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Embedding hypercubes into cylinders, snakes and caterpillars for minimizing wirelength

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ABSTRACT

We consider the problem of embedding hypercubes into cylinders to minimize the wirelength. Further, we show that the edge isoperimetric problem solves the wirelength problem of regular graphs and, in particular, hypercubes into triangular snakes and caterpillars.

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1. Introduction

An interconnection network is modeled by a simple graph whose vertices represent components of the network and whose edges represent physical communication links. Conversely, any graph can also be considered as a topological structure of some interconnection network. Topologically, graphs and interconnection networks are the same things. In interconnection networks, the simulation of one architecture by another is important. The problem of simulating one network by another is modeled as a graph embedding problem. There are several different reasons why such an embedding is important [26,27,29,32].

Let $G(V, E)$ and $H(V, E)$ be finite graphs with n vertices. An embedding f of G into H is defined [3] as follows:

1. f is a bijective map from $V(G) \to V(H)$
2. f is a one-to-one map from $E(G)$ to $\{P_f(f(u), f(v)) \ : \ P_f(f(u), f(v))$ is a path in H between $f(u)$ and $f(v)$ for $(u, v) \in E(G)\}$.

The *edge congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(G, H(e))$ denote the number of edges (u, v) of G such that e is in the path $P_f(f(u), f(v))$ between $f(u)$ and $f(v)$ in H. In other words,

$$EC_f(G, H(e)) = \left|\left\{(u, v) \in E(G) : e \in P_f(f(u), f(v))\right\}\right|$$

where $P_f(f(u), f(v))$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

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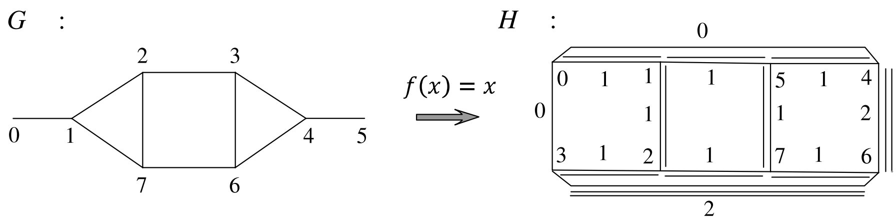

Fig. 1. Wiring diagram of a shuffle-exchange network G into a cube H with $WL_f(G, H) = 12$. The edge congestions are marked on the edges of H.

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G, H)$ is the minimum, over all embeddings $f : V(G) \to V(H)$, of the maximum number of wires that cross any edge of H [4].

The *wirelength* [25] of an embedding f of G into H is given by

$$WL_f(G, H) = \sum_{(u,v)\in E(G)} d_H(f(u), f(v)) = \sum_{e\in E(H)} EC_f(G, H(e))$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(f(u), f(v))$ in H. See Fig. 1. Then, the *wirelength* of G into H is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings f of G into H. The *wirelength problem* [3,4,10,25,27,28] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G, H)$.

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [23,32].

Grid embedding plays an important role in computer architecture. VLSI layout problem [6], crossing number problem [13], edge embedding problem [17] are all a part of grid embedding. Embedding problems have been considered for binary trees into paths [23], binary trees into hypercubes [14], complete binary trees into hypercubes [1], incomplete hypercubes in books [15], tori and grids into twisted cubes [22], meshes into locally twisted cubes [19], meshes into faulty crossed cubes [34], meshes into crossed cubes [16], generalized ladders into hypercubes [8], grids into grids [30], binary trees into grids [27], hypercubes into cycles [10,18], star graph into path [33], snarks into torus [31], generalized wheels into arbitrary trees [28], and hypercubes into grids [25].

Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [3,10]. The embeddings discussed in this paper produce exact wirelength.

2. Edge isoperimetric problem

The following two versions of the edge isoperimetric problem of a graph $G(V, E)$ have been considered in the literature [5], which is NP-complete [17].

Problem 1. Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A\subseteq V, |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u, v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $\theta_G(m) = |\theta_G(A)|$.

Problem 2. Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A\subseteq V, |A|=m} |I_G(A)|$ where $I_G(A) = \{(u, v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$.

For a given m, where $m = 1, 2, \ldots, n$, we consider the problem of finding a subset A of vertices of G such that $|A| = m$ and $|\theta_G(A)| = \theta_G(m)$. Such subsets are called optimal. We say that optimal subsets are nested if there exists a total order $\mathcal{O}$ on the set V such that for any $m = 1, 2, \ldots, n$, the collection of the first m vertices in this order is an optimal subset. In this case, we call the order $\mathcal{O}$ an optimal order [5,20]. This implies that $WL(G, P_n) = \sum_{m=0}^{n} \theta_G(m)$.

Further, if a subset of vertices is optimal with respect to Problem 1, then its complement is also an optimal set. But, it is not true for Problem 2 in general. However for regular graphs, a subset of vertices S is optimal with respect to Problem 1 if and only if S is optimal for Problem 2 [5]. In the literature, Problem 2 is defined as the maximum subgraph problem.

The hypercube is one of the most popular, versatile and efficient topological structures of interconnection networks. The hypercube has many excellent features and thus becomes the first choice of topological structure of parallel processing and computing systems. The machine based on hypercubes such as the Cosmic Cube from Caltech, the iPSC/2 from Intel and Connection Machines have been implemented commercially [12].

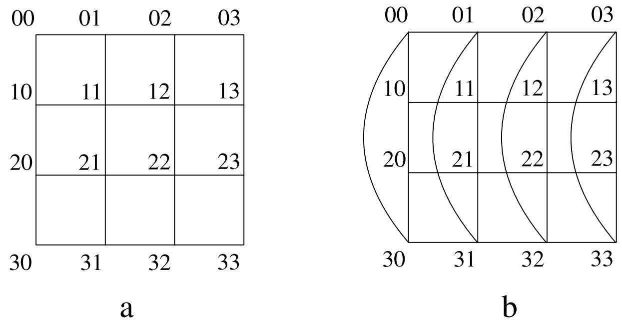

Fig. 2. (a) Grid $P_4 \times P_4$. (b) cylinder $C_4 \times P_4$.

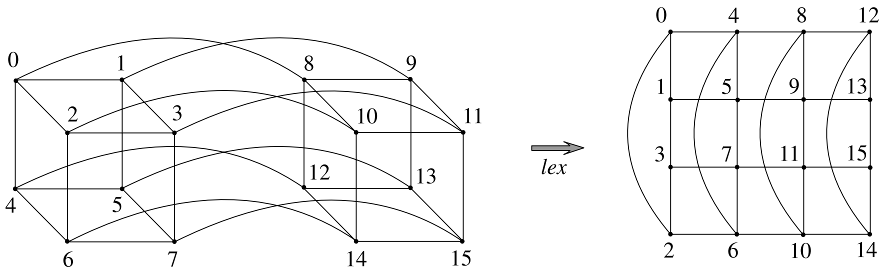

Fig. 3. The lexicographic embedding *lex* of Q^4 into $C_4 \times P_4$.

Definition 1 ([32]). For $r \geq 1$, let Q^r denote the graph of r-dimensional hypercube. The vertex set of Q^r is formed by the collection of all r-dimensional binary strings. Two vertices $x, y \in V(Q^r)$ are adjacent if and only if the corresponding binary strings differ exactly in one bit.

Equivalently, if $n = 2^r$, then the vertices of Q^r can also be identified with integers $0, 1, \ldots, n-1$ so that if a pair of vertices i and j are adjacent, then $i - j = \pm 2^p$ for some $p \geq 0$.

Definition 2 ([7]). A set of m vertices of Q^r is said to be a composite set if the number of edges of the subgraph induced by these m vertices is not less than the number of edges of a subgraph induced by any other set of m vertices of Q^r. A composite hypercube of Q^r is defined to be a subgraph of Q^r, which is induced by some composite set of Q^r.

Definition 3 ([21]). An incomplete hypercube on i vertices of Q^r is the subcube induced by $\{0, 1, \ldots, i-1\}$ and is denoted by L_i, $1 \leq i \leq 2^r$.

Theorem 1 ([7,11,20]). *Let Q^r be an r-dimensional hypercube. For $1 \leq i \leq 2^r$, L_i is a composite set.* □

Lemma 1 ([2,25]). *Let Q^r be an r-dimensional hypercube. Let $m = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_l}$ such that $r \geq t_1 > t_2 > \cdots > t_l \geq 0$. Then $|E(Q^r[L_m])| = [t_1 \cdot 2^{t_1-1} + t_2 \cdot 2^{t_2-1} + \cdots + t_l \cdot 2^{t_l-1}] + [2^{t_2} + 2 \cdot 2^{t_3} + \cdots + (l-1)2^{t_l}]$.* □

3. Wirelength of hypercubes in cylinders

Definition 4. The 2-dimensional grid is defined as $P_{d_1} \times P_{d_2}$, where $d_i \geq 2$ is an integer for each $i = 1, 2$. The cylinder $C_{d_1} \times P_{d_2}$, where $d_1, d_2 \geq 3$ is a $P_{d_1} \times P_{d_2}$ grid with a wraparound edge in each column. See Fig. 2.

It is clear that the vertex set of $P_{d_1} \times P_{d_2}$ is $V = \{x_1x_2 : 0 \leq x_i \leq d_i - 1,\ i = 1, 2\}$ and two vertices $x = x_1x_2$ and $y = y_1y_2$ are linked by an edge, if $|x_1 - y_1| + |x_2 - y_2| = 1$.

Lexicographic embedding. The lexicographic embedding [3] of Q^r with the labeling 0 to $2^r - 1$ into $C_4 \times P_{2^{r-2}}$ is an assignment of label to the vertex x_1x_2 of $C_4 \times P_{2^{r-2}}$ as

$$\begin{aligned} &x_1 + 4x_2 \quad \text{if } x_1 = 0, 1, \\ &3 + 4x_2 \quad \text{if } x_1 = 2, \\ &2 + 4x_2 \quad \text{if } x_1 = 3, \end{aligned}$$

where $0 \leq x_2 \leq 2^{r-2} - 1$. This lexicographic embedding is denoted by *lex*. See Fig. 3.

Lemma 2. *Let*

$$R_1^{lex} = \{0, \quad 1\times 4, \quad 2\times 4, \quad \cdots \quad (2^{r-2}-1)\times 4, \quad 1, \quad 1\times 4+1, \quad 2\times 4+1, \quad \cdots \quad (2^{r-2}-1)\times 4+1\}.$$

Then R_1^{lex} is a composite set in Q^r.

Proof. Define $\varphi : R_1^{lex} \to L_{2^{r-1}}$ by $\varphi(k\times 4+l) = l\times 2^{r-2}+k$. If the binary string of $k\times 4+l$ is $\alpha_1\alpha_2\cdots\alpha_{r-2}\beta_1\beta_2$, then the binary string of $l\times 2^{r-2}+k$ is $\beta_1\beta_2\alpha_1\alpha_2\cdots\alpha_{r-2}$. Thus the binary string of two numbers x and y differ in exactly one bit $\Leftrightarrow$ the binary string of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore (x, y) is an edge in $R_1^{lex} \Leftrightarrow (\varphi(x), \varphi(y))$ is an edge in $L_{2^{r-1}}$. Hence R_1^{lex} and $L_{2^{r-1}}$ are isomorphic. By Theorem 1, R_1^{lex} is a composite set in Q^r. $\square$

Lemma 3. *Let*

$$R_2^{lex} = \{0, \quad 1\times 4, \quad 2\times 4, \quad \cdots \quad (2^{r-2}-1)\times 4, \quad 2, \quad 1\times 4+2, \quad 2\times 4+2, \quad \cdots \quad (2^{r-2}-1)\times 4+2\}.$$

Then R_2^{lex} is a composite set in Q^r.

Proof. Define $\varphi : R_2^{lex} \to L_{2^{r-1}}$ by $\varphi(k\times 4+l\times 2) = k\times 2+l$. If the binary string of $k\times 4+l\times 2$ is $\alpha_1\alpha_2\cdots\alpha_{r-2}\beta_1\beta_2$, then the binary string of $k\times 2+l$ is $\beta_2\alpha_1\alpha_2\cdots\alpha_{r-2}\beta_1$. Thus the binary string of two numbers x and y differ in exactly one bit $\Leftrightarrow$ the binary string of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore (x, y) is an edge in $R_2^{lex} \Leftrightarrow (\varphi(x), \varphi(y))$ is an edge in $L_{2^{r-1}}$. Hence R_2^{lex} and $L_{2^{r-1}}$ are isomorphic. By Theorem 1, R_2^{lex} is a composite set in Q^r. $\square$

As a consequence of Theorem 1, we have the following result.

Lemma 4. *For $j = 1, 2, \ldots, 2^{r-2}$, $C_j^{lex} = \{0, 1, \ldots, 4j-1\}$ is a composite set in Q^r.* $\square$

Notation: $EC_f(G, H(e))$ will be represented by $EC_f(e)$. For any set S of edges of H, $EC_f(S) = \sum_{e\in S} EC_f(e)$.

Lemma 5 (*Congestion Lemma [25]*). *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

(i) For every edge $(a, b) \in G_i$, $i = 1, 2$, *$P_f(f(a), f(b))$ has no edges in S.*
(ii) For every edge (a, b) *in G with $a \in G_1$ and $b \in G_2$, $P_f(f(a), f(b))$ has exactly one edge in S.*
(iii) *G_1 is an optimal set.*

Then $EC_f(S)$ is minimum and $EC_f(S) = r\,|V(G_1)| - 2\,|E(G_1)|$. $\square$

Lemma 6 (*Partition Lemma [25]*). *Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then*

$$WL_f(G, H) = \sum_{i=1}^{p} EC_f(S_i). \quad \square$$

Lemma 7 (*k-partition Lemma [24]*). *Let $f : G \to H$ be an embedding. Let $[kE(H)]$ denote a collection of edges of H with each edge in H repeated exactly k times. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $[kE(H)]$ such that each S_i is an edge cut of H. Then*

$$WL_f(G, H) = \frac{1}{k}\sum_{i=1}^{p} EC_f(S_i). \quad \square$$

Theorem 2 ([20,25]). $WL(Q^r, P_{2^r}) = 2^{2r-1} - 2^{r-1}$. $\square$

Lemma 8. *The lexicographic embedding lex of hypercube Q^r into cylinder $C_4\times P_{2^{r-2}}$ induces a minimum wirelength $WL(Q^r, C_4\times P_{2^{r-2}})$.*

Proof. Let A_i be an edge cut of the cylinder $C_4\times P_{2^{r-2}}$ such that A_i disconnects $C_4\times P_{2^{r-2}}$ into two components X_i and X_i' where $V(X_i)$ is R_i^{lex}, $i = 1, 2$. Let B_j be an edge cut of the cylinder $C_4\times P_{2^{r-2}}$ such that B_j disconnects $C_4\times P_{2^{r-2}}$ into two components Y_j and Y_j' where $V(Y_j)$ is C_j^{lex}, $j = 1, 2, \ldots, 2^{r-2}-1$. See Fig. 4. Let G_i and G_i' be the inverse images of X_i and X_i' under *lex* respectively. The edge cut A_i satisfies conditions (i) and (ii) of the Congestion Lemma. Further, by Lemmas 2 and 3, the subgraph G_i induced by the vertices of R_i^{lex} is a composite set. Thus by the Congestion Lemma, $EC_{lex}(A_i)$ is minimum for $i = 1, 2$. Similarly, let G_j and G_j' be the inverse images of Y_j and Y_j' under *lex* respectively. By Lemma 4, G_j is a composite set induced by the vertices of C_j^{lex}. Thus the edge cut B_j satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore, $EC_{lex}(B_j)$ is minimum for $j = 1, 2, \ldots, 2^{r-2}-1$. The Partition lemma implies that $WL_{lex}(Q^r, C_4\times P_{2^{r-2}})$ is minimum. $\square$

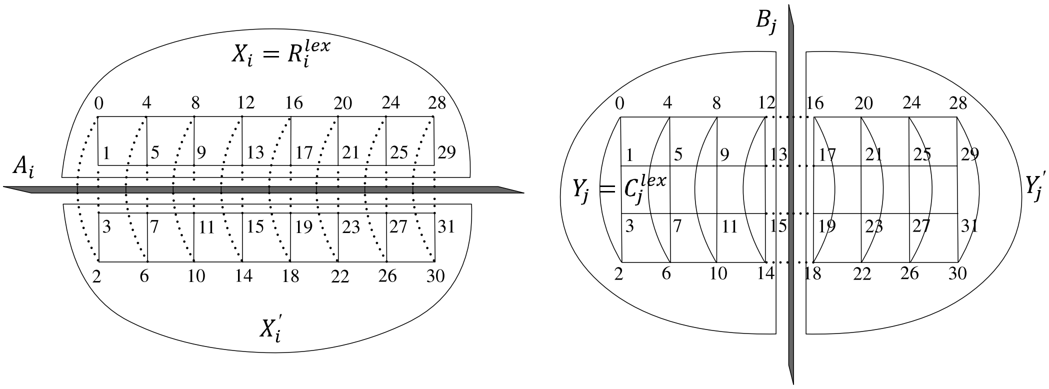

Fig. 4. (a) Each A_i is an edge cut of $C_4 \times P_{2^3}$ which disconnects $C_4 \times P_{2^3}$ into two components X_i and X_i' where $V(X_i)$ is R_i^{lex}. (b) Each B_j is an edge cut of $C_4 \times P_{2^3}$ which disconnects $C_4 \times P_{2^3}$ into two components Y_j and Y_j' where $V(Y_j)$ is C_j^{lex}.

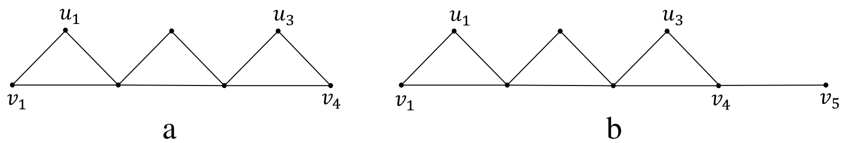

Fig. 5. Triangular snake (a) ΔS_7. (b) ΔS_8.

Theorem 3. *The exact wirelength of Q^r into $C_4 \times P_{2^{r-2}}$ is given by*

$$WL(Q^r, C_4 \times P_{2^{r-2}}) = 2^{2r-3} + 2^{r-1}.$$

Proof. By the symmetric property of the lexicographic embedding, the edges of Q^r are stretched in the cylinder $C_4 \times P_{2^{r-2}}$ either vertically or horizontally. Therefore, each edge of the edge cut A_i has the same edge congestion. Also, each edge of the edge cut B_j has the same edge congestion. Hence the sum of the edge congestions in each row (resp. column) is the same. Since each column is isomorphic to Q^2, the wirelength of each column is 4. By Theorem 2, the wirelength of each row is $2^{2r-5} - 2^{r-3}$. Therefore, $WL(Q^r, C_4 \times P_{2^{r-2}}) = 4(2^{2r-5} - 2^{r-3}) + 4(2^{r-2}) = 2^{2r-3} + 2^{r-1}$. □

It is claimed in Guu's Ph.D. dissertation [18], that the Greycode numbering minimizes cyclic wirelength of hypercubes and $WL(Q^r, C_{2^r}) = 3 \times 2^{2r-3} - 2^{r-1}$. Using the proof techniques followed in this paper, we have the following conjecture.

Conjecture 1. $WL(Q^r, C_{2^{r_1}} \times P_{2^{r_2}}) = 2^{r_1}(2^{2r_2-1} - 2^{r_2-1}) + 2^{r_2}(3 \times 2^{2r_1-3} - 2^{r_1-1})$, *where* $r_1 + r_2 = r, r_1 \le r_2$. □

Conjecture 2. $WL(Q^r, C_{2^{r_1}} \times C_{2^{r_2}}) = 2^{r_1}(3 \times 2^{2r_2-3} - 2^{r_2-1}) + 2^{r_2}(3 \times 2^{2r_1-3} - 2^{r_1-1})$, *where* $r_1 + r_2 = r, r_1 \le r_2$. □

4. Wirelength of hypercubes in triangular snakes

A triangular cactus is a connected graph all of whose blocks are triangles. A triangular snake is a triangular cactus whose block-cutpoint-graph is a path.

Definition 5. A triangular snake ΔS_n with n vertices is a graph obtained from a path $v_1, v_2, \ldots, v_{\lfloor n/2 \rfloor+1}$ by joining v_i and v_{i+1} to a new vertex u_i for $i = 1, 2, \ldots, \lceil n/2 \rceil - 1$. See Fig. 5.

Embedding algorithm A.

Input: An r-regular graph G with optimal order and a triangular snake ΔS_n.

Algorithm: Label the vertices of G using optimal order and the vertices of ΔS_n as follows: Label the vertex v_i as $2i - 2$ for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$, $v_{\lfloor n/2 \rfloor+1}$ as $n - 1$ and u_i as $2i - 1$ for $i = 1, 2, \ldots, \lceil n/2 \rceil - 1$.

Output: An embedding f of G into ΔS_n given by $f(x) = x$ with minimum wirelength.

Proof of correctness. We consider two cases.

Case 1 (*n odd*): Let $S_i = \{(2i-2, 2i-1), (2i-2, 2i)\}$, $S_i' = \{(2i-1, 2i), (2i-2, 2i)\}$ and $S_i'' = \{(2i-2, 2i-1), (2i-1, 2i)\}$, $1 \le i \le \lfloor n/2 \rfloor$. See Fig. 6. The edge set $\{S_i, (2i-1, 2i) : 1 \le i \le \lfloor n/2 \rfloor\}$ constitutes all the edges of ΔS_n exactly once. Similarly, the edge set $\left\{S_i', (2i-2, 2i-1) : 1 \le i \le \lfloor n/2 \rfloor\right\}$ constitutes all the edges of ΔS_n exactly once.

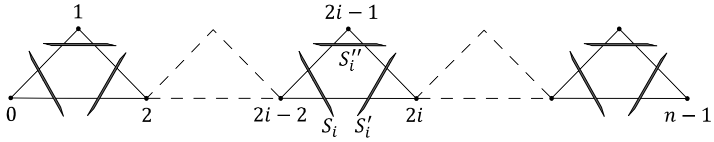

Fig. 6. The edge cuts of triangular snake ΔS_n, n is odd.

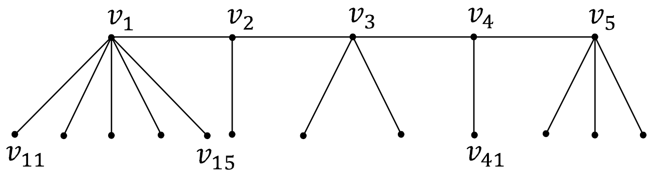

Fig. 7. Caterpillar CAT(5, 1, 2, 1, 3).

Thus, $\{S_i, S_i', S_i'' : 1 \le i \le \lfloor n/2 \rfloor\}$ is a partition of $[2E(\Delta S_n)]$. For each i, $E(\Delta S_n) \setminus S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{0, 1, \ldots, 2i-2\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. Since G_{i1} is an optimal set, each S_i satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum. Similarly $EC_f(S_i')$ is minimum. For each i, $E(\Delta S_n) \setminus S_i''$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{2i-1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. Since G_{i1} is a vertex of G, each S_i'' satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(S_i'')$ is minimum. The 2-partition Lemma implies that the wirelength is minimum.

Case 2 (*n even*): Let $S_i = \{(2i-2, 2i-1), (2i-2, 2i)\}$, $S_i' = \{(2i-1, 2i), (2i-2, 2i)\}$, $S_i'' = \{(2i-2, 2i-1), (2i-1, 2i)\}$, $1 \le i \le \lceil n/2 \rceil - 1$ and $S_{\lceil n/2 \rceil} = S'_{\lceil n/2 \rceil} = \{(n-2, n-1)\}$. The edge set $\{S_i, S_{\lceil n/2 \rceil}, (2i-1, 2i) : 1 \le i \le \lceil n/2 \rceil - 1\}$ constitutes all the edges of ΔS_n exactly once. Similarly, the edge set $\{S_i', S'_{\lceil n/2 \rceil}, (2i-2, 2i-1) : 1 \le i \le \lceil n/2 \rceil - 1\}$ constitutes all the edges of ΔS_n exactly once. Thus $\{S_i, S_i', S_i'', S_{\lceil n/2 \rceil}, S'_{\lceil n/2 \rceil} : 1 \le i \le \lceil n/2 \rceil - 1\}$ is a partition of $[2E(\Delta S_n)]$. For each i, $1 \le i \le \lceil n/2 \rceil - 1$, $E(\Delta S_n) \setminus S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{0, 1, \ldots, 2i-2\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. Since G_{i1} is an optimal set, each S_i satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum. Similarly $EC_f(S_i')$ is minimum. For each i, $1 \le i \le \lceil n/2 \rceil - 1$, $E(\Delta S_n) \setminus S_i''$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{2i-1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. Since G_{i1} is a vertex of G, each S_i'' satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(S_i'')$ is minimum. Similarly $EC_f(S_{\lceil n/2 \rceil}) = EC_f(S'_{\lceil n/2 \rceil})$ is minimum. The 2-partition Lemma implies that the wirelength is minimum. □

Theorem 4. *The exact wirelength of an r-regular graph G with optimal order into a triangular snake ΔS_n is given by*

$$WL(G, \Delta S_n) = \frac{1}{2}\left\{WL(G, P_n) + r\lfloor n/2 \rfloor\right\}.$$

Proof. Following the notation used in Embedding Algorithm A, we divide the proof into two cases.

Case 1 (*n odd*): By edge isoperimetric problem [20], $\sum_{i=1}^{\lfloor n/2 \rfloor}[EC_f(S_i) + EC_f(S_i')] = WL(G, P_n)$ and by Lemma 5, $EC_f(S_i'') = r$. Therefore $WL(G, \Delta S_n) = \frac{1}{2}\{WL(G, P_n) + r\lfloor n/2 \rfloor\}$.

Case 2 (*n even*): By edge isoperimetric problem [20], $\sum_{i=1}^{\lceil n/2 \rceil - 1}[EC_f(S_i) + EC_f(S_i')] + EC_f(S_{\lceil n/2 \rceil}) = WL(G, P_n)$ and by Lemma 5, $EC_f(S_i'') = EC_f(S'_{\lceil n/2 \rceil}) = r$. Therefore $WL(G, \Delta S_n) = \frac{1}{2}\{WL(G, P_n) + r\lceil n/2 \rceil\}$. □

As a consequence of Theorems 2 and 4, we have the following result.

Theorem 5. *The exact wirelength of Q^r into ΔS_{2^r} is given by*

$$WL(Q^r, \Delta S_{2^r}) = 2^{2r-2} + (r-1)2^{r-2}. \quad \square$$

5. Wirelength of hypercubes in caterpillars

A tree is called a caterpillar if the deletion of vertices of degree one leaves a path. This path is called a spine.

Definition 6. Let $m \ge 1$, and k_i $(i = 1, 2, \ldots, m)$ be non-negative integers such that $m + k_1 + \cdots + k_m \ge 3$. A tree which is obtained from a path $v_1, v_2, \ldots, v_m$ by joining v_i to new vertices v_{ij} $(j = 1, 2, \ldots, k_i)$ is called a caterpillar CAT$(k_1, k_2, \ldots, k_m)$. It has $m + k_1 + \cdots + k_m$ vertices. See Fig. 7.

Embedding Algorithm B.

Input: An r-regular graph G with optimal order and a caterpillar $\mathrm{CAT}(k_1, k_2, \ldots, k_m)$.

Algorithm: Label the vertices of G using optimal order and the vertices of $\mathrm{CAT}(k_1, k_2, \ldots, k_m)$ as follows: Label the vertex v_i as $k_1+k_2+\cdots+k_{i-1}+(i-1)$ for $i = 1, 2, \ldots, m$ and v_{ij} as $k_1+k_2+\cdots+k_{i-1}+(i-1)+j$ for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, k_i$, where $k_0 = 0$.

Output: An embedding f of G into $\mathrm{CAT}(k_1, k_2, \ldots, k_m)$ given by $f(x) = x$ with minimum wirelength.

Proof of correctness. Let $S_i = \{(k_1 + k_2 + \cdots + k_{i-1} + i - 1, k_1 + k_2 + \cdots + k_i + i)\}$, $1 \le i \le m - 1$, where $k_0 = 0$. Let $S_i^j = \{(k_1 + k_2 + \cdots + k_{i-1} + i - 1, k_1 + k_2 + \cdots + k_{i-1} + i - 1 + j)\}$, $1 \le i \le m, j = 1, 2, \ldots, k_i$, where $k_0 = 0$. Thus, $\{S_i : 1 \le i \le m - 1\} \cup \{S_i^j : 1 \le i \le m,\ j = 1, 2, \ldots, k_i\}$ is a partition of $E(\mathrm{CAT}(k_1, k_2, \ldots, k_m))$. For each i, $1 \le i \le m-1$, $E(\mathrm{CAT}(k_1, k_2, \ldots, k_m)) \setminus S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{0, 1, \ldots, k_1+k_2+\cdots+k_i+i-1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. Since G_{i1} is an optimal set, each S_i satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore, $EC_f(S_i)$ is minimum. For each i, j, $1 \le i \le m$, $1 \le j \le k_i$, $E(\mathrm{CAT}(k_1, k_2, \ldots, k_m)) \setminus S_i^j$ has two components H_{i1}^j and H_{i2}^j, where $V(H_{i1}^j) = \{k_1 + k_2 + \cdots + k_{i-1} + i - 1 + j\}$, $k_0 = 0$. Let $G_{i1}^j = f^{-1}(H_{i1}^j)$ and $G_{i2}^j = f^{-1}(H_{i2}^j)$. Since G_{i1}^j is a vertex of G, each S_i^j satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore, $EC_f(S_i^j)$ is minimum. The Partition Lemma implies that the wirelength is minimum. □

Theorem 6. *The exact wirelength of an r-regular graph G with optimal order into* $\mathrm{CAT}(k_1, k_2, \ldots, k_m)$ *is given by*

$$WL(G, \mathrm{CAT}(k_1, k_2, \ldots, k_m)) = \sum_{i=1}^{m-1} \theta_G(k_1 + k_2 + \cdots + k_i + i) + r(k_1 + k_2 + \cdots + k_m).$$

Proof. Following the notation used in Embedding Algorithm B, we have $EC_f(S_i) = \theta_G(k_1+k_2+\cdots+k_i+i)$, $1 \le i \le m-1$ and $EC_f(S_i^j) = r$, $1 \le i \le m$, $1 \le j \le k_i$. Therefore, $WL(G, \mathrm{CAT}(k_1, k_2, \ldots, k_m)) = \sum_{i=1}^{m-1} \theta_G(k_1+k_2+\cdots+k_i+i) + r(k_1 + k_2 + \cdots + k_m)$. □

As a consequence of Lemma 1 and Theorem 6, we have the following result.

Theorem 7. *The exact wirelength of Q^r into* $\mathrm{CAT}(k_1, k_2, \ldots, k_m)$, $n = 2^r = m + k_1 + \cdots + k_m$ *is given by*

$$WL(Q^r, \mathrm{CAT}(k_1, k_2, \ldots, k_m)) = \sum_{i=1}^{m-1} \left\{ r(k_1 + \cdots + k_i + i) - 2\left|E(Q^r[L_{k_1+\cdots+k_i+i}])\right| \right\} + r(k_1 + k_2 + \cdots + k_m). \quad \square$$

6. Conclusion

In this paper, we compute the exact wirelength of hypercubes into cylinders. We also prove that the edge isoperimetric problem solves the wirelength problem of regular graphs and, in particular, hypercubes into triangular snakes and caterpillars. Further, this solves the wirelength problem of powers of the Petersen graph [5] and discrete tori [9] into triangular snakes and caterpillars.

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