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ORIGINAL PAPER

Fitted Mesh Method for Singularly Perturbed Robin Type Boundary Value Problem with Discontinuous Source Term

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Abstract In this paper, second order singularly perturbed convection-diffusion Robin type problem with a discontinuous source term is considered. Due to the discontinuity interior layers appears in the solution. A numerical method is constructed for this problem which involves an appropriate piecewise—uniform mesh for the boundary and interior layers. The method is shown to be parameter uniformly convergent with respect to the singular perturbation parameter. Numerical examples are presented to illustrate the theoretical results.

Keywords Singularly perturbed problem · Robin boundary condition · Discontinuous source term · Boundary layer · Interior layer · Finite difference scheme

Mathematics Subject Classification 65L10 CR G1.7

Introduction

Singular perturbation problems (SPPs) model convection–diffusion process in applied mathematics that arise in diverse areas, including linearized Navier–Stokes equation at high Reynolds number and the drift-diffusion equation of semiconductor device modeling, heat and mass transfer at high Péclet number etc [1,2]. The novel aspect of the problem under consideration is that we take a source term in the differential equation which has a jump discontinuity at one or more points in the interior of the domain. This gives rise to an interior layer in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Our goal is to construct an ε uniform numerical method for solving this

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problem, that is a numerical method which generates ε uniformly convergent numerical approximations to the solution and its derivatives. Note that problems with discontinuous data were treated theoretically, in the case of the solution of the convection diffusion with Dirichlet case problem [3,4]. In [5–8] the authors discussed a self-adjoint Dirichlet type problem with discontinuous source term. Shanthi et al. has examined two parameter singularly perturbed BVPs for second order ODEs with discontinuous source term in [9–11]. Singularly perturbed delay differential equation is examined in [12] on an adaptively generated grid.

Motivated by the works of [3–6] we, in the present paper, develop a computational method to solve SPBVPs for second order equations of the type:

$$Ly(x) \equiv \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x), \quad x \in (\Omega^- \cup \Omega^+), \tag{1}$$

$$BC_1 y(0) = \alpha_1 y(0) - \beta_1 \varepsilon y'(0) = p, \quad BC_2 y(1) = \alpha_2 y(1) + \beta_2 y'(1) = q, \tag{2}$$

where $\alpha_1, \beta_1 \geq 0, \ \alpha_1 + \beta_1 > 0, \ \beta_2 \geq 0$, $\alpha_2 > 0$ and $\varepsilon > 0$ is a small parameter. $a(x), b(x)$ are smooth functions $\overline{\Omega}$ such that

$$a(x) \geq \alpha > 0, \tag{3}$$

$$b(x) \geq \beta \geq 0. \tag{4}$$

It is convenient to introduce the notation $\Omega = (0, 1), \ \Omega^- = (0, d), \ \Omega^+ = (d, 1), \quad d \in \Omega$ and to denote the jump at d in any function with $[w](d) = w(d+) - w(d-)$. Further it is assumed that f is sufficiently smooth on $\overline{\Omega} \setminus \{d\}$; a single discontinuity in the source term $f(x)$ occur at a point $d \in \Omega$; $f(x)$ and its derivatives have jump discontinuity at the same point. In general this discontinuity gives rise to an interior layer in the second derivative of the exact solution of the problem. Because f is discontinuous at d the solution y of (1)–(2) does not necessarily have a continuous second derivative at the point d. Thus $y \notin \mathcal{C}^2(\Omega)$. But the first derivative of the solution exists and is continuous. Boundary value problem of the type (1)–(2) model confinement of a plasma column by reaction pressure and geophysical fluid dynamics [13].

The nature of problem discussed in the Dirichlet case $\alpha_1 = 1, \ \beta_1 = 0, \ \alpha_2 = 1$ and $\beta_2 = 0$ and in the Neumann case $\alpha_1 = 0, \ \beta_1 = 1, \ \alpha_2 = 0$ and $\beta_2 = 1$ in [14]. For detailed study one may refer [14,15].

Various methods are available in literature to obtain numerical solution to singularly perturbed differential equation (1) subject to Robin boundary conditions when f is smooth on Ω [14,16–20]. Some recent works have been done in similar type of problem with smooth data as follows. The numerical integration method for general singularly perturbed boundary value problem with mixed boundary condition is presented in [21]. In [22] the authors have shown the advantages of Differential Quadrature Method (DQM) for finding the numerical solution [23]. Pratibhamoy Das et al. discussed on system of reaction diffusion differential equations for Robin or mixed type boundary value problems by a cubic spline approximation [24]. From this investigation the author considered a non-self adjoint Robin type problem with discontinuous source term, and obtained a parameter uniform convergent solution for equation (1)–(2).

"Some Analytical Results" section presents analytic behavior of the solution of the SPP (1)–(2). The present method is described in "Discrete Problem" section. "Error Analysis" section provides error estimates for the numerical solutions. Numerical examples are given in "Numerical Results" section. The paper ends with a discussion.

Throughout this paper, C denotes a generic positive constant that is independent of nodal point (i), mesh size (h) and the singular perturbation parameter (ε). We measure all functions in the maximum point wise norm, which we denote by $\|w\|_D = \sup_{x\in D} |w(x)|$, where D is an open connected set.

Some Analytical Results

In this section, we provide a comparison principle for the following problem. Consequence of this principle gives the stability result for the same problem. By a suitable choice of the barrier function and the procedure adapted from [3,16,25], we can prove the following theorems.

Theorem 1 *The problem* (1)–(2) has a solution $u \in \mathcal{C}^1(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$.

Theorem 2 (Comparison Principle) *Suppose that a function* $u \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega^- \cup \Omega^+)$, *satisfies* $BC_1u(0) \geq 0$, $BC_2u(1) \geq 0$ *and* $Lu(x) \leq 0$, $\forall\, x \in \Omega^- \cup \Omega^+$ *and* $[u'](d) \leq 0$, *then* $u(x) \geq 0$, $\forall\, x \in \overline{\Omega}$.

An immediate consequence of the comparison principle is the following stability result.

Theorem 3 (Stability Result) *Consider the BVP* (1)–(2) subject to the conditions (3)–(4). If $u \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega^- \cup \Omega^+)$ then

$$\|u\|_{\bar{\Omega}} \leq C max\left\{|BC_1u(0)|, |BC_2u(1)|, |Lu|_{\Omega^- \cup \Omega^+}\right\}.$$

Theorem 4 *For each integer* k, *satisfying* $0 \leq k \leq 4$, *the solution* u *of* (1)–(2) satisfy the bounds

$$\left\|u^k\right\|_{\bar{\Omega}\setminus\{d\}} \leq C\varepsilon^{-k}.$$

To establish the parameter-robust properties of the numerical methods involved in this paper, the following decomposition of u into smooth v and singular w components is required. The smooth component v is defined as the solution of

$$av_0' - bv_0 = f, \quad x \in \Omega^- \cup \Omega^+, \quad BC_2v_0(1) = q. \tag{5}$$

$$av_1' - bv_1 = -v_0'', \quad x \in \Omega^- \cup \Omega^+, \quad BC_2v_1(1) = 0. \tag{6}$$

Note that $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2$, where $v_2 \in \mathcal{C}^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ and

$$Lv_2 = -v_1'', \quad BC_1v_2(0) = v_2(d) = BC_2v_2(1) = 0.$$

As in [14,16], we can obtain the following bounds on the derivatives of v for $k = 1, 2, 3$.

$$\|v\| \leq C, \ \left\|v^{(k)}\right\|_{\Omega^- \cup \Omega^+} \leq C(1 + \varepsilon^{(2-k)}). \tag{7}$$

Note also that $|\ [v'](d)\ | \leq\ C$, $|\ [v''](d)\ | \leq\ C$. Define the singular component of the decomposition as follows. Find $w \in \mathcal{C}^0(\Omega)$ such that

$$Lw = 0, \quad x \in \Omega^- \cup \Omega^+,$$

$$BC_1w(0) = y(0) - v(0), \quad BC_2w(1) = 0, \quad [w'](d) = -[v'](d).$$

We can further decompose w as

$$w = w_1 + w_2,$$

where w_1 lois the boundary layer function satisfying

$$w_1 \in C^2(\Omega), \quad Lw_1 = 0, \; x \in \Omega, \tag{8}$$

$$BC_1 w(0) = y(0) - v(0), \quad BC_2 w(1) = 0 \tag{9}$$

and w_2 is the interior layer function satisfying

$$w_2 \in C^0(\Omega), \; Lw_2 = 0, \quad x \in \Omega^- \cup \Omega^+, \tag{10}$$

$$BC_1 w_2(0) = 0, \; BC_2 w_2(1) = 0, \tag{11}$$

$$[w_2'](d) = -[v'](d). \tag{12}$$

The procedure adopted from [16] and suitable barrier function, one can obtain the results of the following Lemma 1, 2 and 3.

Lemma 1 [14,16] *For each integer k satisfying* $0 \le k \le 3$, *the solution* w_1 *of* (8),(9) satisfies the bounds

$$\left| w_1^{(k)}(x) \right| \le C\varepsilon^{-k} e^{-\alpha x/\varepsilon}, \quad x \in \; \Omega^-$$

where C is a constant independent of ε.

Lemma 2 *For each integer* k, *satisfying* $1 \le k \le 3$, *the solution* w_2 *of* (10)–(12) satisfies the bounds

$$|w_2(x)| \le C$$

$$|w_2^k(x)| \le \begin{cases} C(\varepsilon^{(1-k)} e^{-\alpha x/\varepsilon}), & x \in \Omega^- \\ C(\varepsilon^{(1-k)} e^{-\alpha(x-d)/\varepsilon}), & x \in \Omega^+, \end{cases}$$

where C is a constant independent of ε.

Discrete Problem

A fitted mesh method for problem (1)–(2) is now described. On Ω a piecewise uniform mesh of N mesh intervals is constructed as follows. The interval Ω is subdivided into four subintervals.

$$[0, \tau_1] \cup [\tau_1, d] \cup [d, d + \tau_2] \cup [d + \tau_2, 1]$$

for some τ_1, τ_2 that satisfies $0 < \tau_1 \le d/2, \; 0 \le \tau_2 \le \dfrac{1-d}{2}$. On each subinterval a uniform mesh with $\dfrac{N}{4}$ mesh-intervals is placed. The interior points of the mesh are denoted by

$$\Omega_\varepsilon^N = \left\{ x_i : 1 \le i \le \frac{N}{2} - 1 \right\} \cup \left\{ x_i : \frac{N}{2} + 1 \le i \le N - 1 \right\}.$$

Clearly $x_{N/2} = d$ and $\overline{\Omega}_\varepsilon^N = \{x_i\}_0^N$. Note that this mesh is a uniform mesh when $\tau_1 = \dfrac{d}{2}$ and $\tau_2 = \dfrac{1-d}{2}$. It is fitted to the singular perturbation problem (13)–(14) by choosing τ_1 and τ_2 to be the following functions of N and ε

$$\tau_1 = \min\left\{\frac{d}{2}, (\varepsilon/\alpha)\ln N\right\} \quad \text{and} \quad \tau_2 = \min\left\{\frac{1-d}{2}, (\varepsilon/\alpha)\ln N\right\}.$$

We introduce the following notation for four mesh widths

$$h_1 = 4\tau_1/N, \quad H_1 = 4(d-\tau_1)/N, \quad h_2 = 4\tau_2/N, \quad H_2 = 4(1-d-\tau_2)/N.$$

On the piecewise-uniform mesh $\overline{\Omega}_\varepsilon^N$ a standard centered finite difference operator is used. Then the fitted mesh method for (1)–(2) is

$$L_i^y \equiv \varepsilon\delta^2 y_i + a(x_i)D^+ y_i - b(x_i)y_i = f(x_i), \quad \forall\ x_i \in \Omega_\varepsilon^N \tag{13}$$

$$BC_1^N y_0 = \alpha_1 y_0 - \beta_1\varepsilon D^+ y_0 = r, \quad BC_2^N y_N = \alpha_2 y_N + \beta_2 D^- y_N = s,$$
$$D^- y_\varepsilon(x_{N/2}) = D^+ y_\varepsilon(x_{N/2}), \tag{14}$$

where,

$$\delta^2 Z_i = \left(\frac{Z_{i+1}-Z_i}{x_{i+1}-x_i} - \frac{Z_i - Z_{i-1}}{x_i - x_{i-1}}\right)\frac{2}{x_{i+1}-x_{i-1}},$$
$$D^+ Z_i = \frac{Z_{i+1}-Z_i}{x_{i+1}-x_i}, \quad \text{and} \quad D^- Z_i = \frac{Z_i - Z_{i-1}}{x_i - x_{i-1}}.$$

Lemma 3 *Suppose that a mesh function* Z *satisfies* $BC_1^N Z(x_0) \geq 0, BC_2^N Z(x_N) \geq 0, L^N Z(x_i) \geq 0$ *for all* $x_i \in \Omega_\varepsilon^N$*, and* $D^+ Z(x_{N/2}) - D^- Z(x_{N/2}) \leq 0$*, then* $Z(x_i) \geq 0$ *for all* $x_i \in \overline{\Omega}_\varepsilon^N$.

Error Analysis

Define the function V to be the solution of

$$L^N V = f(x_i), \quad \forall\ x_i \in \Omega_\varepsilon^N$$
$$BC_1^N V(0) = BC_1 v(0), \quad V(d) = v(d), \quad BC_2^N V(1) = BC_2 v(1).$$

Using a standard stability and consistency argument coupled with the obvious barrier functions on each interval $[0, d]$, $[d, 1]$ separately, by [14], one can deduce that

$$|V(x_i) - v(x_i)| \leq \begin{cases} CN^{-1}(d - x_i), & x_i \leq d, \\ CN^{-1}(1 - x_i), & x_i \geq d. \end{cases} \tag{15}$$

We define W to be the solution of

$$L^N W = 0, \quad \forall\ x_i \in \Omega_\varepsilon^N$$
$$BC_1^N W(0) = BC_1 w(0), \quad BC_2^N W(1) = BC_2 w(1), \quad [DW(d)] = -[DV(d)],$$

where throughout this section, we denote the jump in the discrete derivative of mesh function Z at the point $x_i = d$ by

$$[DZ(d)] = D^+ Z(d) - D^- Z(d).$$

Analogously to the continuous case we can further decompose W as

$$W = W_1 + W_2,$$

where W_1 (the discrete analogue of the boundary layer function w_1) is defined as solution of

$$L^N W_1 = 0, \quad \forall \ x_i \in \Omega_\varepsilon^N \cup \{d\} \tag{16}$$

$$BC_1^N W_1(0) = BC_1 w_1(0), \quad BC_2^N W_1(1) = 0 \tag{17}$$

and W_2 (the discrete analogue of the interior layer function w_2) is defined as solution of

$$L^N W_2 = 0, \quad \forall \ x_i \in \Omega_\varepsilon^N \tag{18}$$

$$BC_1^N W_2(1) = 0, \quad BC_2^N W_2(1) = 0 \tag{19}$$

$$[DW_2(d)] = -[DV(d)] - [DW_1(d)] \tag{20}$$

As in [14],

$$|W_1 - w_1| \leq CN^{-1} \ln N, \quad x_i \in \Omega_\varepsilon^N \cup \{d\}. \tag{21}$$

And $| \ W_1 \ | \leq CN^{-1}$ for all $x_i \geq d$. Note that the jump at $x = d$ in the derivative of weak interior layer function w_2 is bounded. The next lemma establishes the discrete counterpart of this.

The procedure given in page. 140 of [3] is also valid for $BC_1^N y_0$ and $BC_2^N y_N$ with the same barrier function. Therefore, from Lemma 4 and Lemma 5 of [3], we obtain

Lemma 4 *Assuming $N \geq 8$ then*

$$|W_2(x_i) - w_2(x_i)| \leq C(\varepsilon + N^{-1}) \leq CN^{-1} \ln N.$$

Let us first consider the magnitude of the truncation error at the point of discontinuity $x = d$. Using the procedure following in [3] and the derivative estimate established in [14] the following result can be obtain

$$|[D(V - v)(d)]| \leq C(\varepsilon N)^{-1} \quad \text{and} \quad |[D(W_2 - w_2)(d)]| \ \leq CN^{-1} \ln N/\varepsilon.$$

Define the barrier function ϕ_i [3] to be

$$\phi_i = CN^{-1} \ln N \begin{cases} 1, & x_i \leq d \\ \psi, & x_i \geq d \end{cases} + CN^{-1} \ln N(1 - x_i),$$

where ψ is the solution of

$$\varepsilon\delta^2\psi + \alpha D^+\psi - b\psi = 0, \quad x_i \in \Omega^+ \cap \Omega_\varepsilon^N$$
$$\psi(d) = 1, \quad BC_2^N \psi(1) = 0.$$

Theorem 5 *Let U, u be the solution of L^N, L respectively. Assume that $N \geq 8$, then*

$$\|\tilde{U} - u\|_{\overline{\Omega}} \leq CN^{-1} \ln N,$$
$$\varepsilon \ \|D^+\tilde{U} - u'\|_{\overline{\Omega}} \leq CN^{-1} \ln N,$$

where $\tilde{U}$ is the piecewise linear interpolant of U on $\overline{\Omega}$ and C is a constant independent of N and ε.

Proof Using (15), (21) and Lemma 4 it is easy to derive

Table 1 Maximum pointwise error E_ε^N and order of convergence ρ^N for the Example 1

ε	Number of mesh points					
	2^6	2^7	2^8	2^9	2^{10}	2^{11}
2^0	1.1329E−02	5.5712E−03	2.6949E−03	1.2574E−03	5.3885E−04	1.7961E−04
2^{-1}	6.8319E−03	3.2397E−03	1.5367E−03	7.0975E−04	3.0258E−04	1.0059E−04
2^{-2}	8.7403E−03	4.4761E−03	2.2085E−03	1.0406E−03	4.4812E−04	1.4973E−04
2^{-3}	2.1626E−02	1.0694E−02	5.1847E−03	2.4216E−03	1.0382E−03	3.4615E−04
2^{-4}	2.0405E−02	1.0005E−02	4.8332E−03	2.2538E−03	9.6554E−04	3.2179E−04
2^{-5}	1.7981E−02	1.0249E−02	5.6613E−03	2.2538E−03	1.4016E−03	5.1662E−04
2^{-6}	2.0072E−02	1.1136E−02	6.0306E−03	3.1191E−03	1.4580E−03	5.2160E−04
2^{-7}	2.2061E−02	1.2045E−02	6.4104E−03	3.2729E−03	1.5180E−03	5.4072E−04
2^{-8}	2.3563E−02	1.2830E−02	6.7717E−03	3.4147E−03	1.5689E−03	5.5587E−04
2^{-9}	2.4610E−02	1.3410E−02	7.0775E−03	3.5532E−03	1.6202E−03	5.6995E−04
2^{-10}	2.5198E−02	1.3801E−02	7.3129E−03	3.6779E−03	1.6744E−03	5.8585E−04
2^{-11}	2.5506E−02	1.4019E−02	7.4585E−03	3.7681E−03	1.7215E−03	6.0295E−04
2^{-12}	2.5658E−02	1.4128E−02	7.5359E−03	3.8225E−03	1.7536E−03	6.1686E−04
E^N	2.5658E−02	1.4128E−02	7.5359E−03	3.8225E−03	1.7536E−03	6.1686E−04
ρ	8.6085E−01	9.0671E−01	9.7927E−01	1.1242E+00	1.5073E+00	

$$\|\tilde{U} - u\|_{\overline{\Omega}} \;\; \leq CN^{-1} \ln N$$

To extend this one can follow the procedure adopted in [14] and [15]. To obtain the bound on the errors in approximations to the scaled derivative, we apply the arguments from [15] and the procedure adopted in [3] separately on each subdomain $[0, d]$ and $[d, 1]$ to get

$$\begin{aligned}
\varepsilon \; \|\bar{D}^+ y - y'\|_{\overline{\Omega}} &\leq CN^{-1} \ln N \\
\varepsilon \; \|\bar{D}^+ V - v'\|_{\overline{\Omega}} &\leq CN^{-1} \ln N \\
\varepsilon \; \|\bar{D}^+(W_1 - w_1)\|_{[0,d)} &\leq CN^{-1} \ln N
\end{aligned}$$

and from the proof of Lemma 4 of [3]

$$\varepsilon \; \|\bar{D}^+(W_1 - w_1)\|_{[d,1)} \leq CN^{-1} \ln N.$$

On the subdomain $[d, 1]$, we also apply these arguments to get

$$\varepsilon \; \|\bar{D}^+(W_2 - w_2)(x_i)\| \leq CN^{-1} \ln N, \quad x_i \geq d.$$

Based on the simple argument given in [3]. We can prove

$$\varepsilon||\bar{D}^+ W_2 - w_2(x_i)|| \leq CN^{-1} lnN, \; x_i < d.$$

which completes the proof. □

Table 2 Maximum pointwise error E_ε^N and order of convergence ρ^N for the Example 2

ε	Number of mesh points					
	2^6	2^7	2^8	2^9	2^{10}	2^{11}
2^0	6.6163E−02	3.2658E−02	1.5827E−02	7.3916E−03	3.1691E−03	1.0566E−03
2^{-1}	6.4196E−02	3.1037E−02	1.4876E−02	6.9081E−03	2.9532E−03	9.8316E−04
2^{-2}	2.1351E−01	1.0537E−01	5.1051E−02	2.3838E−02	1.0219E−02	3.4070E−03
2^{-3}	3.4614E−01	1.6986E−01	8.2076E−02	3.8274E−02	1.6397E−02	5.4647E−03
2^{-4}	4.7760E−01	2.4152E−01	1.1853E−01	5.5717E−02	2.3969E−02	8.0047E−03
2^{-5}	8.4233E−01	4.5042E−01	2.2634E−01	1.0795E−01	4.6763E−02	1.5672E−02
2^{-6}	9.0232E−01	5.4579E−01	3.0795E−01	1.6300E−01	7.7457E−02	2.8493E−02
2^{-7}	9.3284E−01	5.6182E−01	3.1678E−01	1.6732E−01	7.8777E−02	2.8312E−02
2^{-8}	9.4993E−01	5.7117E−01	3.2238E−01	1.7024E−01	8.0147E−02	2.8806E−02
2^{-9}	9.5888E−01	5.7609E−01	3.2530E−01	1.7176E−01	8.0859E−02	2.9063E−02
2^{-10}	9.6347E−01	5.7862E−01	3.2680E−01	1.7253E−01	8.1223E−02	2.9195E−02
2^{-11}	9.6580E−01	5.7991E−01	3.2756E−01	1.7293E−01	8.1407E−02	2.9261E−02
2^{-12}	9.6698E−01	5.8056E−01	3.2795E−01	1.7313E−01	8.1501E−02	2.9295E−02
E^N	9.6698E−01	5.8056E−01	3.2795E−01	1.7313E−01	8.1501E−02	2.9295E−02
ρ	7.3604E−01	8.2399E−01	9.2161E−01	1.0870E+00	1.4762E+00	

Numerical Results

In this section, two examples are given to illustrate the computational methods discussed in this paper.

Consider the singularly perturbed BVPs with discontinuous source term:

Example 1

$$\varepsilon y''(x) + y'(x) = f(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$y(0) - \varepsilon y'(0) = 1, \quad y(1) + y'(1) = -1,$$

where

$$f(x) = \begin{cases} 0.7, & \text{for } 0 \le x \le 0.5, \\ -0.6, & \text{for } 0.5 < x \le 1. \end{cases}$$

Example 2

$$\varepsilon y''(x) + \frac{1}{1+x} y'(x) = f(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$y(0) - \varepsilon y'(0) = 1, \quad y(1) + y'(1) = 1,$$

where

$$f(x) = \begin{cases} 1 + x, & \text{for } 0 \le x \le 0.5, \\ 4.0, & \text{for } 0.5 < x \le 1. \end{cases}$$

which validate the theoretical results established in the previous section. The nodal errors and orders of convergence are estimated using the interpolation principle. Define the nodal

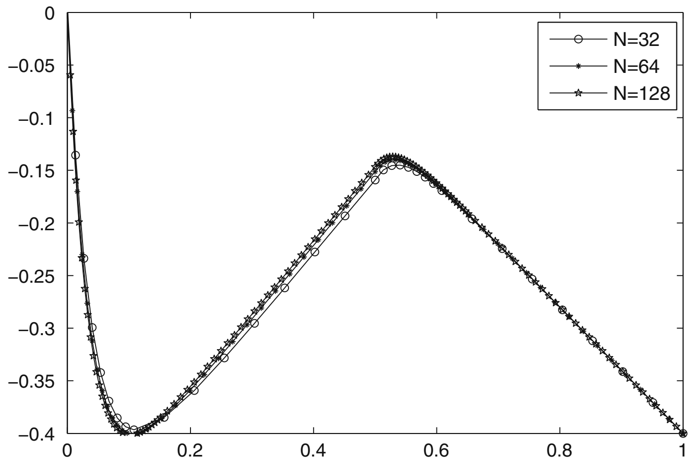

Fig. 1 Solution plot of $N = 32, 64, 128$ and $\varepsilon = 2^{-5}$ for the Problem 1

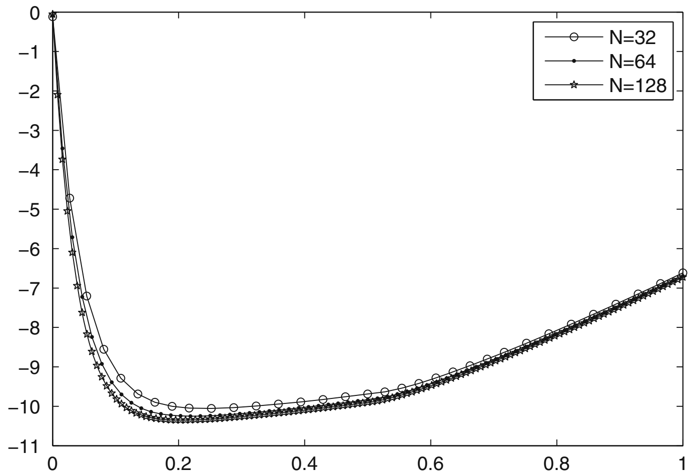

Fig. 2 Solution plot of $N = 32, 64, 128$ and $\varepsilon = 2^{-5}$ for the Problem 2

error of interpolation principle, the difference between the numerical solutions for various values of N and N=*4096*.

$$E_\varepsilon^N \equiv \max_{x_i \in \overline{\Omega}_\varepsilon^N} |U_\varepsilon^N(x_i) - \bar{U}_\varepsilon^{4096}(x_i)| \quad \text{and} \quad E^N = \max_\varepsilon E_\varepsilon^N.$$

From these quantities the parameter-robust order of convergence are computed from

$$\rho^N = log_2\left(\frac{E^N}{E^{2N}}\right).$$

Tables 1 and 2 present values of the maximum pointwise error E_ε^N and order of convergence ρ^N for the Examples 1 and 2 respectively. The numerical solutions of the Example 1 and 2 are presented in Figs. 1 and 2.

Conclusion

A singularly perturbed second-order ODEs of Robin type BVPs with discontinuous source term is considered. Due to discontinuity in the source term there is an interior layer occurring. To fit the interior and boundary layer a suitable piecewise uniform mesh is constructed. The numerical method generates ε-uniform convergence in the global maximum norm of the approximations. The numerical approximation coincides with theoretical result.

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