	[67]	0.59	98.48	3.18	[66]
0.1993	75	5	[59]	0.8754	125	17	[59]	0.36	79.94	3.38	[66]	0.60	87.9	6.1	[69]
0.200	72.9	29.6	[60]	0.880	90	40	[57]	0.38	81.5	1.9	[68]	0.61	97.3	2.1	[68]
0.270	77	14	[58]	0.900	117	23	[58]	0.40	82.04	2.03	[66]	0.64	98.82	2.98	[66]
0.280	88.8	36.6	[60]	1.037	154	20	[59]	0.43	86.45	3.97	[64]	0.73	97.3	7.0	[69]
0.3519	83	14	[59]	1.300	168	17	[58]	0.44	82.6	7.8	[69]	2.30	224	8.6	[72]
0.3802	83	13.5	[61]	1.363	160	33.6	[63]	0.44	84.81	1.83	[66]	2.33	224	8	[73]
0.400	95	17	[58]	1.430	177	18	[58]	0.48	87.79	2.03	[66]	2.34	222	8.5	[74]
0.4004	77	10.2	[61]	1.530	140	14	[58]	0.51	90.4	1.9	[68]	2.36	226	9.3	[75]
0.4247	87.1	11.2	[61]	1.750	202	40	[58]								
0.4497	92.8	12.9	[61]	1.965	186.5	50.4	[63]								
0.470	89	34	[62]												

B. SN Ia datasets

The first indication for the accelerating expansion of the Universe is due to observations of supernovae of type *Ia*. Since then, several new SN *Ia* datasets have been published. In this analysis, the Union 2.1 compilation supernovae datasets is considered containing 580 points from [78] which provides the estimated values of the distance moduli $\mu_i = \mu_i^{obs}$ for a particular redshift z_i in the interval $0 < z_i \leq 1.41$. The model parameters of the models are to be fitted with, comparing the observed μ_i^{obs} value to the theoretical μ_i^{th} value of the distance moduli which are the logarithms $\mu_i^{th} = \mu(D_L) = m - M = 5 \log_{10}(D_L) + \mu_0$, where m and M are the apparent and absolute magnitudes and $\mu_0 = 5 \log(H_0^{-1}/Mpc) + 25$ is the nuisance parameter that has been marginalized. The luminosity distance is defined by,

$$D_L(z) = \frac{c(1+z)}{H_0} S_k \left(H_0 \int_0^z \frac{1}{H(z^*)} dz^* \right),$$

$$\text{where } S_k(x) = \begin{cases} \sinh(x\sqrt{|\Omega_k|})/\Omega_k, & \Omega_k > 0 \\ x, & \Omega_k = 0 \\ \sin x\sqrt{|\Omega_k|}/|\Omega_k|, & \Omega_k < 0 \end{cases}.$$

Here, $\Omega_k = 0$ (flat space-time). For our cosmological models M1 and M2 with theoretical value $H(z)$ which are depending on the model parameters α & β , the distance $D_L(z)$ is calculated and the corresponding chi square function measuring differences between the SN *Ia* observational data and values predicted by the models is given by,

$$\chi_{SN}^2(\mu_0, \alpha, \beta) = \sum_{i=1}^{580} \frac{[\mu^{th}(\mu_0, z_i, \alpha, \beta) - \mu^{obs}(z_i)]^2}{\sigma_{\mu(z_i)}^2}, \quad (14)$$

$\sigma_{\mu(z_i)}^2$ is the standard error in the observed value. Following [79] after marginalizing μ_0 , the chi square function is written as,

$$\chi_{SN}^2(\alpha, \beta) = A(\alpha, \beta) - [B(\alpha, \beta)]^2 / C(\alpha, \beta)$$

where

$$A(\alpha, \beta) = \sum_{i=1}^{580} \frac{[\mu^{th}(\mu_0=0, z_i, \alpha, \beta) - \mu^{obs}(z_i)]^2}{\sigma_{\mu(z_i)}^2},$$

$$B(\alpha, \beta) = \sum_{i=1}^{580} \frac{[\mu^{th}(\mu_0=0, z_i, \alpha, \beta) - \mu^{obs}(z_i)]}{\sigma_{\mu(z_i)}^2},$$

$$C(\alpha, \beta) = \sum_{i=1}^{580} \frac{1}{\sigma_{\mu(z_i)}^2}.$$

C. BAO datasets

Baryonic acoustic oscillations is an analysis dealing with the early Universe. It is known that the early Universe filled with baryons, photons and dark matter. Moreover, baryons and photons together act as single fluid (coupled tightly through the Thompson scattering) and can not collapse under gravity rather oscillate due to the large pressure of photons. These oscillations are termed a Baryonic acoustic oscillations (BAO). The characteristic scale of BAO is governed by the sound horizon r_s at the photon decoupling epoch z_* is given as,

$$r_s(z_*) = \frac{c}{\sqrt{3}} \int_0^{\frac{1}{1+z_*}} \frac{da}{a^2 H(a) \sqrt{1 + (3\Omega_{0b}/4\Omega_{0\gamma})a}},$$

where Ω_{0b} stands for the baryon density and $\Omega_{0\gamma}$ stands for the photon density at present time.

The BAO sound horizon scale is also used to derive the angular diameter distance D_A and the Hubble expansion rate H as a function of z . If $\Delta\theta$ be the measured angular separation of the BAO feature in the 2 point correlation function of the galaxy distribution on the sky and the Δz be the measured redshift separation of the BAO feature in the 2 point correlation function along the line of sight then,

$$\Delta\theta = \frac{r_s}{d_A(z)} \text{ where } d_A(z) = \int_0^z \frac{dz'}{H(z')} \text{ and } \Delta z = H(z)r_s.$$

In this work, BAO datasets of $d_A(z_*)/D_V(z_{BAO})$ from the references [80–85] is considered where the photon decoupling redshift is $z_* \approx 1091$ and $d_A(z)$ is the co-moving angular diameter distance and $D_V(z) = (d_A(z)^2 z/H(z))^{1/3}$ is the dilation scale. The data used for this analysis is given in the Table-3

z_{BAO}	0.106	0.2	0.35	0.44	0.6	0.73
$\frac{d_A(z_*)}{D_V(z_{BAO})}$	30.95 ± 1.46	17.55 ± 0.60	10.11 ± 0.37	8.44 ± 0.67	6.69 ± 0.33	5.45 ± 0.31

The chi square function for BAO is given by [85]

$$\chi_{BAO}^2 = X^T C^{-1} X, \quad (15)$$

where

$$X = \begin{pmatrix} \frac{d_A(z_*)}{D_V(0.106)} - 30.95 \\ \frac{d_A(z_*)}{D_V(0.2)} - 17.55 \\ \frac{d_A(z_*)}{D_V(0.35)} - 10.11 \\ \frac{d_A(z_*)}{D_V(0.44)} - 8.44 \\ \frac{d_A(z_*)}{D_V(0.6)} - 6.69 \\ \frac{d_A(z_*)}{D_V(0.73)} - 5.45 \end{pmatrix},$$

and C^{-1} is the inverse covariance matrix defined in [85].

$$C^{-1} = \begin{pmatrix} 0.48435 & -0.101383 & -0.164945 & -0.0305703 & -0.097874 & -0.106738 \\ -0.101383 & 3.2882 & -2.45497 & -0.0787898 & -0.252254 & -0.2751 \\ -0.164945 & -2.454987 & 9.55916 & -0.128187 & -0.410404 & -0.447574 \\ -0.0305703 & -0.0787898 & -0.128187 & 2.78728 & -2.75632 & 1.16437 \\ -0.097874 & -0.252254 & -0.410404 & -2.75632 & 14.9245 & -7.32441 \\ -0.106738 & -0.2751 & -0.447574 & 1.16437 & -7.32441 & 14.5022 \end{pmatrix}$$

With the above samples of Hubble (H_z), supernovae of type Ia (SN) and baryon acoustic oscillations (BAO) datasets, the chi square functions (13), (14) and (15) are minimized to get the average values of the model parameters α & β . The maximum likelihood contours for the model parameters α & β are shown in the following figures FIG. 3, FIG. 4, FIG. 5, and FIG. 6 for independent H_z datasets and combined $H_z + SN$, $SN + BAO$ and $H_z + SN + BAO$ datasets respectively with 1- σ , 2- σ and 3- σ error contours in the α - β plane.

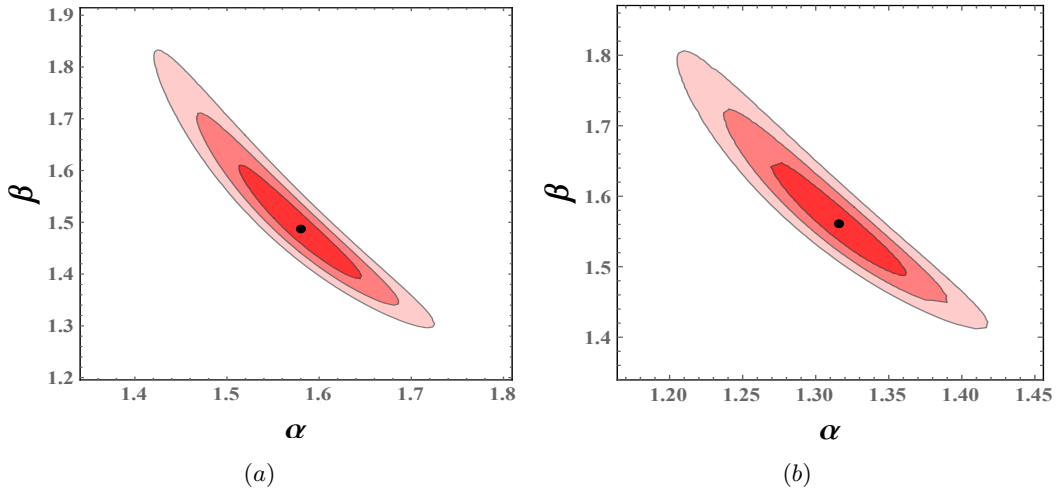


FIG. 3: Figures (a) and (b) are contour plots for Hubble datasets (H_z) for models M1 and M2 respectively.

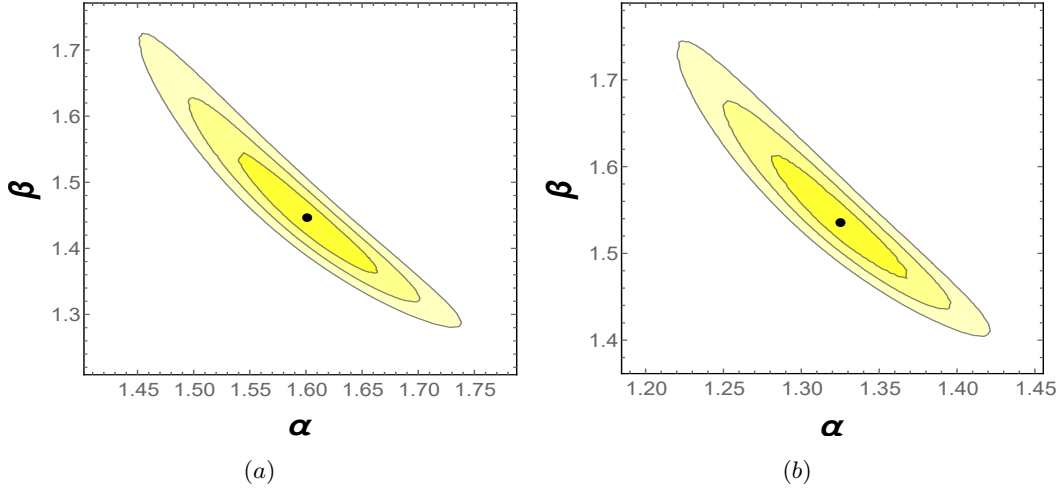


FIG. 4: Figures (a) and (b) are contour plots for combined $H_z + SN$ datasets for models M1 and M2 respectively.

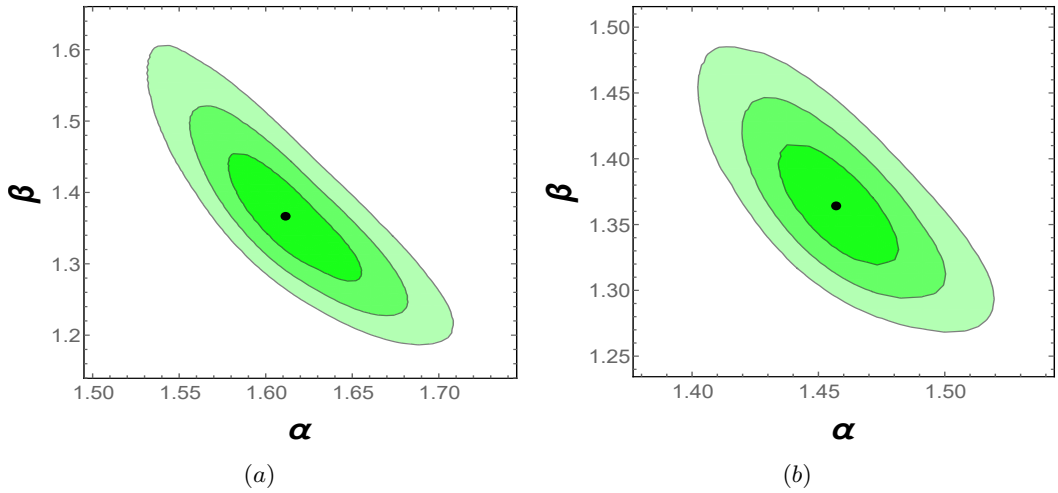


FIG. 5: Figures (a) and (b) are contour plots for combined $SN + BAO$ datasets for models M1 and M2 respectively.

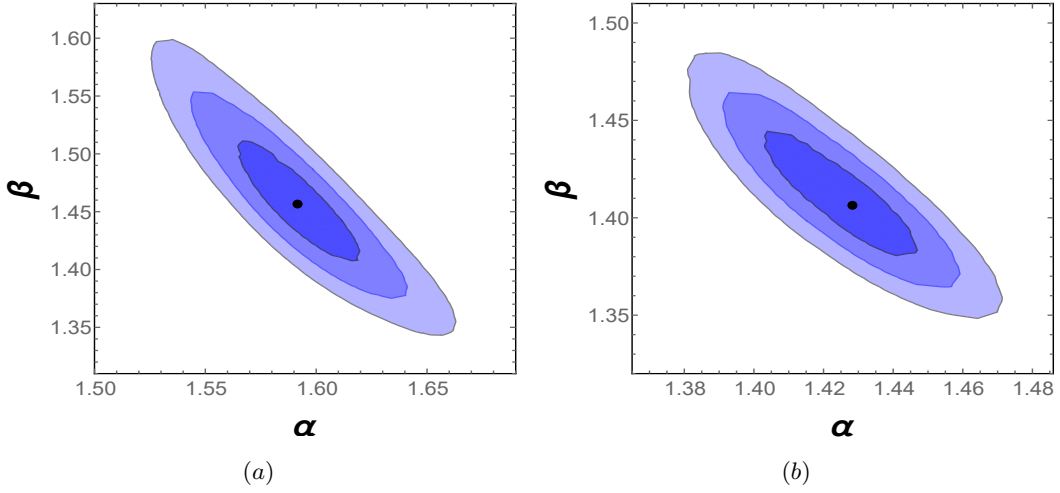


FIG. 6: Figures (a) and (b) are contour plots for combined $H_z + SN + BAO$ datasets for models M1 and M2 respectively.

The average mean values (constrained values) of the model parameters and the minimum chi square values are tabulated in Table-4 for independent H_z datasets and combined $H_z + SN$, $SN + BAO$ and $H_z + SN + BAO$ datasets.

Datasets	Models	α	β	χ^2_{\min}	χ^2/dof
$H(z)$	M1	1.58064	1.48729	31.329529	0.56962
	M2	1.31611	1.56124	29.972660	0.54495
$H(z) + SN$	M1	1.60094	1.44572	596.49325	0.93935
	M2	1.32551	1.53587	595.02853	0.93705
$SN + BAO$	M1	1.61116	1.36647	564.45777	0.96653
	M2	1.45677	1.36451	566.44641	0.96994
$H(z) + SN + BAO$	M1	1.59173	1.45678	599.07805	0.93459
	M2	1.42829	1.40637	614.40132	0.95850

The error bar plots of the 57 points of H_z and 580 points of Union 2.1 compilation datasets are plotted and shown in the following figures FIG. 7 and FIG. 8, using the constrained values of the model parameters as in Table-4 for both the models M1 and M2.

VI. GEOMETRICAL DYNAMICS OF THE MODELS

A. Deceleration parameter & Phase transition

The expressions for the deceleration parameter can be written in terms of redshift z as:

$$q(z) = -1 + \alpha - 2\alpha [1 + \{\beta(1+z)\}^\alpha]^{-1} \quad (16)$$

for model M1 and

$$q(z) = -1 + \alpha - 3\alpha [1 + \{\beta(1+z)\}^{2\alpha}]^{-1} \quad (17)$$

for model M2. The behavior of q is shown in the following FIG. 9 and the important values assumed by the deceleration parameter q in the course of evolution are tabulated in the following Table-5 for different sets of α & β as obtained.

The initial value of the deceleration parameter (q_i as $z \rightarrow \infty$), the present value of the deceleration parameter (q_0 as $z \rightarrow 0$) and the far future value of the deceleration parameter (q_f as $z \rightarrow -1$) are calculated together with the

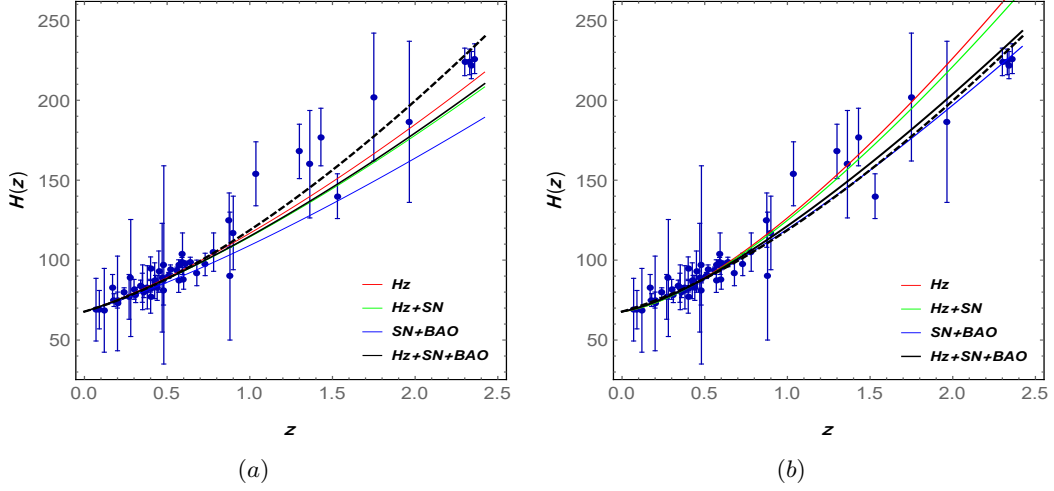


FIG. 7: Figures (a) and (b) are the error bar plots for 57 data points from Hubble datasets together with the models M1 and M2 shown in solid red lines respectively. The dashed lines in both the figures are Λ CDM model shown for comparison.

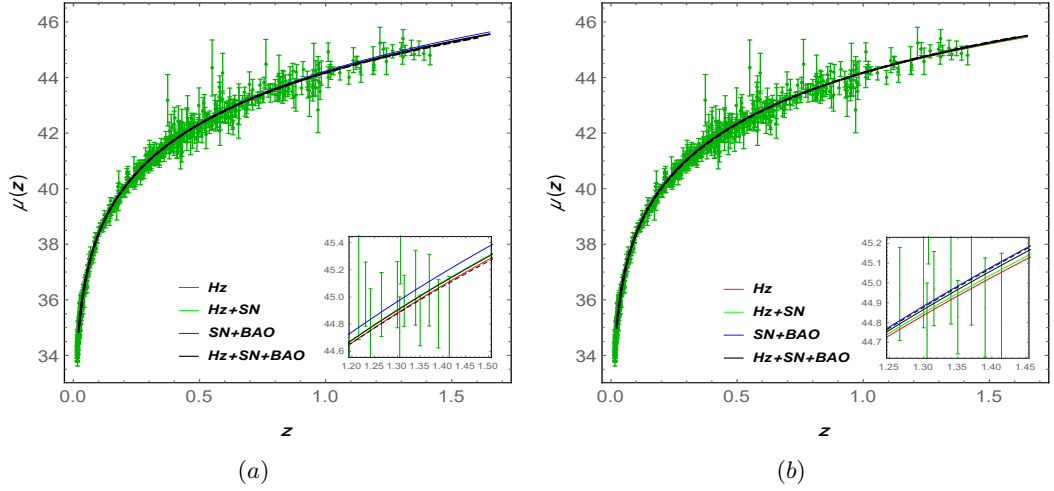


FIG. 8: Figures (a) and (b) are the error bar plots for Union 2.1 compilation supernovae datasets together with the models M1 and M2 shown in solid red lines respectively. The dashed lines in both the figures are Λ CDM model shown for comparison.

phase transition redshift (z_{tr} for which $q = 0$, the redshift at which the Universe transitioned from decelerating expansion to accelerating one) using the constrained (numerical) values of the model parameters α & β (see Table-4) for both the models M1 and model M2 in the following Table-5 and Table-6 respectively.

redshift	formula	H_z	$H_z + SN$	$SN + BAO$	$H_z + SN + BAO$
$z \rightarrow \infty (q_i)$	$q_i = -1 + \alpha$	0.58064	0.60094	0.61116	0.59173
$z \rightarrow 0 (q_0)$	$q_0 = -1 + \alpha - \frac{2\alpha}{1+\beta\alpha}$	-0.51977	-0.54087	-0.60308	-0.53714
$z \rightarrow -1 (q_f)$	$q_f = -1 - \alpha$	-2.58064	-2.60094	-2.61116	-2.59173
$z_{tr} (q = 0)$	$z_{tr} = -1 + \frac{1}{\beta} \left(\frac{\alpha+1}{\alpha-1} \right)^{\frac{1}{\alpha}}$	0.72763	0.72730	0.80236	0.73622

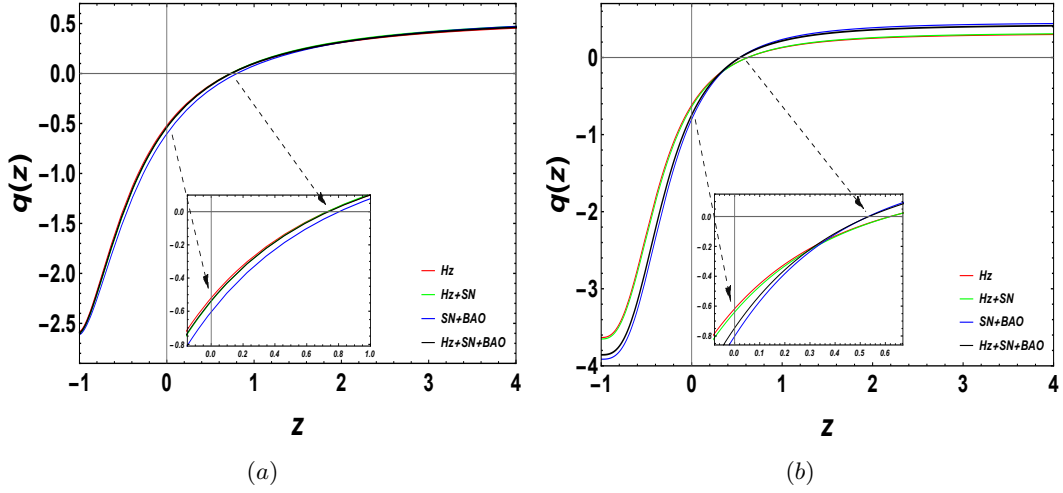


FIG. 9: Figures (a) and (b) show the evolution of deceleration parameter from past ($z = 4$) to far future with a phase transition for models M1 and M2 respectively.

redshift	formula	H_z	$H_z + SN$	$SN + BAO$	$H_z + SN + BAO$
$z \rightarrow \infty (q_i)$	$q_i = -1 + \alpha$	0.31611	0.32551	0.45677	0.42829
$z \rightarrow 0 (q_0)$	$q_0 = -1 + \alpha - \frac{3\alpha}{1+\beta^{2\alpha}}$	-0.61721	-0.63987	-0.80152	-0.74601
$z \rightarrow -1 (q_f)$	$q_f = -1 - 2\alpha$	-3.63222	-3.65102	-3.91354	-3.85658
$z_{tr} (q = 0)$	$z_{tr} = -1 + \frac{1}{\beta} \left(\frac{2\alpha+1}{\alpha-1} \right)^{\frac{1}{2\alpha}}$	0.61941	0.62055	0.53182	0.53471

From the above Table-5 and Table-6, it is observed that in both the models M1 and M2, the Universe begins smoothly with deceleration (means no inflationary phase in these models) and transit to accelerating phase at around $z_{tr} \approx 0.72$ in model M1 for all numerical constrained values of model parameters α & β while in model M2, the phase transition occurs at $z_{tr} \approx 0.62$ H_z and $H_z + SN$ constrained values of α & β and $z_{tr} \approx 0.53$ $SN + BAO$ and $H_z + SN + BAO$ constrained values of α & β . The present values of the deceleration parameter q_0 found in both models can be seen in the above tables consistent with predicted values. In the future, the Universe enter into super acceleration phase ($q < -1$) for both the models M1 and M2 and finally attains maximum values $q_f < -2.58$ in model M1 and $q_f < -3.61$ in model M2 for all the values of α & β .

B. Statefinder diagnostics

Statefinder diagnostics [86–89] is a technique generally used to distinguish various dark energy models and compare their behavior using the higher order derivatives of the scale factor. The parameters are s & r and calculated using the relations:

$$r = \frac{\ddot{a}}{aH^3}, \quad s = \frac{r-1}{3(q-\frac{1}{2})}. \quad (18)$$

The statefinder diagnostics pairs are constructed as $\{s, r\}$ and $\{q, r\}$ wherein different trajectories in the s - r and q - r planes are plotted to see the temporal evolutions of different dark energy models. The fixed points in this contexts are generally considered as $\{s, r\} = \{0, 1\}$ for Λ CDM model and $\{s, r\} = \{1, 1\}$ for SCDM (standard cold dark matter) model in FLRW background and the departures of any dark energy model from these fixed points are analyzed. The other diagnostic pair is $\{q, r\}$ and the fixed points considered are $\{q, r\} = \{-1, 1\}$ for Λ CDM model and $\{q, r\} = \{0.5, 1\}$ for SCDM model. The statefinder parameters for the considered model M1 are calculated as,

$$r(z) = 1 + \alpha(2\alpha - 3) + \frac{6\alpha}{1 + \{\beta(1+z)\}^\alpha} \left[1 - \alpha + \frac{\alpha}{1 + \{\beta(1+z)\}^\alpha} \right] \quad (19)$$

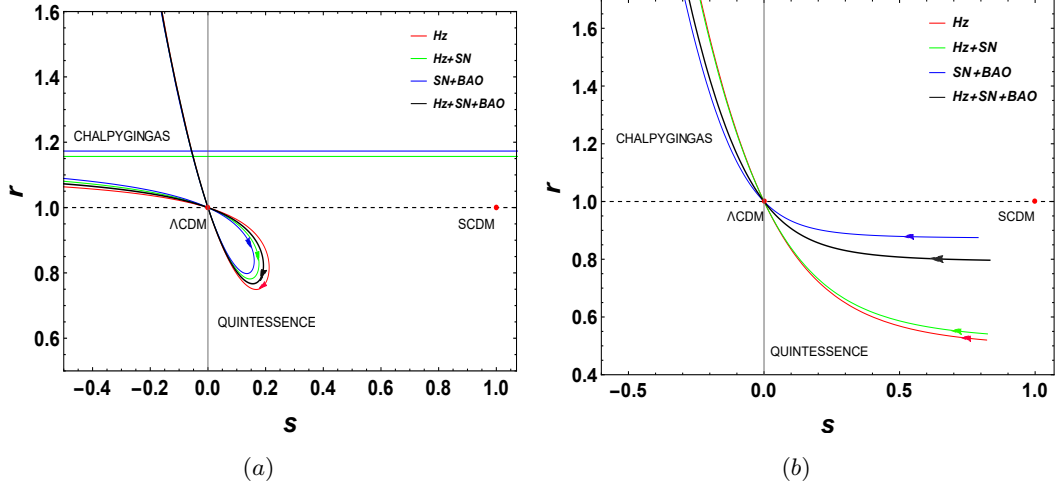


FIG. 10: Figures (a) and (b) are the s - r plots for model M1 and M2 respectively showing the different trajectories of the models.

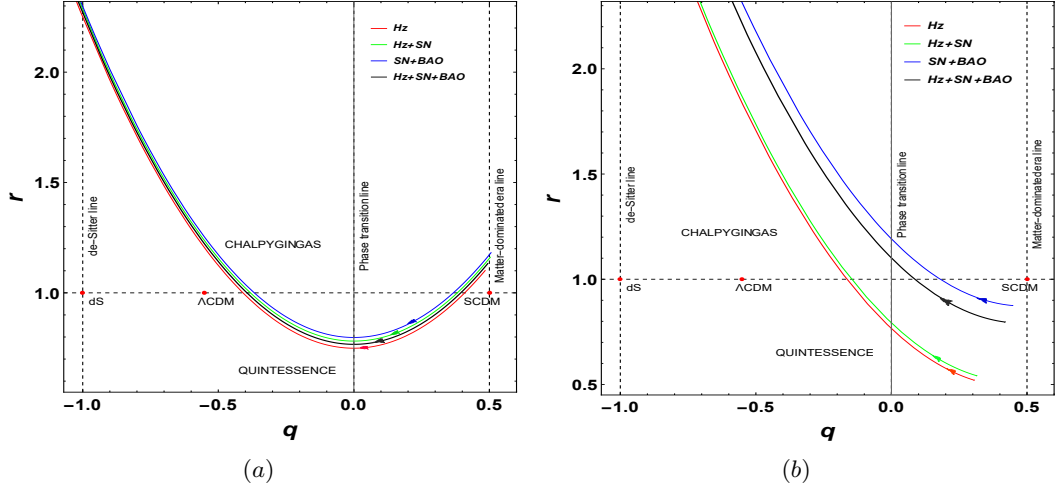


FIG. 11: Figures (a) and (b) are q - r plots for models M1 and M2 respectively showing different trajectories of the models.

$$s(z) = \frac{2\alpha}{3} - \frac{\alpha}{1 + \{\beta(1+z)\}^\alpha} + \frac{\alpha(3+2\alpha)}{3[-3-2\alpha+(2\alpha-3)\{\beta(1+z)\}^\alpha]} \quad (20)$$

and for model M2

$$r(z) = 1 - 3\alpha + 2\alpha^2 + \frac{12\alpha^2}{[1 + \{\beta(1+z)\}^{2\alpha}]^2} + \frac{3\alpha(3-2\alpha)}{1 + \{\beta(1+z)\}^{2\alpha}} \quad (21)$$

$$s(z) = \frac{2}{3} \left[\alpha - \frac{2\alpha}{1 + \{\beta(1+z)\}^{2\alpha}} + \frac{\alpha(3+4\alpha)}{[-3-4\alpha+(2\alpha-3)\{\beta(1+z)\}^{2\alpha}]} \right] \quad (22)$$

In the above figure FIG. 10, one can see the diverge evolutions of the model M1 and model M2 in the s - r and q - r planes. Both the models showing distinctive features as compared to the other standard models. One can observe

that at early times, model M1 presumes values in the range $r > 1$ and $s < 0$ representing Chaplygin gas type DE model and evolves to quintessence region and again reverts to Chaplygin gas region at late times by crossing the intermediate Λ CDM fixed point $\{0, 1\}$ during evolution. But, the model M2 is different and evolves from quintessence region in the past and goes to Chaplygin gas region intermediating the Λ CDM fixed point $\{0, 1\}$ during its evolution for all cases. The figure FIG. 11 depicts the temporal evolution of the models M1 and M2 in the $\{q, r\}$ plane providing additional information about the models M1 and M2 wherein the dashed lines describe the evolution of the Λ CDM model below which quintessence region and the upper one is Chaplygin gas region are shown. The evolution of model M1 and M2 are clearly observed. Both the models M1 and M2 deviates from de Sitter point $(-1, 1)$.

C. Om diagnostic

Om diagnostic is another tool introduced in [90–93], using the Hubble parameter and serving the purpose of providing a null test of the Λ CDM model. Like, statefinder diagnostic, Om diagnostic is also an effective method to discriminate various DE models from Λ CDM model according to the slope variation of $Om(z)$. Positive slope of diagnostic implies a Quintessence nature ($\omega > -1$), Negative slope of diagnostic implies a Phantom nature ($\omega < -1$) and Constant slope with respect to redshift tells the nature of dark energy coincide with that of the cosmological constant ($\omega = -1$).

The $Om(z)$ for a flat Universe is defined as:

$$Om(z) = \frac{\left(\frac{H(z)}{H_0}\right)^2 - 1}{(1+z)^3 - 1}, \quad (23)$$

which can also be represented as $Om(z) = \Omega_{m0} + (1 - \Omega_{m0}) \frac{(1+z)^{3(1+\omega)} - 1}{(1+z)^3 - 1}$. For a constant EoS parameter ω imply $Om(z) = \Omega_{m0}$ and different values of $Om(z)$ suggest whether the model is a Λ CDM model or quintessence or phantom models. For the models of consideration here, the expressions for $Om(z)$ for models M1 and M2 are obtained as,

$$Om(z) = \frac{\frac{[1+\{\beta(1+z)\}^\alpha]^4}{(1+\beta^\alpha)^4(1+z)^{2\alpha}} - 1}{(1+z)^3 - 1} \quad (24)$$

$$Om(z) = \frac{\frac{[1+\{\beta(1+z)\}^{2\alpha}]^3}{(1+\beta^{2\alpha})^3(1+z)^{4\alpha}} - 1}{(1+z)^3 - 1} \quad (25)$$

The slope variation of $Om(z)$ vs. z are shown in the following figure FIG. 12 for models M1 and M2. For both the models M1 and M2 and for all values of α & β , the $Om(z)$ values is less than Ω_{m0} in the redshift range $z > 0$ showing the models are in quintessence region in the past and for the redshift range $z < 0$, $Om(z)$ values decreases sharply and becomes negative implying the both the models enter into phantom region.

D. Jerk, Snap and Lerk parameters

Likewise, the Hubble and the deceleration parameters, the other cosmographic parameters, jerk, snap and lerk parameters also play significant roles in analyzing a cosmological model. The cosmic jerk $j(z=0) \simeq 1$ signify a cosmic acceleration. The evolution of jerk parameter for the models M1 and M2 are shown in FIG. 13 showing that for all numerical constrained values of model parameters α & β , $j_0 \in (1.1, 1.4)$ for model M1 and $j_0 \in (0.6, 1.2)$ for model M2. The increasing values of jerk, snap and lerk parameters in the future ($z < 0$) showing the deviation from the Λ CDM model which can also be interpreted from the statefinder diagrams FIG. 9 and FIG. 10. Similarly, the evolution of snap and lerk parameters are shown in FIG. 14 and FIG. 15 respectively for both the models M1 and M2. From the figures, it can be seen that for all values of model parameters α & β , $s_0 \in (1.2, 2.2)$ for model M1 and $s_0 \in (-1.5, 0 - 0.7)$ for model M2 and $l_0 \in (6, 9)$ for model M1 and $l_0 \in (5, 13)$ for model M2. These values of j_0 , s_0 , l_0 are in good agreement with the expected values. One can also interpret that the model M2 has better fit to the observational datasets as compared to model M1 which can also be seen from FIG. 14 and FIG. 15.

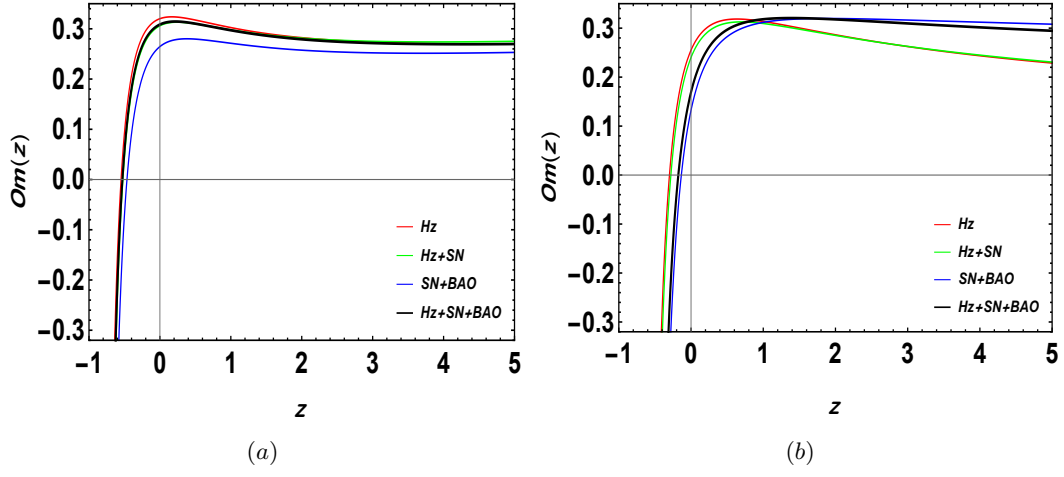


FIG. 12: Figures (a) and (b) show the plots for $Om(z)$ vs. z for models M1 and M2 respectively.

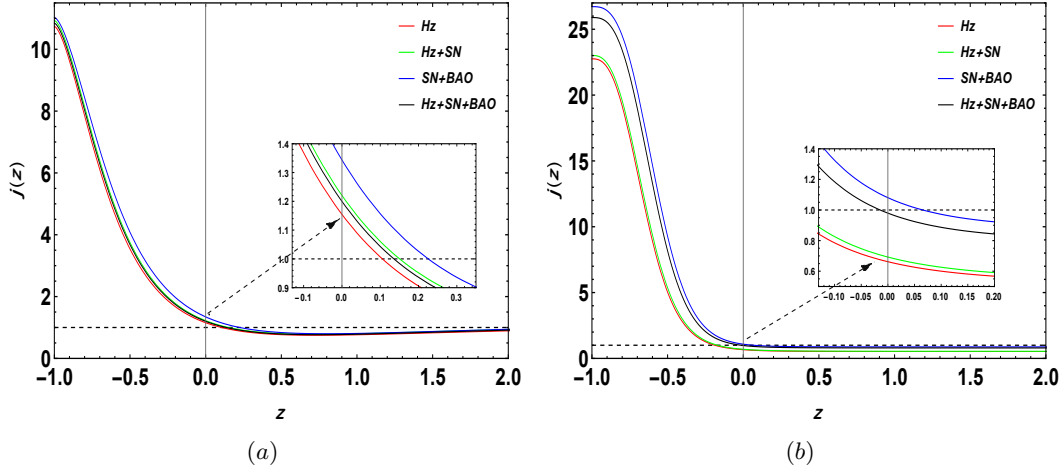


FIG. 13: Figures (a) and (b) show the plots for jerk parameter $j(z)$ vs. z for models M1 and M2 respectively.

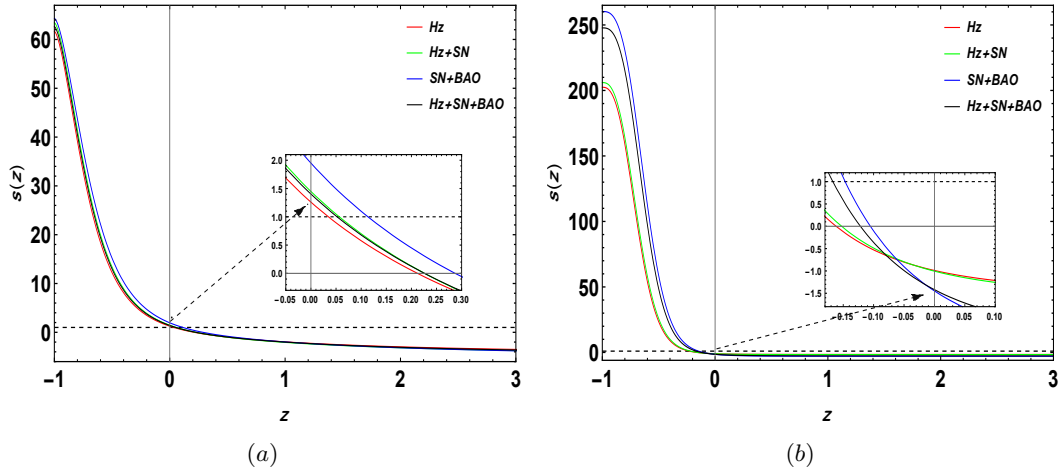


FIG. 14: Figures (a) and (b) show the plots for snap parameter $s(z)$ vs. z for models M1 and M2 respectively.

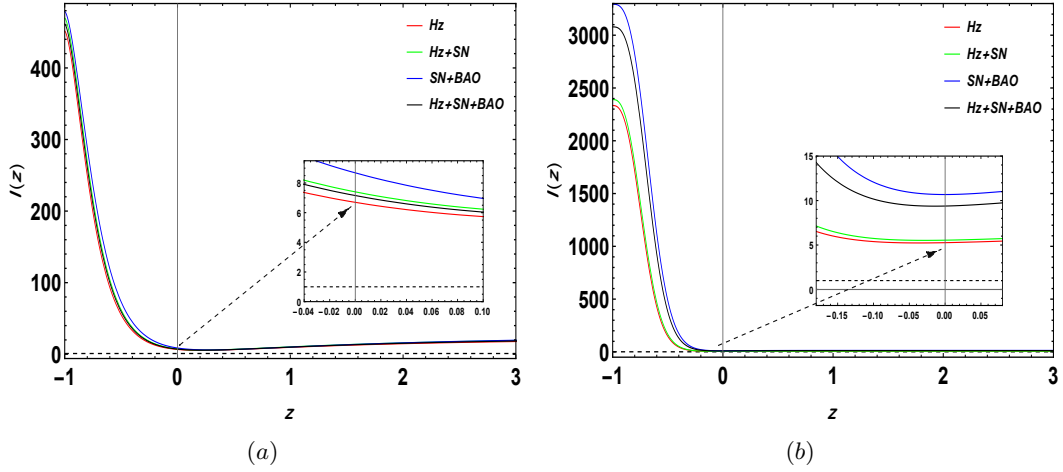


FIG. 15: Figures (a) and (b) show the plots for lerk parameter $l(z)$ vs. z for models M1 and M2 respectively.

VII. PHYSICAL DYNAMICS OF THE MODELS

The geometrical part of the Einstein field equations is discussed elaborately and now the physical interpretations can be discussed for the obtained models once the matter content of the Universe is specified. In the introduction, it is mentioned that the candidate of dark energy is still unknown and it is a matter of speculation only to choose any candidate described in literature. However, the most discussed candidate and having best fit with some observations is the Einstein's cosmological constant. So, in the following, the cosmological constant will be considered as a candidate of dark energy for further analysis.

So, let us consider the two fluid Universe, cold dark matter and dark energy only, since the radiation contribution at present is negligible. The matter pressure is $p = p_m = 0$ for cold dark matter and for dark energy the equation of state is $p_{DE} = \omega_{DE}\rho_{DE}$. In the following, the physical behavior of the matter and dark energy densities and pressures are found out and their evolutions are shown graphically.

A. Cosmological constant

When the candidate of dark energy is the cosmological constant implying $\rho_{DE} = \rho_{\Lambda} = M_{pl}^{-2}\Lambda$ and for which the equation of state parameter ω_{DE} reduces to -1 . Solving equations (4) and (5), it is easy to obtain the explicit expressions for the matter energy density and the energy density of cosmological constant as,

$$\frac{\rho_m}{M_{pl}^2 H_0^2} = \frac{2\alpha [1 + \{\beta(1+z)\}^{\alpha}]^4 - 4\alpha [1 + \{\beta(1+z)\}^{\alpha}]^3}{(1 + \beta\alpha)^4 (1+z)^{2\alpha}}, \quad (26)$$

$$\frac{\rho_{\Lambda}}{M_{pl}^2 H_0^2} = \frac{\Lambda}{H_0^2} = \frac{(3 - 2\alpha) [1 + \{\beta(1+z)\}^{\alpha}]^4 + 4\alpha [1 + \{\beta(1+z)\}^{\alpha}]^3}{(1 + \beta\alpha)^4 (1+z)^{2\alpha}}. \quad (27)$$

for model M1 and

$$\frac{\rho_m}{M_{pl}^2 H_0^2} = \frac{2\alpha [1 + \{\beta(1+z)\}^{2\alpha}]^3 - 6\alpha [1 + \{\beta(1+z)\}^{2\alpha}]^2}{(1 + \beta^{2\alpha})^3 (1+z)^{4\alpha}}, \quad (28)$$

$$\frac{\rho_{\Lambda}}{M_{pl}^2 H_0^2} = \frac{\Lambda}{H_0^2} = \frac{(3 - 2\alpha) [1 + \{\beta(1+z)\}^{2\alpha}]^3 + 6\alpha [1 + \{\beta(1+z)\}^{2\alpha}]^2}{(1 + \beta^{2\alpha})^3 (1+z)^{4\alpha}}. \quad (29)$$

for model M2. The evolution of these physical parameters are shown in the FIG. 16 and FIG. 17.

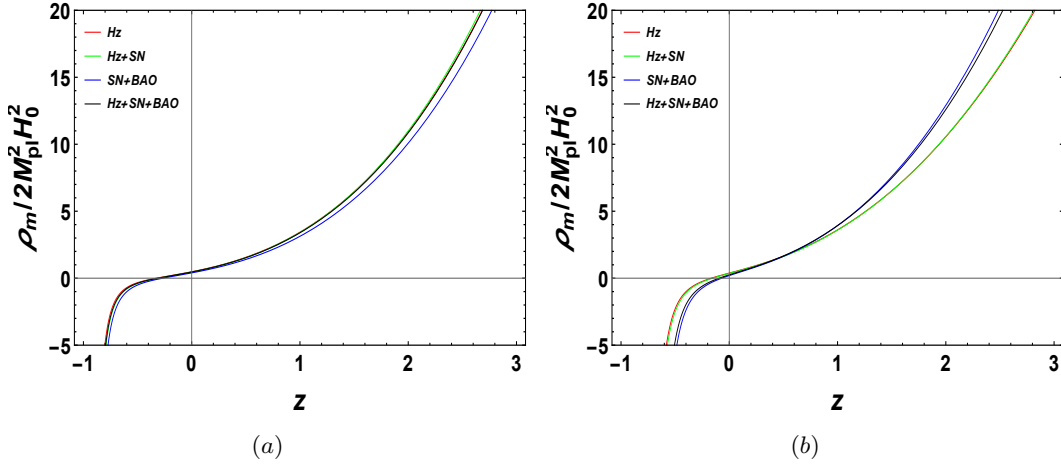


FIG. 16: Figures (a) and (b) show the evolution of the matter energy densities (ρ_m) for models M1 and M2 respectively.

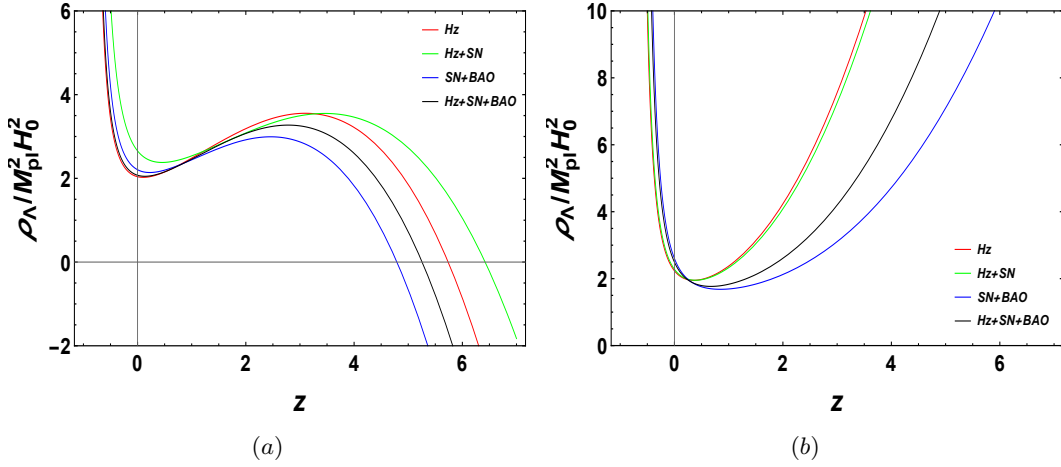


FIG. 17: Figures (a) and (b) show the evolution of the energy densities of the cosmological constant (ρ_Λ) for models M1 and M2 respectively.

The density parameters for matter ($\Omega_m = \frac{\rho_m}{3M_{pl}^2 H^2}$) and density parameter for cosmological constant ($\Omega_\Lambda = \frac{\rho_\Lambda}{3H^2}$) can also be computed for both the models M1 and M2 as,

$$\Omega_m = \frac{2\alpha}{3} - \frac{4\alpha}{3[1 + \{\beta(1+z)\}^\alpha]}, \quad \Omega_\Lambda = 1 - \frac{2\alpha}{3} + \frac{4\alpha}{3[1 + \{\beta(1+z)\}^\alpha]} \quad (30)$$

for model M1 and

$$\Omega_m = \frac{2\alpha}{3} - \frac{2\alpha}{[1 + \{\beta(1+z)\}^{2\alpha}]}, \quad \Omega_\Lambda = 1 - \frac{2\alpha}{3} + \frac{2\alpha}{[1 + \{\beta(1+z)\}^{2\alpha}]} \quad (31)$$

for model M2. One can see from the above expressions that the sum total of the density parameters with these components is equal to 1. The evolution of the density parameters are shown in FIG. 18.

B. Scalar field

Since the equation of state for cosmological constant is non dynamical and observations reveal it's dynamical characteristics, other candidates such as a general scalar field came into picture for a suitable candidate of dark energy. For an ordinary scalar field ϕ for the action can be represented as,

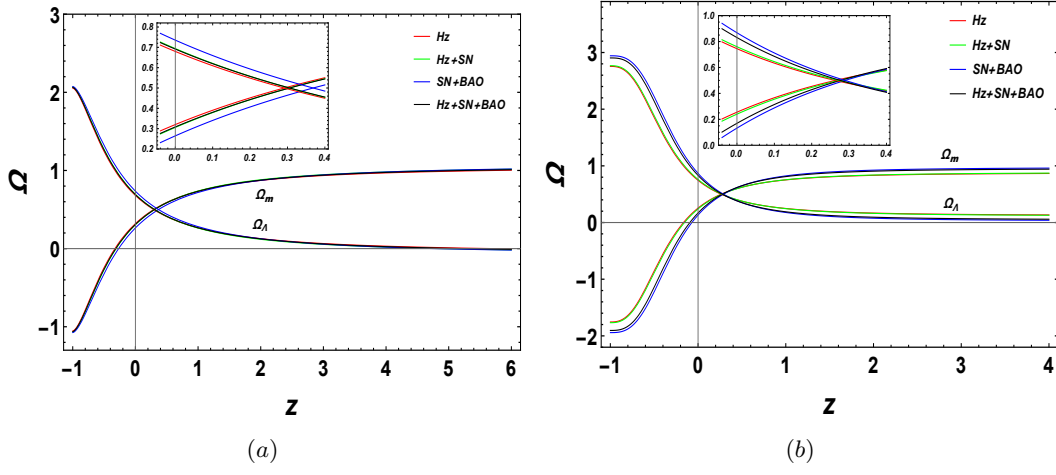


FIG. 18: Figures (a) and (b) show the evolution of the density parameters for matter (Ω_m) and cosmological constant (Ω_Λ) for models M1 and M2 respectively.

$$S = \int d^4x \sqrt{-g} \left\{ \frac{M_p^2}{2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + L_{Matter} \right\}. \quad (32)$$

The term $V(\phi)$ is the potential function for the scalar field ϕ . In the considered FLRW background the energy density ρ_ϕ will take the form $\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi)$ and pressure p_ϕ will take the form $p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi)$. For a two component Universe, scalar field and cold dark matter with minimal interaction between them (i.e. they conserve separately giving $\rho = ca^{-3} = c(1+z)^3$, c is a constant of integration), then the solutions obtained from Eqs. (4) and (5) are,

$$\frac{V(\phi)}{M_{pl}^2 H_0^2} = \frac{(3-\alpha)[1 + \{\beta(1+z)\}^{\alpha}]^4 + 2\alpha[1 + \{\beta(1+z)\}^{\alpha}]^3}{(1+\beta\alpha)^4 (1+z)^{2\alpha}} - \frac{c}{2M_{pl}^2 H_0^2} (1+z)^3 \quad (33)$$

$$\frac{\rho_\phi}{M_{pl}^2 H_0^2} = \frac{3[1 + \{\beta(1+z)\}^{\alpha}]^4}{(1+\beta\alpha)^4 (1+z)^{2\alpha}} - \frac{c}{M_{pl}^2 H_0^2} (1+z)^3 \quad (34)$$

and the expression for the scalar field $\phi(z)$ can be calculated by integrating,

$$\frac{\phi - \phi_0}{\sqrt{2} M_{pl}} = - \int \left[\frac{\alpha[1 + \{\beta(1+z)\}^{\alpha}]^4 - 2\alpha[1 + \{\beta(1+z)\}^{\alpha}]^3}{(1+\beta\alpha)^4 (1+z)^{2\alpha}} - \frac{c}{2M_{pl}^2 H_0^2} (1+z)^3 \right]^{\frac{1}{2}} \frac{(1+\beta\alpha)^2 (1+z)^{\alpha-1}}{[1 + \{\beta(1+z)\}^{\alpha}]^2} dz \quad (35)$$

for model M1. Here, ϕ_0 is an integrating constant. Similarly, the potential and energy densities for model M2 are obtained as,

$$\frac{V(\phi)}{M_{pl}^2 H_0^2} = \frac{(3-\alpha)[1 + \{\beta(1+z)\}^{2\alpha}]^3 + 3\alpha[1 + \{\beta(1+z)\}^{2\alpha}]^2}{(1+\beta^2\alpha)^3 (1+z)^{4\alpha}} - \frac{c}{2M_{pl}^2 H_0^2} (1+z)^3 \quad (36)$$

$$\frac{\rho_\phi}{M_{pl}^2 H_0^2} = \frac{3[1 + \{\beta(1+z)\}^{2\alpha}]^3}{(1+\beta^2\alpha)^3 (1+z)^{4\alpha}} - \frac{c}{M_{pl}^2 H_0^2} (1+z)^3 \quad (37)$$

and the expression for the scalar field $\phi(z)$ can be calculated by integrating,

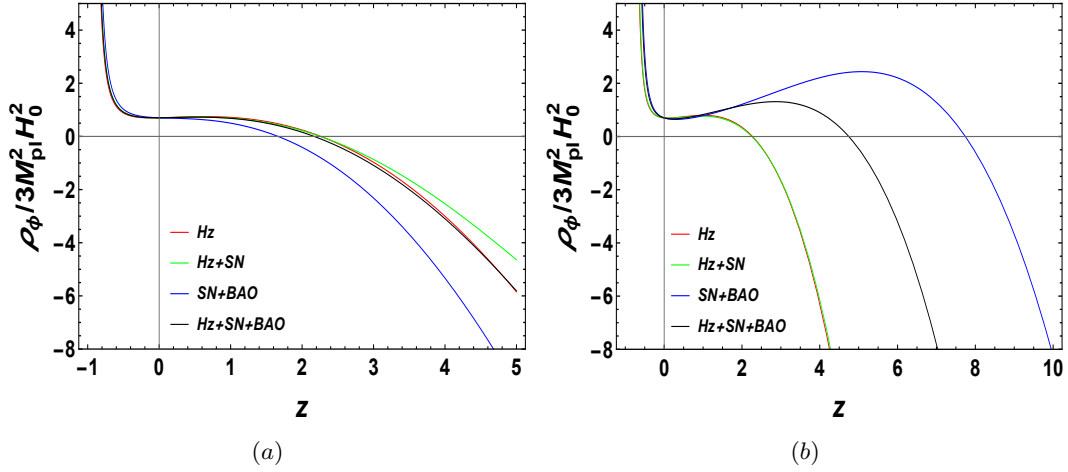


FIG. 19: Figures (a) and (b) show the evolution of the scalar field energy density (ρ_ϕ) for models M1 and M2 respectively.

$$\frac{\phi - \phi_0}{\sqrt{2}M_{pl}} = - \int \left[\frac{\alpha \left[1 + \{\beta(1+z)\}^{2\alpha} \right]^3 - 3\alpha \left[1 + \{\beta(1+z)\}^{2\alpha} \right]^2}{(1 + \beta^{2\alpha})^3 (1+z)^{4\alpha}} - \frac{c}{2M_{pl}^2 H_0^2} (1+z)^3 \right]^{\frac{1}{2}} \frac{(1 + \beta^{2\alpha})^{\frac{3}{2}} (1+z)^{2\alpha-1}}{\left[1 + \{\beta(1+z)\}^{2\alpha} \right]^{\frac{3}{2}}} dz. \quad (38)$$

The density parameters for matter ($\Omega_m = \frac{\rho_m}{3M_{pl}^2 H^2}$) and density parameter for the scalar field ($\Omega_\phi = \frac{\rho_\phi}{3M_{pl}^2 H^2} = \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{3M_{pl}^2 H^2}$) can be computed for both the models M1 and M2 as,

$$\Omega_\phi = 1 - \Omega_m, \quad \Omega_m = \frac{c(1+z)^3 (1 + \beta^\alpha)^4 (1+z)^{2\alpha}}{3M_{pl}^2 H_0^2 [1 + \{\beta(1+z)\}^\alpha]^4} \quad (39)$$

for model M1 and

$$\Omega_\phi = 1 - \Omega_m, \quad \Omega_m = \frac{c(1+z)^3 (1 + \beta^{2\alpha})^3 (1+z)^{4\alpha}}{3M_{pl}^2 H_0^2 [1 + \{\beta(1+z)\}^{2\alpha}]^3} \quad (40)$$

for model M2. From equations (39) and (40), one obtains $\Omega_{m0} = \frac{c}{3M_{pl}^2 H_0^2} \implies c = \Omega_{m0} 3M_{pl}^2 H_0^2$. The equations of state parameter ($\omega_\phi = \frac{p_\phi}{\rho_\phi}$) are given by,

$$\omega_\phi^{eff} = \frac{1}{3} \frac{(2\alpha - 3) [1 + \{\beta(1+z)\}^\alpha]^4 - 4\alpha [1 + \{\beta(1+z)\}^\alpha]^3}{[1 + \{\beta(1+z)\}^\alpha]^4 - \Omega_{m0} (1 + \beta^\alpha)^4 (1+z)^{2\alpha+3}} \quad (41)$$

$$\omega_\phi^{eff} = \frac{1}{3} \frac{(2\alpha - 3) [1 + \{\beta(1+z)\}^{2\alpha}]^3 - 6\alpha [1 + \{\beta(1+z)\}^{2\alpha}]^2}{[1 + \{\beta(1+z)\}^{2\alpha}]^3 - \Omega_{m0} (1 + \beta^{2\alpha})^3 (1+z)^{4\alpha+3}} \quad (42)$$

The evolution of the Scalar field energy density, scalar field potential and the density parameters are shown in FIG. 19, FIG. 20 and FIG. 21 respectively for models M1 and M2.

The evolution of the equation of state parameter ($\omega_\phi(z)$) vs. redshift z is plotted by neglecting the matter contribution and shown in the FIG. 22 for models M1 and M2.

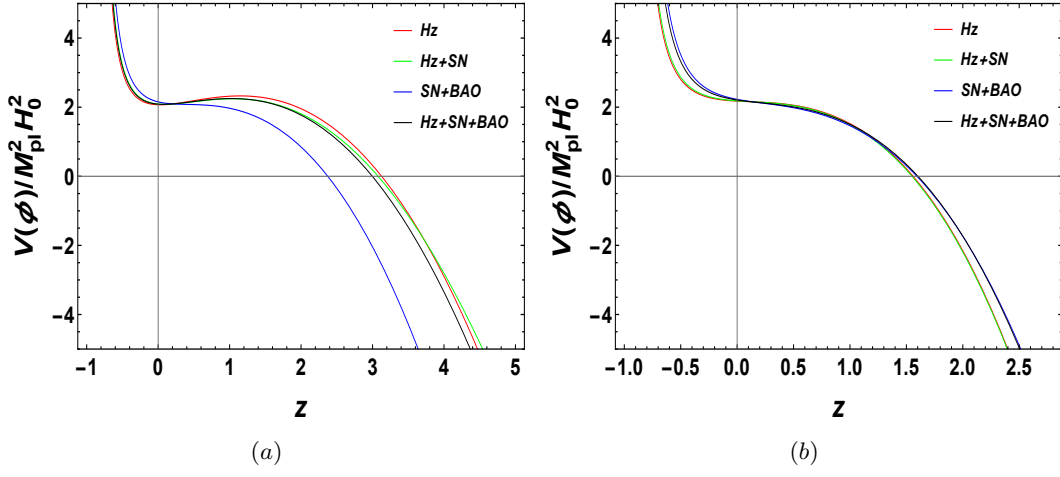


FIG. 20: Figures (a) and (b) show the evolution of the scalar field potential $V(\phi) \sim z$ for models M1 and M2 respectively.

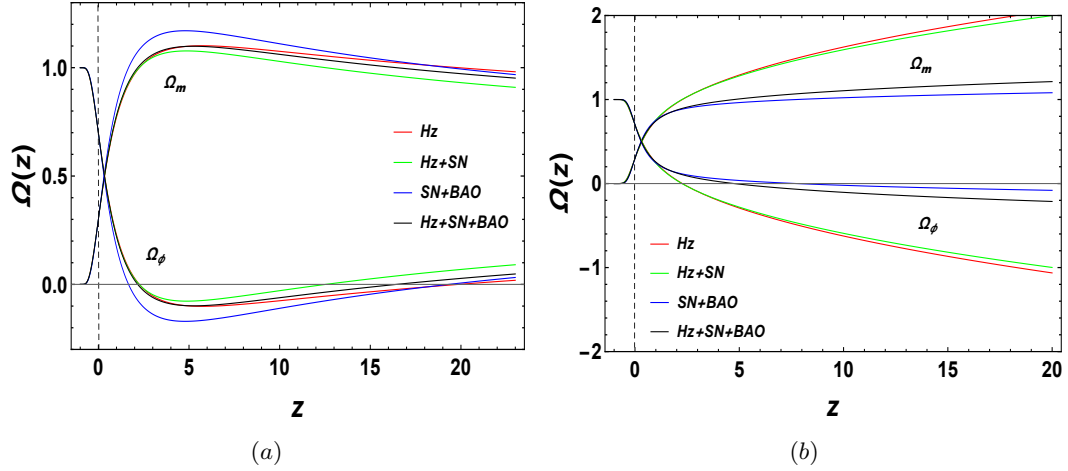


FIG. 21: Figures (a) and (b) show the evolution of the density parameters Ω_ϕ & Ω_m w.r.t. redshift z for models M1 and M2 respectively.

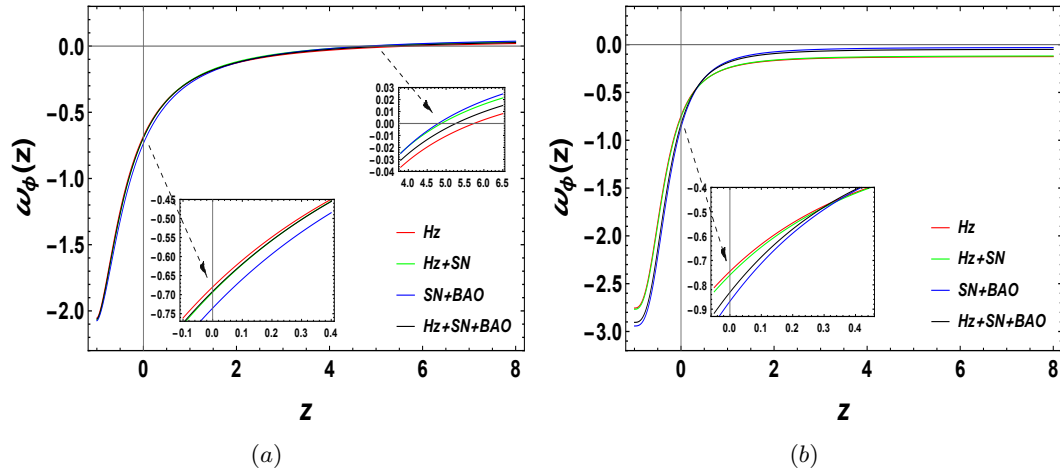


FIG. 22: Figures (a) and (b) show the evolution of equation of state parameter vs. redshift ($\omega_\phi(z) \sim z$) for models M1 and M2 respectively.

VIII. AGE OF THE UNIVERSE

The calculation of the age of the Universe is associated to the values of the cosmological parameters, specifically the Hubble parameter. In general, using the Friedmann equation one can obtain the relation as $t_0 = \frac{1}{H_0} F(\Omega_x)$, $x =$ radiation, matter, dark energy, neutrino etc. The functional F contributes a fraction and largely the term $1/H_0$ in the age calculation e.g. for $H_0 = 69 \text{ km/s/Mpc}$, one obtains $1/H_0 \approx 14.5 \text{ Gyr}$ (Giga years) and the factor $F = 0.956$ for Λ CDM model with $(\Omega_m, \Omega_\Lambda) = (0.3086, 0.6914)$ giving pretty good estimate of t_0 and $F = 0.666$ for Einstein-de-Sitter model with $(\Omega_m, \Omega_\Lambda) = (1, 0)$ giving much smaller value of t_0 . So, the introduction of cosmological constant is significant as matter-only Universe was not enough to explain the globular clusters in the Milky Way which appeared to be older than the age of the Universe calculated then. According to the Planck2015 results age of the universe is estimated to be $13.799 \pm 0.021 \text{ Gyr}$ with $H_0 = 67.74 \pm 0.46$ within 68% confidence limits for Λ CDM model constrained by combined CMB power spectra, Planck polarization data, CMB lensing reconstruction and external data of BAO, JLA (Joint light curve analysis) and Hubble datasets.

Here, the present work is a model independent study wherein the geometrical parameter H is parametrized for which the calculation of the age is unaffected by the matter content and solely depend on the functional form of the Hubble parameter $H(t)$. We have already established the t - z relationships for models M1 and M2 which can be rewritten as,

$$t(z) = \frac{(1 + \beta^\alpha)^2}{\alpha\beta^\alpha [1 + \{\beta(1+z)\}^\alpha]} \frac{1}{H_0} \quad \text{and} \quad t(z) = \frac{(1 + \beta^{2\alpha})^{\frac{3}{2}}}{\alpha\beta^{2\alpha} [1 + \{\beta(1+z)\}^{2\alpha}]^{\frac{1}{2}}} \frac{1}{H_0}$$

respectively. By considering the present value of the Hubble parameter, $H_0 = 67.8 \text{ Km/Sec/Mpc}$, the terms multiplied to $1/H_0$ are calculated and for both the models are greater than 0.96 for all constrained values of α & β and give pretty good estimate for the present age of the Universe and is larger than the standard model. The age calculation is tabulated in the following Table-7 for all the constrained numerical values of α & β for both the models M1 and M2.

Models	M1			M2		
Datasets	(α, β)	Factor	Age (in Gyr)	(α, β)	Factor	Age (in Gyr)
H_z	(1.58064, 1.48729)	0.97046	14.0068	(1.31611, 1.56124)	0.99502	14.3613
$H_z + SN$	(1.60094, 1.44572)	0.97084	14.0123	(1.32551, 1.53587)	0.99630	14.3797
$SN + BAO$	(1.61116, 1.36647)	0.99597	14.3751	(1.45677, 1.36451)	0.96400	13.9136
$H_z + SN + BAO$	(1.59173, 1.45678)	0.97343	14.0496	(1.42829, 1.40637)	0.96445	13.9201

IX. RESULTS AND CONCLUSION

To summarize the results, the philosophy behind writing this present paper is to discuss the phenomenology of cosmological parametrization to obtain exact solutions of Einstein field equations. As an exemplification, a simple parametrization of Hubble parameter is considered with some model parameters which reduce to some known models (see [55]) for some specific values of the model parameters involved. Two models discussed here in details and both the models M1 and M2 exhibit a phase transition from deceleration to acceleration. Also, both the models diverges in finite time and show big rip singularity. For consistency of the models obtained here, some observational datasets namely, $H(z)$ datasets with updated 57 data points, Supernovae datasets from union 2.1 compilation datasets containing 580 data points and BAO datasets with 6 data points are considered and compared with the standard Λ CDM model. Both the models M1 and M2 contain two model parameters α & β which are constrained through these datasets and some numerical values are obtained in pairs with independent H_z , combined $H_z + SN$, $SN + BAO$ and $H_z + SN + BAO$ datasets which are then used for further analysis for geometrical and physical interpretations of the models. The present values of the deceleration parameters obtained for these constrained values of model parameters α & β are calculated which are tabulated in Table-5 and Table-6 together with the phase transition redshifts and are in certain standard estimated range. In the future the Universe in both the models enters into super acceleration phases and diverges in finite times. The other geometrical parameters such as jerk, snap and lerk parameters are also discussed and their evolutions are shown graphically. The statefinder diagnostics and om diagnostics are also presented to compare the obtained models with the standard Λ CDM model and the models behavior are shown in plots compared

with the standard Λ CDM model and SCDM model. After the brief cosmographic analysis, the physical interpretation of the models are discussed by considering the cosmological constant and scalar field as candidates of dark energy. The matter energy density (ρ_m), energy density of (ρ_Λ), the density parameters Ω_Λ and Ω_m are also calculated and their dynamical behavior w.r.t. redshift z are shown graphically for both the models M1 and M2 using the numerical constrained values of the model parameters α & β . Similarly, the evolution of energy density (ρ_ϕ), the potential ($V(\phi)$) of the scalar field (ϕ), density parameters Ω_Λ and Ω_m and also the equation of state parameter ω_ϕ of the scalar field are shown graphically for both the models M1 and M2. The geometrical and physical analysis for both the models M1 and M2 interpret that both models M1 and M2 have the quintessence behavior in the past and phantom like behavior in the future. Finally, the age of the Universe for both the models M1 and M2 are calculated for the constrained numerical values of model parameters α & β . It is found that the age found for both the models are greater than the standard model and consistent with the age constraints of Λ CDM model.

The conclusion is that the model M2 which is a quadratic varying deceleration parameter model has better fit to the observational datasets (see FIG. 7 and FIG. 8) and shows better approximation to the present cosmological scenario on geometrical as well as physical grounds as compared to the model M1 which is a linearly varying deceleration parameter model. The presented study is an example of doing a comprehensive analysis of any cosmological model that describe a simple methodology of finding exact solution of the Einstein field equations, comparing to the observations and estimating model parameters from the observational datasets. A brief list of various schemes of parametrization of different geometrical and physical parameters used in the past few decades to obtain the exact solutions of EFEs are also summarized here which will help the readers for their studies in cosmological modelling.

X. APPENDIX

A brief list of various parametrization schemes of parametrization of geometrical and physical parameters used in the past few decades to find exact solutions of Einstein Field Equations is given below.

A. PARAMETRIZATIONS OF GEOMETRICAL PARAMETERS

Scale factor $a(t)$

Given below a list of different expansion laws of the scale factor those have been extensively studied in different contexts.

$$a(t) = \text{constant} \quad [30] \quad (\text{Static model})$$

$$a(t) = ct \quad [40, 94] \quad (\text{Milne model or Linear expansion})$$

$$a(t) \sim \exp(H_0 t) \quad [95] \quad (\Lambda\text{CDM model or Exponential expansion})$$

$$a(t) \sim \exp\left[-\alpha t \ln\left(\frac{t}{t_0}\right) + \beta t\right] \quad [96] \quad (\text{Inflationary model})$$

$$a(t) \sim \exp[-\alpha t - \beta t^n] \quad [96] \quad (\text{Inflationary model})$$

$$a(t) \sim [\exp(\alpha t) - \beta \exp(-\alpha t)]^n \quad [96] \quad (\text{Inflationary model})$$

$$a(t) \sim \exp\left(\frac{t}{M}\right) \left[1 + \cos\left(\frac{t}{N}\right)\right] \quad [97] \quad (\text{quasi steady state cosmology, Cyclic Universe})$$

$$a(t) \sim t^\alpha \quad [98] \quad (\text{Power law Cosmology})$$

$$a(t) \sim t^n \exp(\alpha t) \quad [99] \quad (\text{Hybrid expansion})$$

$$a(t) \sim \exp[n(\log t)^m] \quad [100] \quad (\text{Logamediate expansion})$$

$$a(t) \sim \cosh \alpha t \quad [95] \quad (\text{Hyperbolic expansion})$$

$$a(t) \sim (\sinh \alpha t)^{\frac{1}{n}} \quad [101] \quad (\text{Hyperbolic expansion})$$

$$a(t) \sim \left(\frac{t}{t_s - t}\right)^n \quad [102] \quad (\text{Singular model})$$

$$a(t) \sim t^n \exp[\alpha(t_s - t)] \quad [102] \quad (\text{Singular model})$$

$$a(t) \sim \exp\left(\alpha \frac{t^2}{t_*^2}\right) \quad [103] \quad (\text{Bouncing Model})$$

$$a(t) \sim \exp\left(\frac{\beta}{\alpha+1}(t - t_s)^{\alpha+1}\right) \quad [103] \quad (\text{Bouncing Model})$$

$$a(t) \sim \left(\frac{3}{2}\rho_{cr}t^2 + 1\right)^{\frac{1}{3}} \quad [103] \quad (\text{Bouncing Model})$$

$$a(t) \sim \left(\frac{t_s - t}{t_*}\right) \quad [103] \quad (\text{Bouncing Model})$$

$$a(t) \sim \sin^2\left(\alpha \frac{t}{t_*}\right) \quad [103] \quad (\text{Bouncing Model})$$

Hubble parameter $H(t)$ or $H(a)$

$$H(a) = Da^{-m} \quad [104]$$

$$H(a) = e^{\frac{1-\gamma a^2}{\alpha a}} \quad [105]$$

$$H(a) = \alpha(1 + a^{-n}) \quad [106]$$

$$H(t) = \frac{m}{\alpha t + \beta} \quad [107]$$

$$H(t) = \frac{16\alpha t}{15[1+(8\alpha t^2)/5]} \quad [108]$$

$$H(t) = m + \frac{n}{t} \quad [109]$$

$$H(t) = \frac{\alpha t_R}{t(t_R - t)} \quad [110]$$

$$H(t) = \frac{\alpha}{3} (t + T_0)^3 - \beta (t + T_0) + \gamma \quad [111]$$

$$H(t) = \alpha e^{\lambda t} \quad [112]$$

$$H(t) = \alpha + \beta(t_s - t)^n \quad [112]$$

$$H(t) = \alpha - \beta e^{-nt} \quad [113]$$

$$H(t) = f_1(t) + f_2(t)(t_s - t)^n \quad [114]$$

$$H(t) = \frac{\beta t^m}{(t^n + \alpha)^p} \quad [55]$$

$$H(t) = n\alpha \tanh(m - nt) + \beta \quad [115]$$

$$H(t) = \alpha \tanh\left(\frac{t}{t_0}\right) \quad [15]$$

$$H(z) = [\alpha + (1 - \alpha)(1 + z)^n]^{\frac{3}{2n}} \quad [116]$$

Deceleration parameter $q(t)$ or $q(a)$, $q(z)$

$$q(t) = m - 1 \quad [117]$$

$$q(t) = -\alpha t + m - 1 \quad [118]$$

$$q(t) = \alpha \cos(\beta t) - 1 \quad [119]$$

$$q(t) = -\frac{\alpha t}{1+t} \quad [120]$$

$$q(t) = -\frac{\alpha(1-t)}{1+t} \quad [120]$$

$$q(t) = -\frac{\alpha}{t^2} + \beta - 1 \quad [121]$$

$$q(t) = (8n^2 - 1) - 12nt + 3t^2 \quad [122]$$

$$q(a) = -1 - \frac{\alpha a^\alpha}{1+a^\alpha} \quad [123]$$

$$q(z) = q_0 + q_1 z \quad [124]$$

$$q(z) = q_0 + q_1 z(1+z)^{-1} \quad [125]$$

$$q(z) = q_0 + q_1 z(1+z)(1+z^2)^{-1} \quad [126]$$

$$q(z) = \frac{1}{2} + q_1(1+z)^{-2} \quad [127]$$

$$q(z) = q_0 + q_1[1 + \ln(1+z)]^{-1} \quad [128]$$

$$q(z) = \frac{1}{2} + (q_1 z + q_2)(1+z)^{-2} \quad [129]$$

$$q(z) = -1 + \frac{3}{2} \left(\frac{(1+z)^{q_2}}{q_1 + (1+z)^{q_2}} \right) \quad [131]$$

$$q(z) = -\frac{1}{4} \left[3q_1 + 1 - 3(q_1 + 1) \left(\frac{q_1 e^{q_2(1+z)} - e^{-q_2(1+z)}}{q_1 e^{q_2(1+z)} + e^{-q_2(1+z)}} \right) \right] \quad [132]$$

$$q(z) = -\frac{1}{4} + \frac{3}{4} \left(\frac{q_1 e^{\frac{q_2 z}{\sqrt{1+z}}} - e^{-\frac{q_2 z}{\sqrt{1+z}}}}{q_1 e^{\frac{q_2 z}{\sqrt{1+z}}} + e^{-\frac{q_2 z}{\sqrt{1+z}}}} \right) \quad [132]$$

$$q(z) = q_f + \frac{q_i - q_f}{1 - \frac{q_i}{q_f} \left(\frac{1+z_i}{1+z} \right)^{\frac{1}{\tau}}} \quad [133]$$

$$q(z) = q_0 - q_1 \left(\frac{(1+z)^{-\alpha} - 1}{\alpha} \right) \quad [134]$$

$$q(z) = q_0 + q_1 \left[\frac{\ln(\alpha+z)}{1+z} - \beta \right] \quad [135]$$

$$q(z) = q_0 - (q_0 - q_1)(1+z) \exp[z_c^2 - (z + z_c)^2] \quad [136]$$

JerK parameter $j(z)$

$$j(z) = -1 + j_1 \frac{f(z)}{E^2(z)}, \text{ where } f(z) = z, \frac{z}{1+z}, \frac{z}{(1+z)^2}, \log(1+z) \text{ and } E(z) = \frac{H(z)}{H_0} \quad [137]$$

$$j(z) = -1 + j_1 \frac{f(z)}{h^2(z)}, \text{ where } f(z) = 1, 1+z, (1+z)^2, (1+z)^{-1} \text{ and } h(z) = \frac{H(z)}{H_0} \quad [138]$$

B. PARAMETRIZATIONS OF PHYSICAL PARAMETERS

Pressure $p(\rho)$, $p(z)$

The matter content in the Universe is not properly known but it can be categorized with its equations of states $p = p(\rho)$. Following is a list of some cosmic fluid considerations with their EoS. Also, some dark energy pressure parametrization are listed.

$$p(\rho) = w\rho \text{ (Perfect fluid EoS)}$$

$$p(\rho) = w\rho - f(H) \text{ [139] (Viscous fluid EoS)}$$

$$p(\rho) = w\rho + k\rho^{1+\frac{1}{n}} \text{ [140] (Polytropic gas EoS)}$$

$$p(\rho) = \frac{8w\rho}{3-\rho} - 3\rho^2 \text{ [141] (Vanderwaal gas EoS)}$$

$$p(\rho) = -(w+1)\frac{\rho^2}{\rho_F} + w\rho + (w+1)\rho_\Lambda \text{ [142] (EoS in quadratic form)}$$

$$p(\rho) = -\frac{B}{\rho} \text{ [143] (Chaplygin gas EoS)}$$

$$p(\rho) = -\frac{B}{\rho^\alpha} \text{ [144] (Generalized Chaplygin gas EoS)}$$

$$p(\rho) = A\rho - \frac{B}{\rho^\alpha} \text{ [145] (Modified Chaplygin gas EoS)}$$

$$p(\rho) = A\rho - \frac{B(a)}{\rho^\alpha} \text{ [146] (Variable modified Chaplygin gas EoS)}$$

$$p(\rho) = A(a)\rho - \frac{B(a)}{\rho^\alpha} \text{ [147] (New variable modified Chaplygin gas EoS)}$$

$$p(\rho) = -\rho - \rho^\alpha \text{ [148] (DE EoS)}$$

$$p(z) = \alpha + \beta z \text{ [149] (DE EoS)}$$

$$p(z) = \alpha + \beta \frac{z}{1+z} \text{ [149] (DE EoS)}$$

$$p(z) = \alpha + \beta \left(z + \frac{z}{1+z} \right) \text{ [150] (DE EoS)}$$

$$p(z) = \alpha + \beta \ln(1+z) \text{ [151] (DE EoS)}$$

Equation of state parameter $w(z)$

$$w(z) = w_0 + w_1 z \text{ [152] (Linear parametrization)}$$

$$w(z) = w_0 + w_1 \frac{z}{(1+z)^2} \text{ [153] (JBP parametrization)}$$

$$w(z) = w_0 + w_1 \frac{z}{(1+z)^n} \text{ [154] (Generalized JBP parametrization)}$$

$$w(z) = w_0 + w_1 \frac{z}{1+z} \text{ [155] (CPL parametrization)}$$

$$w(z) = w_0 + w_1 \left(\frac{z}{1+z} \right)^n \text{ [154] (Generalized CPL parametrization)}$$

$$w(z) = w_0 + w_1 \frac{z}{\sqrt{1+z^2}} \text{ [156] (Square-root parametrization)}$$

$$w(z) = w_0 + w_1 \sin(z) \text{ [157] (Sine parametrization)}$$

$$w(z) = w_0 + w_1 \ln(1+z) \text{ [158] (Logarithmic parametrization)}$$

$$w(z) = w_0 + w_1 \ln \left(1 + \frac{z}{1+z} \right) \text{ [159] (Logarithmic parametrization)}$$

$$w(z) = w_0 + w_1 \frac{z(1+z)}{1+z^2} \text{ [160] (BA parametrization)}$$

$$w(z) = w_0 + w_1 \left(\frac{\ln(2+z)}{1+z} - \ln 2 \right) \text{ (MZ parametrization)}$$

$$w(z) = w_0 + w_1 \left(\frac{\sin(1+z)}{1+z} - \sin 1 \right) \text{ [161] (MZ parametrization)}$$

$$w(z) = w_0 + w_1 \frac{z}{1+z^2} \text{ (FSSL parametrization)}$$

$$w(z) = w_0 + w_1 \frac{z^2}{1+z^2} \text{ [162] (FSSL parametrization)}$$

$$w(z) = -1 + \frac{1+z}{3} \frac{\alpha+2\beta(1+z)}{\gamma+2\alpha(1+z)+\beta(1+z)^2} \text{ [163] (ASSS parametrization)}$$

$$w(z) = \frac{1+\left(\frac{1+z}{1+z_s}\right)^\alpha}{w_0+w_1\left(\frac{1+z}{1+z_s}\right)^\alpha} \text{ [164] (Hannestad Mortsell parametrization)}$$

$$w(z) = -1 + \alpha(1+z) + \beta(1+z)^2 \text{ [165] (Polynomial parametrization)}$$

$$w(z) = -1 + \alpha[1+f(z)] + \beta[1+f(z)]^2 \text{ [166] (Generalized Polynomial parametrization)}$$

$$w(z) = w_0 + z \left(\frac{dw}{dz} \right)_0 \text{ [167]}$$

$$w(z) = \frac{-2(1+z)d'_c - 3d'_c}{3[d'_c - \Omega_M(1+z)^3(d'_c)^3]} \text{ where } d'_c = \int_0^z \frac{H_0 dz}{H(z)} \text{ [168]}$$

$$w_x(a) = w_0 \exp(a-1) \text{ [169]}$$

$$w_x(a) = w_0 a(1 - \log a) \text{ [169]}$$

$$w_x(a) = w_0 a \exp(1-a) \text{ [169]}$$

$$w_x(a) = w_0 a(1 + \sin(1-a)) \text{ [169]}$$

$$w_x(a) = w_0 a(1 + \arcsin(1-a)) \text{ [169]}$$

$$w_{de}(z) = w_0 + w_1 q \text{ [170]}$$

$$w_{de}(z) = w_0 + w_1 q(1+z)^\alpha \text{ [170]}$$

$$w_{de}(z) = \frac{w_0}{[1+b \ln(1+z)]^2} \text{ [171]}$$

$$w_x(z) = w_0 + b \{1 - \cos[\ln(1+z)]\} \text{ [172]}$$

$$w_x(z) = w_0 + b \sin[\ln(1+z)] \text{ [172]}$$

$$w_x(z) = w_0 + b \left[\frac{\sin(1+z)}{1+z} - \sin 1 \right] \text{ [172]}$$

$$w_x(z) = w_0 + b \left(\frac{z}{1+z} \right) \cos(1+z) \text{ [172]}$$

$$w(z) = w_0 + w_a \left[\frac{\ln(2+z)}{1+z} - \ln 2 \right] \quad [173]$$

$$w(z) = w_0 + w_a \left[\frac{\ln(\alpha+1+z)}{\alpha+z} - \frac{\ln(\alpha+1)}{\alpha} \right] \quad [174]$$

Energy density ρ

$$\rho = \rho_c \quad [175], [176]$$

$$\rho \sim \theta^2 \quad [177]$$

$$\rho = \frac{A}{a^4} \sqrt{a^2 + b} \quad [178]$$

$$(\rho + 3p) a^3 = A \quad [179]$$

$$\rho + p = \rho_c \quad [180]$$

$$\rho_{de}(z) = \rho_{de}(0) \left[1 + \alpha \left(\frac{z}{1+z} \right)^n \right] \quad [181]$$

$$\rho_{de}(z) = \frac{1}{\rho_\phi} \left(\frac{d\rho_\phi}{d\phi} \right) = -\frac{\alpha a}{(\beta+a)^2} \quad [182]$$

$$\rho_{de}(z) = \alpha H(z) \quad [183]$$

$$\rho_{de}(z) = \alpha H(z) + \beta H^2(z) \quad [183]$$

$$\rho_{de}(z) = \frac{3}{\kappa^2} [\alpha + \beta H^2(z)] \quad [183]$$

$$\rho_{de}(z) = \frac{3}{\kappa^2} \left[\alpha + \frac{2}{3} \beta \dot{H}(z) \right] \quad [183]$$

$$\rho_{de}(z) = \frac{3}{\kappa^2} \left[\alpha H^2(z) + \frac{2}{3} \beta \dot{H}(z) \right] \quad [183]$$

$$\rho_{de}(z) = \frac{3}{\kappa^2} \left[\alpha + \beta H^2(z) + \frac{2}{3} \gamma \dot{H}(z) \right] \quad [183]$$

$$\rho_{de}(z) = \rho_{\phi 0} (1+z)^\alpha e^{\beta z} \quad [184]$$

Cosmological constant (Λ)

In order to resolve the long standing cosmological constant problem, authors have considered some variation laws for the cosmological constant in the past forty years, commonly known as “ Λ -varying cosmologies” or “Decaying vacuum cosmologies”. Later the idea was adopted to explain the accelerated expansion of the Universe considering varying Λ . Following is list of such decay laws of Λ .

$$\Lambda \sim a^{-n} \quad [185]$$

$$\Lambda \sim H^n \quad [186]$$

$$\Lambda \sim \rho \quad [185]$$

$$\Lambda \sim t^n \quad [186]$$

$$\Lambda \sim q^n \quad [186]$$

$$\Lambda \sim e^{-\beta a} \quad [187]$$

$$\Lambda = \Lambda(T) \quad [188] \quad T \text{ is Temperature}$$

$$\Lambda \sim C + e^{-\beta t} \quad [189]$$

$$\Lambda = 3\beta H^2 + \alpha a^{-2} \quad [190]$$

$$\Lambda = \beta \frac{\ddot{a}}{a} \quad [191]$$

$$\Lambda = 3\beta H^2 + \alpha \frac{\ddot{a}}{a} \quad [192]$$

$$\frac{d\Lambda}{dt} \sim \beta \Lambda - \Lambda^2 \quad [193]$$

Scalar field Potentials $V(\phi)$

$$V(\phi) = V_0 \phi^n \quad [14] \quad (\text{Power law})$$

$$V(\phi) = V_0 \exp \left[-\frac{\alpha \phi}{M_{pl}} \right] \quad [14] \quad (\text{exponential})$$

$$V(\phi) = \frac{V_0}{\cosh[\phi/\phi_0]} \quad [14]$$

$$V(\phi) = V_0 [\cosh(\alpha \phi / M_{pl})]^{-\beta} \quad (\text{hyperbolic}) \quad [14]$$

$$V(\phi) = \frac{\alpha}{\phi^n} \quad (\text{Inverse power law}) \quad [194]$$

$$V(\phi) = \frac{V_0}{1 + \beta \exp(-\alpha \kappa \phi)} \quad (\text{Woods-Saxon potential}) \quad [195]$$

$$V(\phi) = \alpha c^2 \left[\tanh \frac{\phi}{\sqrt{6}\alpha} \right]^2 \quad (\alpha\text{-attractor}) \quad [196]$$

$$V(\phi) = V_0 (1 + \phi^\alpha)^2 \quad [197]$$

$$V(\phi) = V_0 \exp(\alpha \phi^2) \quad [197]$$

$$V(\phi) = \frac{1}{4} (\phi^2 - 1)^2 \quad [198]$$

Note: All the parametrization listed above contain some arbitrary constants such as $\alpha, \beta, \gamma, m, n, p, q_0, q_1, q_2, w_0, w_1, A, B$ are model parameters which are generally constrained through observational datasets or through any

analytical methods and also some arbitrary functions $f_1(t)$, $f_2(t)$. t_s denote the bouncing time or future singularity time and t_* some arbitrary time.

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