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Fixed Point Theorems Concerning Hausdorff F-PGA Contraction in Complete Metric Space

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Abstract.
Harandi Amini-Harandi [2012], in 2012 established the existence of a fixed point by using the concept of set-valued contraction. In the present paper, authors have generalized this concept by considering Hausdorff F-PGA contraction and assured the existence of a fixed point. Hence, it is interesting to note that in a complete Hausdorff metric space, the fixed point exists with a lighter contraction map.

1. Introduction
Fixed point theory, has always been an important branch of Mathematics, the concepts of fixed point theory play a crucial role in solving various mathematical problems Zhao and Li [2011], Hussain et al. [2014]. Researchers consider various spaces Mishraa et al., Mishra et al. [2020, 2015b], and study existence and uniqueness of the fixed points in these space by applying different contraction mappings Sanatee et al. [2020], Mishra et al. [2020, 2015a]. Nadler Nadler et al. [1969] generalized the notable work of Banach Banach [1922] by proposing the concept of multi-valued contraction mappings. The concept of multi-valued contraction mappings was further studied by Wardowski Wardowski [2012] who introduced a new concept of contraction called the F-contraction and given a benchmark theorem, which generalized the Banach contraction principle. Harandi Amini-Harandi [2012] used this F-contraction map and established some important results concerning the generalization of the Banach contraction principle in context to the F-contraction. It may be noted that we have previously defined the F-PGA contraction map Powar et al. [2018], which was found to be a generalized form of the F-contraction map. In this paper, we have taken a complete metric space, with a Hausdorff metric defined on it. Further, we define the F-PGA contraction map over this complete metric space, called the Hausdorff F-PGA contraction map. Using this Hausdorff F-PGA contraction map, we have generalized the averments of Harandi Amini-Harandi [2012] and established the existence of a fixed point for Hausdorff F-PGA contraction.

2. Preliminaries
In this section, we list some of the basic definitions and examples which are being used in this paper and are required, to get an insight into the concept.

Definition 1 *Piri et al. [2017] Let F_r be the family of all functions $F:(0,\infty)\to R$ such that*
(f_1) *F is strictly increasing, i.e. for all $x,y\in(0,\infty)$ and $x<y\Rightarrow F(x)<F(y)$;*
(f_2) *For each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty}\alpha_n=0$ if and only if $\lim_{n\to\infty}F(\alpha_n)=-\infty$*
(f_3) *there exists $k\in(0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha)=0$.*

Definition 2 *Powar et al. [2018] Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be an* ***F-contraction*** *on (X,d) if there exist $F\in F_r$ and $\tau\in(0,\infty)$ such that*

$$\forall x,y\in X, d(Tx,Ty)>0\Rightarrow \tau+F(d(Tx,Ty))=F(d(x,y)).$$

Definition 3 *Powar et al. [2018] Let (X,d)be the metric space. A mapping $T:X\to X$ is said to be* ***F-PGA contraction*** *if there exists $\tau\in(0,\infty)$ and $F\in F_r$ such that*

$$\forall x,y\in X,\ d(Tx,Ty)>0\Rightarrow \tau+F(d(Tx,Ty))=F(max\{G(x,y),d(x,y)\}).$$

Where

$$G(x,y)\ =\begin{cases}\frac{(d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx))}{max\{d(x,Ty),d(Tx,y)\}} & if\ \ max\{d(x,Ty),d(Tx,y)\}\neq\ 0\\ 0 & if\ \ max\{d(x,Ty),d(Tx,y)\}=\ 0\end{cases}$$

Definition 4 *Amini-Harandi [2012] Let (X,d) be a complete metric space and let $CB(X)$ be the collection of all non-empty compact subset of X. Let h be the Hausdorff metric for d, that is,*

$$h(A,B)=max\{\sup d(a,B)_{a\in A},\ \sup d(b,A)_{b\in B}\}$$

for all $A,B\in CB(X)$, where $d(a,B)=\inf\{d(a,b):b\in B\}$

Example 1 *Let (R,d) be a metric space, where $d(x,y)=|x-y|$. Let A= $[0,21]$, B= $[22,32]$ are compact subsets of R. Then $(a,B)_{a\in A}=sup\{d(a,22):a\in A\}=d(0,22)=22$*
$(b,A)_{b\in B}=sup\{d(b,21):b\in B\}=d(32,21)=11$

$$h(A,B)\ =\ \max\{22,11\}=\ 22.$$

Definition 5 *Nadler et al. [1969] Let (X,d) be a complete metric space. A function $T:X\to CB(X)$ is called a multi-valued contraction mapping $(m.v.c.m)$ if there exists real number $\alpha<1$ such that $h(T(x),T(y))=\alpha d(x,y),\ \forall x,y\in X$. A point x is said to be a fixed point of multi-valued mapping T if $x\in Tx$.*

Example 2 *Let I= $[0,1]\subset R$ with usual metric d and let f:$I\to I$ defined by*

$$f(x)=\begin{cases}\frac{x+1}{2}, & if\ \ 0=\ \ x=\ \ \frac{1}{2}\\ \frac{-x}{2}+1, & if\ \ \frac{1}{2}=\ \ x=\ \ 1\end{cases}$$

and T:$I\to CB(I)$ by $T(x)=\{0\}\cup\{f(x)\}$ for each $x\in I$. It is easy to verify that T is a multi-valued contraction mapping and the set of a fixed point of T is $\{0,\frac{2}{3}\}$.

Definition 6 *Let (X,d) be a complete metric space. Let T:$X\to CB(X)$ be a set-valued map and $F\in F_r$, then T is said to be Hausdorff F-PGA contraction if there exists $\tau\in(0,\infty)$ such that*

$$\forall x,y\in X, d(Tx,Ty)>0\Rightarrow \tau+F(h(Tx,Ty))=F(\max\{G(x,y),\ d(x,y)\}).$$

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Example 3 *Let F: $(0,\infty) \to R$ defined by $F(x) = \log x$. A mapping T:$R \to CB(R)$ which satisfy the condition of Hausdorff F-PGA contraction then*

$$\tau+\log(h(Tx,Ty)) = \log\left(max\{\begin{array}{ll} \frac{(d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx))}{max\{d(x,Ty),d(Tx,y)\}} & if\ \ max\{d(x,Ty),d(Tx,y)\} \neq\ 0 \\ 0 & if\ \ max\{d(x,Ty),d(Tx,y)\} = \ 0 \end{array}\right. ,\ d(x,y)\}$$

$$\log e^{\tau}.(h(Tx,Ty)) =$$

$$\log\left(max\{\begin{array}{ll} \frac{(d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx))}{max\{d(x,Ty),d(Tx,y)\}} & if\ \ max\{d(x,Ty),d(Tx,y)\} \neq\ 0 \\ 0 & if\ \ max\{d(x,Ty),d(Tx,y)\} = \ 0 \end{array}\right. ,d(x,y)\}$$

If the maximum is $\frac{(d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx))}{max\{d(x,Ty),d(Tx,y)\}}$ *then*

$$h(Tx,Ty) = e^{-\tau}\frac{(d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx))}{max\{d(x,Ty),d(Tx,y)\}}$$

*If the maximum is $d(x,y)$ then, $h(Tx,Ty = e^{-\tau}d(x,y)$. A **multi-valued contraction mapping (m.v.c.m.)** is a special case of Hausdorff F-PGA contraction in this case.*

Example 4 *Let $X= [0,1] \cup \{2,3,...\}$ and*

$$d(x,y) = \begin{cases} 0, & if\ \ x{=}y \\ |x-y|, & if\ \ x,y\ \in\ [0,1] \\ |x+y|, & if\ \ one\ \ of\ \ x,y\ \notin\ [0,1] \end{cases}$$

Then (X,d) is a complete metric space. Define the mapping T:$X \to CB(X)$ by

$$Tx= \begin{cases} \{\frac{3}{4}\}, & if\ \ x{=}\ 0 \\ \{1\}, & if\ \ x\ \in\ (0,1] \\ \{1,x-1\}, & if\ \ x\in\ \{2,3,...\} \end{cases}$$

for $y= 1$ and $x> 2$, since $d(x,y)$ =$x+1$ and $h(Tx,Ty)$ =x, we get

$$\lim_{x\to\infty}\frac{h(Tx,Ty)}{d(x,y)} = \lim_{x\to\infty}\frac{x}{x+1} = 1.$$

we can not find $\alpha<1$ satisfying

$$h(Tx,Ty) = \alpha d(x,y)$$

hence, T is not a multi-valued contraction. Since

$$h(T0,T\frac{1}{4})\ = d(0,\frac{1}{4}) = \frac{1}{4}$$

and let

$$F\colon (0,\infty) \to R$$

defined by $F(x) = \log x$, then by definition of Hausdorff F-PGA contraction,

$$\tau + \log h(Tx,Ty) = \log max\{G(x,y), d(x,y)\}$$

$$\Rightarrow\quad \log e^{\tau} + \log h(Tx,Ty) = d(x,y)$$

i.e. $h(Tx,Ty) = e^{-\tau}d(x,y)$ where $\tau>0$. It is clear that T satisfies the condition of Hausdorff F-PGA contraction but it is not multi-valued contraction mapping.

3. Main Result

It has been observed that there exists a Hausdorff F-PGA unique fixed point for the complete metric space. The following theorem generalizes the claims of Harandi:

Theorem 1. *Let (X,d) be a complete metric space, and $T : X \to CB(X)$ be a Hausdorff F-PGA contraction. Then T has a unique fixed point.*

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. If Tx_0=Tx_1 then $x_1 \in Tx_1$.

$\Rightarrow x_1$ is a fixed point of T.

So, we assume that $Tx_0 \to Tx_1$. Since $Tx_0, Tx_1 \in CB(X)$, there is a point $x_1 \in Tx_0$ such that

$$\tau + F(h(Tx_0,Tx_1)) = F(max\{G(x_0,x_1),\, d(x_0,x_1)\}),\ \tau > 0 \tag{1}$$

since $Tx_1, Tx_2 \in CB(X)$ and $x_1 \in Tx_0$ so, there is a point $x_2 \in Tx_1$ such that

$$F(h(Tx_1,Tx_2)) = F(max\{G(x_1,x_2),\, d(x_1,x_2)\}) - \tau$$

$$= F[d(x_1,x_2)] - \tau$$

$$= F[h(Tx_0,Tx_1)] - \tau$$

$$= [F(max\{G(x_0,x_1),d(x_0,x_1)\}) - \tau] - \tau$$

$$= [F(max\{G(x_0,x_1),d(x_0,x_1)\}) - 2\tau]$$

$$= \dots\dots\dots$$

$$F\left(h\left(Tx_n,Tx_{n+1}\right)\right) = F(max\{G(x_0,x_1),d(x_0,x_1)\}) - (n+1)\tau. \tag{2}$$

continuing in this fashion we produce a sequence $\{x_n\}_{n=1}^{\infty}$ of points of X such that $x_{n+1} \in Tx_n$. Now, we shall show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence.

By Cauchy criterion for convergence, for every ε>0 there, exist $n_0 \in N$ such that

$$F(h(Tx_n,Tx_{n+k})) = F(h(Tx_n,Tx_{n+1})) + F(h(T(x_{n+1},Tx_{n+2})) + \dots + F(h(T(x_{n+k-1},Tx_{n+k})).$$

$$= (k+1)F(max(G(x_0,x_1),d(x_0,x_1))) - (n+1)\tau - (n+2)\tau - \dots - (n+k)\tau.$$

$$= (k+1)F(max(G(x_0,x_1),d(x_0,x_1))) - [(n+1) + (n+2) + \dots + (n+k)]\tau.$$

$$= (k+1)F(max(G(x_0,x_1),d(x_0,x_1))) - [\frac{(n+k)(n+k+1)}{2} - \frac{n(n+1)}{2}]\tau < \varepsilon$$

It follows that the sequence $\{x_n\}_1^{\infty}$ is Cauchy sequence.

Letting $n \to \infty$ we get $F(h(Tx_n,Tx_{n+k})) = -\infty$.

In particular for k= 1 we get $lim_{n\longrightarrow\infty}F(h(Tx_n,Tx_{n+1})) = -\infty$.

Since $F \in F_r$, by using (f_2) of Definition 2, we arrive at

$$\lim_{n\to\infty} h(Tx_n,Tx_{n+1})) = 0.$$

Since X is complete, then every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ is convergent to some point $x_0 \in X$ i.e $x_n \to x_0$ as $n \to \infty$. So,

$$h(Tx_0,Tx_1) = 0 \Rightarrow Tx_0 \in Tx_1.$$

Since $x_1 \in Tx_0$ then $x_1 \in Tx_1$

$\Rightarrow x_1 \in Tx_1$ i.e. $x_0 \in Tx_0$.

$\Rightarrow x_0$ is a fixed point of T.

Claim: x_0 is unique.

Let if possible there exist two fixed point x_1 and x_2 such that $x_1 \neq x_2$. Then by definition of fixed point $x_1 \in Tx_1$ and $x_2 \in Tx_2$. Now by the Hausdorff F-PGA contraction, we have

$$\tau \ \leq F(max\{G(x_1,x_2),d(x_1,x_2)\}) - F(h(Tx_1,Tx_2))$$

$$\tau \leq F(d(x_1,x_2)) - F(h(x_1,x_2))$$

$\tau \leq 0$ (since F is strictly increasing)

This is a contradiction because $\tau>0$. Hence T has a unique fixed point.

4. Conclusion

The concept of F-PGA contraction maps has been applied in the complete Hausdorff metric space. The Hausdorff F-PGA contraction map is a generalization of the F- contraction, the existence, and uniqueness of the fixed point assure the extension and generalization of the work by Harandi.

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