Discrete Applied Mathematics 169 (2014) 9–14

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Embedding of hypercubes into sibling trees

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ARTICLE INFO

Article history:
Received 21 April 2012
Received in revised form 21 December 2013
Accepted 7 January 2014
Available online 28 January 2014

Keywords:
Hypercubes
Embedding
Congestion
Wirelength
Sibling trees

ABSTRACT

The aim of this paper is to generalize the Congestion Lemma, which has been considered an efficient tool to compute the minimum wirelength (Manuel et al., 2009) and thereby obtain the minimum wirelength of embedding hypercubes into sibling trees.

1. Introduction

Embeddings are of great importance in the applications of parallel computing. Every parallel application has its intrinsic communication pattern. The communication pattern graph is mapped onto the topology of multiprocessor structures so that the corresponding application can be executed with minimal communication overhead.

A graph embedding [21] of a *guest* graph G into a *host* graph H is defined by a bijective mapping $f : V(G) \to V(H)$ together with a mapping P_f which assigns to each edge (u, v) of G a path between $f(u)$ and $f(v)$ in H. Let $EC_f(e)$ denote the number of edges (u, v) of G such that e is in the path $P_f((u, v))$ between $f(u)$ and $f(v)$ in H [15]. In other words, $EC_f(e) = \left|\left\{(u, v) \in E(G) : e \in E(P_f((u, v)))\right\}\right|$. See Fig. 1. Let S be a subset of the edge set of H. Then $EC_f(S) = \sum_{e\in S} EC_f(e)$.

The wirelength [9,15] of an embedding f of G into H is given by

$$WL_f(G, H) = \sum_{(u,v)\in E(G)} \left|E(P_f((u, v)))\right| = \sum_{e\in E(H)} EC_f(e) = \sum_{i=1}^{p} EC_f(S_i)$$

where $\{S_1, S_2, \ldots, S_p\}$ is a partition of $E(H)$.

The *minimum wirelength* of embedding G into H is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings f of G into H. Since our aim is to construct embeddings of minimum wirelength, we will take P_f to be a mapping that assigns to each edge (u, v) of G a shortest path between vertices $f(u)$ and $f(v)$ in H.

The wirelength of a graph embedding is used in the study of VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture and

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http://dx.doi.org/10.1016/j.dam.2014.01.002

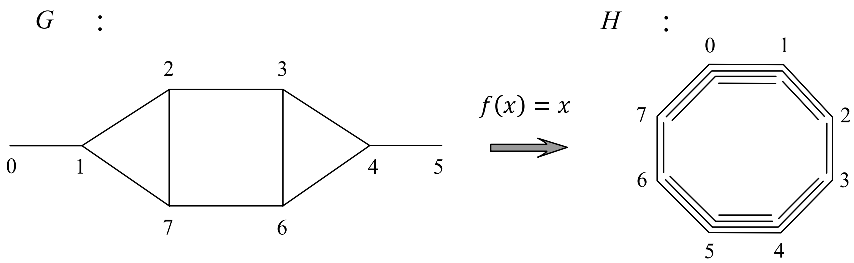

Fig. 1. Wiring diagram of a shuffle–exchange network G into a cycle H with the edge congestions $EC_f((0,1)) = 3$, $EC_f((1,2)) = 2$, $EC_f((2,3)) = 1$, $EC_f((3,4)) = 2$, $EC_f((4,5)) = 3$, $EC_f((5,6)) = 2$, $EC_f((6,7)) = 1$ and $EC_f((7,0)) = 2$.

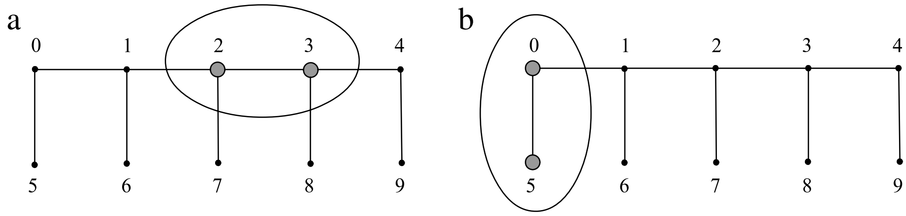

Fig. 2. (a) $A = \{2, 3\}$ is an optimal set with respect to Problem 2 whereas it is not an optimal set to Problem 1 (b) $A = \{0, 5\}$ is an optimal set with respect to Problems 1 and 2.

structural engineering [12,22]. Embedding problems have been considered for complete binary trees into hypercubes [1], tori and grids into twisted cubes [11], meshes into locally twisted cubes [8], meshes into faulty crossed cubes [23], meshes into crossed cubes [6], generalized ladders into hypercubes [3], hypercube into cycles [4], hypercubes into grids [15], hypercubes into cylinders, snakes and caterpillars [14], hypercubes into certain trees [18], m-sequential k-ary trees into hypercubes [20], enhanced and augmented hypercubes into complete binary trees [13], folded hypercubes into grids [16] and circulant into certain graphs [17]. In this paper we generalize the Congestion Lemma [15] and obtain the minimum wirelength of hypercubes into sibling trees.

2. Edge isoperimetric problem

The edge isoperimetric problem [9] is used to solve the wirelength problem. The following two versions of the edge isoperimetric problem of a graph $G(V, E)$ have been considered in the literature [2] and are NP-complete [7].

Problem 1. Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A\subseteq V,\,|A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u, v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ and $|A| = m$ such that $\theta_G(m) = |\theta_G(A)|$.

Problem 2. Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A\subseteq V,|A|=m} |I_G(A)|$ where $I_G(A) = \{(u, v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ and $|A| = m$ such that $I_G(m) = |I_G(A)|$.

We call such a set A optimal [2,9]. If a subset of vertices is optimal with respect to Problem 1, then its complement is also an optimal set. However, it is not true for Problem 2 in general. See Fig. 2. The two discrete problems mentioned above are closely related and for k-regular graphs are equivalent due to the equation $|\theta_G(A)| = k \times |A| - 2 \times |I_G(A)|$, which implies $\theta_G(m) = k \times m - 2 \times I_G(m)$, $m = 1, 2, \ldots, |V|$ [2]. In the literature, Problem 2 is called the maximum subgraph problem [7].

Definition 1 ([22]). For $r \geq 1$, let Q_r denote the r-dimensional hypercube. The vertex set of Q_r is the set of all r-dimensional binary representations. Two vertices $x, y \in V(Q_r)$ are adjacent if and only if the corresponding binary representations differ exactly in one bit.

Definition 2 ([10]). An incomplete hypercube on i vertices of Q_r is the subcube induced by $\{0, 1, \ldots, i-1\}$ and is denoted by L_i, $1 \leq i \leq 2^r$.

Theorem 1 ([9]). *Let Q_r be an r-dimensional hypercube. For $1 \leq i \leq 2^r$, L_i is an optimal set.*

Lemma 1 ([15]). *Let Q_r be an r-dimensional hypercube. Let $m = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_l}$ such that $r > t_1 > t_2 > \cdots > t_l \geq 0$. Then $|E(Q_r[L_m])| = [t_1 \cdot 2^{t_1-1} + t_2 \cdot 2^{t_2-1} + \cdots + t_l \cdot 2^{t_l-1}] + [2^{t_2} + 2 \cdot 2^{t_3} + \cdots + (l-1)2^{t_l}]$.*

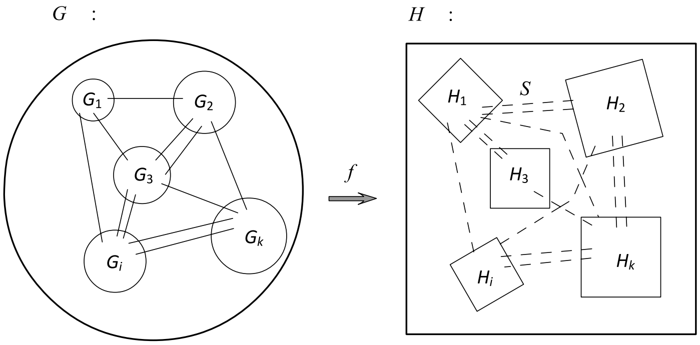

Fig. 3. $f : G \to H$ is an embedding and the dotted lines represent edge cut S in H.

3. Generalized congestion lemma

The wirelength problem of hypercube on a grid has been solved by Manuel et al. [15], using the Congestion Lemma and is given below.

Lemma 2 (*Congestion Lemma [15]*)**.** *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S splits H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also assume that S satisfies the following conditions:*

(i) *For every edge $(a, b) \in G_i, i = 1, 2, P_f((a, b))$* has no edges in S.
(ii) *For every edge (a, b)* in G with $a \in G_1$ and $b \in G_2, P_f((a, b))$ has exactly one edge in S.
(iii) *G_1 is a maximum subgraph on k* vertices where $k = |V(G_1)|$.

Then $EC_f(S)$ is minimum and $EC_f(S) = \sum_{e\in S} EC_f(e) = r\,|V(G_1)| - 2\,|E(G_1)|$.

Although the congestion lemma has been considered an efficient tool in the computation of wirelength, this lemma fails when the edge cut leaves more than two components. This motivates the following result.

Lemma 3 (*Generalized Congestion Lemma*)**.** *Let f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S splits H into k components H_i, $1 \le i \le k$. Let $G_i = G[f^{-1}(H_i)]$, $1 \le i \le k$, be such that the sets G_i are optimal and S satisfies the following conditions:*

(i) *For every edge $(u, v) \in G_i$, $1 \le i \le k, P_f((u, v))$* has no edges in S.
(ii) *For every edge (u, v)* in G with $u \in G_i$ and $v \in G_j$ for $i < j, P_f((u, v))$ has exactly one edge in S.

Then $EC_f(S)$ is minimum over all possible embeddings and $EC_f(S) = \frac{1}{2}\sum_{i=1}^{k} \theta_G(m_i)$ where $m_i = |V(G_i)|$. Further when G is an r-regular graph $EC_f(S) = \frac{r}{2}\,|V(G)| - \sum_{i=1}^{k} |E(G_i)|$.

Proof. Let $X = \{(u, v) \in E(G) : u \in G_i, v \in G_j \text{ for } i < j\}$. By condition (i), no edge of G_i, $1 \le i \le k$, contributes to $EC_f(S)$. By condition (ii), every edge (u, v) of S increments $EC_f(S)$ by 1. Therefore $EC_f(S) = |X|$. For $1 \le i \le k$, G_i is an optimal set and hence $EC_f(S)$ is minimum. We compute $|X|$ in the following way. Let $E(G_i \wedge G_j)$ denote the set of edges in G with one end in G_i and the other end in G_j. See Fig. 3. Then $|X| = \theta_G(m_1) + \theta_G(m_2) - |E(G_1 \wedge G_2)| + \theta_G(m_3) - |E(G_1 \wedge G_3)| - |E(G_2 \wedge G_3)| + \cdots + \theta_G(m_i) - \sum_{j=1}^{i-1} \left|E(G_j \wedge G_i)\right| + \cdots + \theta_G(m_k) - \sum_{j=1}^{k-1} \left|E(G_j \wedge G_k)\right|$ where $m_i = |V(G_i)|$. This implies that $|X| = \sum_{i=1}^{k} \theta_G(m_i) - |X|$. Therefore $|X| = \frac{1}{2}\sum_{i=1}^{k} \theta_G(m_i)$. Further when G is an r-regular graph, $\theta_G(m_i) = r\,|V(G_i)| - 2\,|E(G_i)|$, implies $|X| = \frac{r}{2}\,|V(G)| - \sum_{i=1}^{k} |E(G_i)|$. □

4. Wirelength of hypercubes into sibling trees

The most common type of tree is the binary tree. It is so named because each node can have at most two descendants. A binary tree is said to be a complete binary tree if each internal node has exactly two descendants. These descendants are described as left and right children of the parent node. Binary trees are widely used in data structures because they are easily stored, easily manipulated, and easily retrieved [22].

For any non-negative integer r, the complete binary tree of height r, denoted by T_r, is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. Clearly, a complete binary tree T_r has r levels and level i, $1 \le i \le r$, contains 2^{i-1} vertices. Thus T_r has exactly $2^r - 1$ vertices. The 1-rooted complete binary tree T_r^1 is obtained from a complete binary tree T_r by attaching to its root a pendant edge. The new vertex is called the root of T_r^1 and is considered to be at level 0. A *sibling tree* ST_r is obtained from the 1-rooted complete binary tree T_r^1 by adding edges (sibling edges) between left and right children of the same parent node. See Fig. 4.

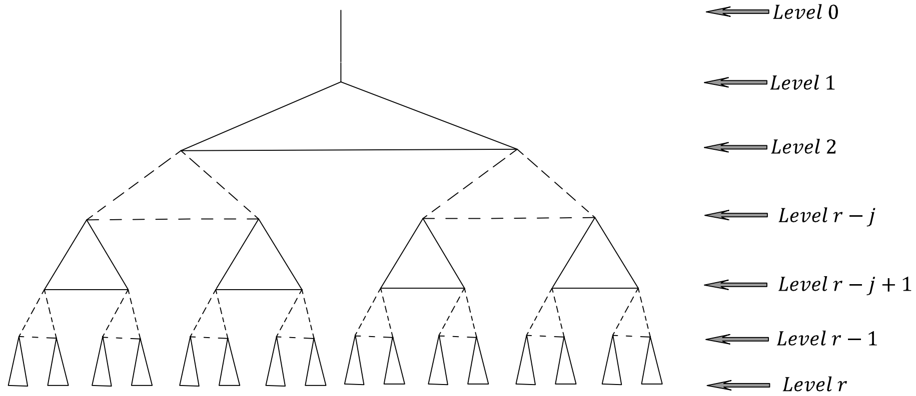

Fig. 4. Sibling tree with levels.

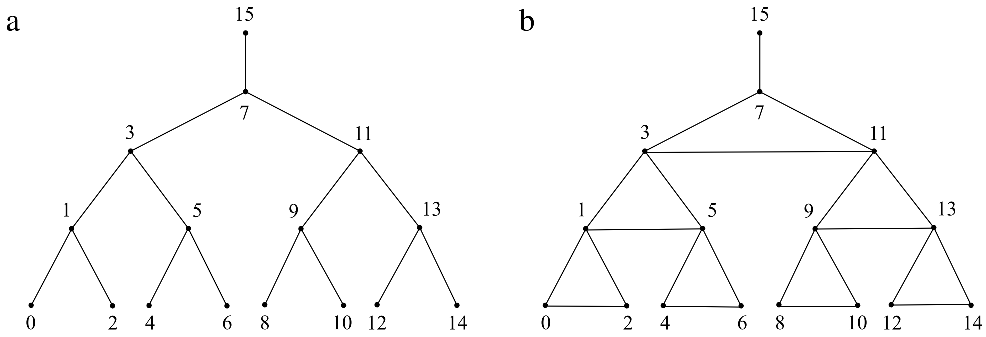

Fig. 5. (a) 1-rooted complete binary tree T_4^1 with inorder labeling (b) Sibling tree ST_4.

There are several useful ways in which we can systematically order all nodes of a tree [5,19]. The three most important ordering are called *preorder*, *inorder* and *postorder*. To achieve these orderings the tree is traversed in a particular fashion. Starting from the root, the tree is traversed counter clockwise staying as close to the tree as possible. For preorder, we list a node the first time we pass it. For inorder, we list a leaf the first time we pass it, but list an interior node the second time we pass it. For postorder, we list a node the last time we pass it.

Embedding Algorithm (LexIn)

Input: The r-dimensional hypercube Q_r and the sibling tree ST_r on 2^r vertices.

Algorithm: Label the vertex $x_1x_2 \ldots x_r$ of Q_r as $\sum_{i=1}^{r} x_i \cdot 2^{r-i}$ and label the vertices of T_r^1 by inorder traversal from 0 to $2^r - 1$. As $V(T_r^1) = V(ST_r)$, we consider the vertex label of ST_r is same as T_r^1. See Fig. 5.

Output: An embedding f of Q_r into ST_r given by $f(x) = x$ with minimum wirelength.

Theorem 2. *The minimum wirelength of Q_r into ST_r is given by*

$$WL(Q_r, ST_r) = 2^{r-1}(r^2 - 4r + 10) - r - 5.$$

Proof. For $j = 1, 2, \ldots, r-1$ and $i = 1, 2, \ldots, 2^{r-j-1}$, let S_{ij} be the edge cut of the sibling tree ST_r consisting of triangles induced by the ith parent vertex from left to right in level $r - j$ with its left and right child such that S_{ij} disconnects ST_r into three components H_{ij}^1, H_{ij}^2 and H_{ij}^3.

We prove that for $j = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, 2^{r-j}$, $Tcut_i^{2^j-1} = \{2^j(i-1), 2^j(i-1)+1, 2^j(i-1)+2, \ldots, 2^j(i-1)+(2^j-2)\}$ is an optimal set in Q_r. Further for $j = 1, 2, \ldots, r-1$ and $i = 1, 2, \ldots, 2^{r-j-1}$, the set $TTcut_i^{2(2^j-1)} = Tcut_{2i-1}^{2^j-1} \cup Tcut_{2i}^{2^j-1}$ is an optimal set in Q_r and also $V(Q_r) \setminus TTcut_i^{2(2^j-1)}$ is an optimal set in Q_r.

Define $\varphi : Tcut_i^{2^j-1} \to L_{2^j-1}$ by $\varphi(2^j(i-1)+k) = k$. If the binary representation of $2^j(i-1)+k$ is $\alpha_1\alpha_2 \ldots \alpha_r$ then the binary representation of k is $\underbrace{00\ldots00}_{r-j \text{ times}}\alpha_{r-j+1}\alpha_{r-j+2} \ldots \alpha_r$. Thus the binary representations of two numbers x and y differ in exactly one bit if and only if the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore (x, y) is an edge in $Tcut_i^{2^j-1}$ if and only if $(\varphi(x), \varphi(y))$ is an edge in L_{2^j-1}. Hence $Tcut_i^{2^j-1}$ and L_{2^j-1} are isomorphic. By Theorem 1, $Tcut_i^{2^j-1}$ is an optimal set in Q_r. By Lemma 1, we have for $j = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, 2^{r-j}$, $\left|E(Q_r[Tcut_i^{2^j-1}])\right| = j(2^{j-1}-1)$. Further, we have

$$\begin{aligned} TTcut_i^{2(2^j-1)} = \{&2^j(2i-2), \quad 2^j(2i-2)+1, \quad 2^j(2i-2)+2, \quad \ldots \quad 2^j(2i-2)+2^j-2, \\ &2^j(2i-1), \quad 2^j(2i-1)+1, \quad 2^j(2i-1)+2, \quad \ldots \quad 2^j(2i-1)+2^j-2\}. \end{aligned}$$

The sets $\{2^j(2i-2), 2^j(2i-2)+1, 2^j(2i-2)+2, \ldots, 2^j(2i-2)+2^j-2\}$ and $\{2^j(2i-1), 2^j(2i-1)+1, 2^j(2i-1)+2, \ldots, 2^j(2i-1)+2^j-2\}$ are isomorphic to L_{2^j-1}. Also the binary representation of $2^j(2i-2)$ and $2^j(2i-1)$ differ exactly in one bit. Therefore $\left|E(Q_n[TTcut_i^{2(2^j-1)}])\right| = 2\left|E(Q_n[L_{2^j-1}])\right| + 2^j - 1 = 2j(2^{j-1}-1) + 2^j - 1 = (j+1)2^j - 2j - 1$. But by Lemma 1, $\left|E(Q_n[L_{2(2^j-1)}])\right| = (j+1)2^j - 2j - 1$ and hence by Theorem 1, $TTcut_i^{2(2^j-1)}$ is an optimal set in Q_r. Since Q_r is an r-regular graph, $V(Q_r) \setminus TTcut_i^{2(2^j-1)}$ is an optimal set in Q_r.

Clearly $V(H_{ij}^1) = Tcut_{2i-1}^{2^j-1}$, $V(H_{ij}^2) = Tcut_{2i}^{2^j-1}$ and $V(H_{ij}^3) = V(Q_r) \setminus TTcut_i^{2(2^j-1)}$. Let $G_{ij}^k = G[f^{-1}(H_{ij}^k)]$, $1 \le k \le 3$. Then the sets G_{ij}^k are optimal in Q_r. Thus the edge cut S_{ij} satisfies the conditions of the Generalized Congestion Lemma. Therefore $EC_f(S_{ij})$ is minimum for $j = 1, 2, \ldots, r-1$ and $i = 1, 2, \ldots, 2^{r-j-1}$. Let S_{1r} be the set containing only the cut edge of the sibling tree ST_r that disconnects ST_r into two components H_{1r} and H_{2r} where $V(H_{1r}) = L_{2^r-1}$ and $V(H_{2r}) = \{2^r-1\}$. Let $G_{1r} = G[f^{-1}(H_{1r})]$ and $G_{2r} = G[f^{-1}(H_{2r})]$. By Theorem 1, G_{1r} is an optimal set in Q_r. Thus S_{1r} satisfies the conditions of the Generalized Congestion Lemma. Therefore $EC_f(S_{1r})$ is minimum. Hence

$$WL(Q_r, ST_r) = \frac{1}{2}\left\{\sum_{j=1}^{r-1}\sum_{i=1}^{2^{r-j-1}}\{2\theta_{Q_r}(2^j-1) + \theta_{Q_r}(2^r-2^{j+1}+2)\} + \theta_{Q_r}(2^r-1) + \theta_{Q_r}(1)\right\}.$$

Since $\theta_{Q_r}(2^r-2^{j+1}+2) = \theta_{Q_r}(2^{j+1}-2)$ and $\theta_{Q_r}(2^r-1) = \theta_{Q_r}(1)$.

We have

$$\begin{aligned} WL(Q_r, ST_r) &= \frac{1}{2}\left\{\sum_{j=1}^{r-1}\sum_{i=1}^{2^{r-j-1}}\{2\theta_{Q_r}(2^j-1) + \theta_{Q_r}(2^{j+1}-2)\} + 2\theta_{Q_r}(1)\right\} \\ &= \frac{1}{2}\left\{\sum_{j=1}^{r-1}\sum_{i=1}^{2^{r-j-1}}\{2r(2^j-1) - 4j(2^{j-1}-1) + 2r(2^j-1) - 2[(j+1)2^j - 2j - 1]\} + 2r\right\} \\ &= 2^{r-1}(r^2 - 4r + 10) - r - 5. \quad \square \end{aligned}$$

5. Concluding remarks

We have obtained the minimum wirelength of hypercubes into sibling trees by generalizing the Congestion Lemma [15] and the interested readers may refer to our earlier publication [18] for alternative proof. Also it is known that the set of vertices $L_i = \{0, 1, \ldots, i-1\}$, $1 \le i \le 2^r$, is optimal in folded hypercubes [16] and hence it is easy to compute the minimum wirelength of folded hypercubes into sibling trees.

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