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Fixed point theorems for fuzzy mappings and applications to ordinary fuzzy differential equations

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Abstract

Ran and Reurings (Proc. Am. Math. Soc. 132(5):1435-1443, 2004) proved an analog of the Banach contraction principle in metric spaces endowed with a partial order and discussed some applications to matrix equations. The main novelty in the paper of Ran and Reurings involved combining the ideas in the contraction principle with those in the monotone iterative technique. Motivated by this, we present some common fixed point results for a pair of fuzzy mappings satisfying an almost generalized contractive condition in partially ordered complete metric spaces. Also we give some examples and an application to illustrate our results.

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1 Introduction

The Banach contraction principle [1] is a very popular tool in solving existence problems in many branches of mathematical analysis. This famous theorem can be stated as follows.

Theorem 1.1 ([1]) *Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying*

$$d(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in X,$$

where k is a constant in $(0, 1)$. Then T has a unique fixed point $x^* \in X$.

There is a great number of generalizations of the Banach contraction principle. In fact, existence theorems of fixed points have been established for mappings defined on various types of spaces and satisfying different types of contractive inequalities. Tasković [2] presented a comprehensive survey of such results in metric spaces. A new category of contractive fixed point problems was addressed by Khan *et al.* [3] that introduced the concept of altering distance function, which is a control function that alters distance between two points in a metric space (see also [4–6] and references therein).

Definition 1.2 ([3]) $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

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(i) φ is continuous and nondecreasing,

(ii) $\varphi(t) = 0 \Leftrightarrow t = 0$.

Another generalization of the Banach contraction principle was suggested by Alber and Guerre-Delabriere [7] in Hilbert spaces by introducing the concept of weakly contractive mappings as follows.

Definition 1.3 Let (X, d) be a metric space. A mapping $T : X \to X$ is called weakly contractive if and only if:

$$d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big) \quad \text{for all } x, y \in X,$$

where φ is an altering distance function.

Rhoades [6] showed that most results of [7] are still valid for any Banach space. Weak inequalities of the above type have been used to establish fixed point results in a number of subsequent works (see [4, 5, 8–10] and references therein).

Recently, many results appeared related to fixed points in complete metric spaces endowed with a partial ordering $\preceq$. Most of them are hybrids of two fundamental principles: the Banach contraction principle and the monotone iterative technique. In fact, these results deal with a monotone (either order-preserving or order-reversing) self-mapping T satisfying, with some restrictions, a classical contractive condition and such that for some $x_0 \in X$, either $x_0 \preceq Tx_0$ or $Tx_0 \preceq x_0$. The first result in this direction was given by Ran and Reurings [11, Theorem 2.1]. In their paper, Ran and Reurings proved an analog of the Banach contraction principle in a metric space endowed with a partial ordering and gave applications to matrix equations. Subsequently, Nieto and Rodríguez-López [12] extended the result of Ran and Reurings [11] for nondecreasing mappings and applied to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Thereafter, many works related to fixed point problems have also been considered in partially ordered probabilistic metric spaces [13], partially ordered G-metric spaces [14, 15], partially ordered cone metric spaces [16], partially ordered fuzzy metric spaces [17–22] and partially ordered non-Archimedean fuzzy metric spaces [23, 24]. For other related works one is referred to [11, 22, 25–31].

On the other hand, in the year 1965, Zadeh [32] introduced the concept of fuzzy set which motivated a lot of mathematical activities on generalization of the notion of fuzzy set. Heilpern [33] introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings, which was successively generalized by Estruch and Vidal [34]. Afterward, a number of papers appeared in which fixed points of fuzzy mappings satisfying contractive inequalities have been discussed (see [35–37] and references therein). Recently, many authors studied fixed point results for application to partial differential equation and integral equations (see [38–43]).

Now, we briefly describe our reasons for being interested in results of this kind. The applications of fixed point theorems are remarkable in different disciplines of mathematics, engineering and economics in dealing with problems arising in approximation theory, game theory and many others (see [44] and references therein).

Motivated by this, we prove a common fixed point theorem for a pair of fuzzy mappings without taking into account any commutativity condition in complete ordered metric spaces. The key feature of our theorem is that the contractive condition is only assumed to hold on elements that are comparable in respect to the partial ordering. We show that under such conditions, the conclusions of previous fixed point theorems of fuzzy mappings still hold. The main result is based on an almost generalized contractive condition and generalizes, improves and extends many known results in the comparable literature [4–6, 17, 35] in the sense of fuzziness under ordered metric spaces. At the end of the paper, we remark that some of the ideas existing in the literature can also be used to extend our result.

2 Preliminaries

For the sake of completeness, we briefly recall some basic concepts used in the sequel.

Throughout the rest of the paper unless otherwise stated (X, d) stands for a complete metric space. A fuzzy set in X is a function with domain X and values in $[0,1]$. If A is a fuzzy set on X and $x \in X$, then the functional value Ax is called the grade of membership of x in A. The α-level set of A, denoted by A_α, is defined by

$$A_\alpha = \{x : Ax \geq \alpha\} \quad \text{if } \alpha \in (0,1], \qquad A_0 = \overline{\{x : Ax > 0\}},$$

where $\overline{A}$ denotes the closure of the set A. For any two subsets A and B of X we denote by $H(A, B)$ the Hausdorff distance.

Definition 2.1 A fuzzy set A in a metric linear space is said to be an approximate quantity iff A_α is compact and convex in X for each $\alpha \in [0,1]$ and $\sup_{x \in X} Ax = 1$.

Let $I = [0,1]$ and $W(X) \subset I^X$ be the collection of all approximate quantities in X. For $\alpha \in [0,1]$, the family $W_\alpha(X)$ is given by $\{A \in I^X : A_\alpha \text{ is nonempty and compact}\}$.

For a metric space (X, d) we denote by $V(X)$ the collection of fuzzy sets A in X for which A_α is compact and $\sup Ax = 1$ for all $\alpha \in [0,1]$. Clearly, when X is a metric linear space, $W(X) \subset V(X)$.

Definition 2.2 Let $A, B \in V(X)$, $\alpha \in [0,1]$. Then

$$p_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x, y), \qquad D_\alpha(A, B) = H(A_\alpha, B_\alpha),$$

where H is the Hausdorff distance.

Definition 2.3 Let $A, B \in V(X)$. Then A is said to be more accurate than B (or B includes A), denoted by $A \subset B$, if and only if $Ax \leq Bx$ for each $x \in X$.

According to [45], for $x \in X$ we write $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of X. For $\alpha \in (0,1]$ the fuzzy point x_α of X is the fuzzy set of X given by $x_\alpha(x) = \alpha$ and $x_\alpha(z) = 0$ if $z \neq x$. Then we give the following definition.

Definition 2.4 Let x_α be a fuzzy point of X. We will say that x_α is a fixed fuzzy point of the fuzzy mapping F over X if $x_\alpha \subset Fx$ (*i.e.*, the fixed degree of x for F, say $(Fx)(x)$, is at

least α) [34]. In particular, and according to [33], if $\{x\} \subset Fx$, we say that x is a fixed point of F.

To complete the proof of our main result, we need the following lemma.

Lemma 2.5 ([33]) *Let (X,d) be a metric space, $x, y \in X$ and $A, B \in W(X)$:*
   (1) *if $p_\alpha(x, A) = 0$, then $x_\alpha \subset A$,*
   (2) *$p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$,*
   (3) *if $x_\alpha \subset A$, then $p_\alpha(x, B) \leq D_\alpha(A, B)$.*

Definition 2.6 Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space if and only if:
   (i) (X, d) is a metric space,
   (ii) $(X, \preceq)$ is partially ordered.

Definition 2.7 Let $(X, \preceq)$ be a partially ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

3 Main results

Denote with Φ, the family of nondecreasing functions $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$. The next lemma is obvious.

Lemma 3.1 *If $\varphi \in \Phi$, then $\varphi(0) = 0$ and $\varphi(t) < t$ for each $t > 0$.*

Our first result is the following common fixed point theorem involving an almost generalized contractive condition.

Theorem 3.2 *Let $(X, d, \preceq)$ be a complete ordered metric space and $T_1, T_2 : X \to W_\alpha(X)$ be two fuzzy mappings satisfying*

$$D_\alpha(T_1 x, T_2 y) \leq \varphi\big(M(x, y)\big) + \mathbf{L} \min\big\{p_\alpha(x, T_1 x), p_\alpha(y, T_2 y), p_\alpha(x, T_2 y), p_\alpha(y, T_1 x)\big\} \qquad (1)$$

for all comparable elements $x, y \in X$, where $\mathbf{L} \geq 0$ and

$$M(x, y) = \max\left\{d(x, y), p_\alpha(x, T_1 x), p_\alpha(y, T_2 y), \frac{1}{2}\big[p_\alpha(x, T_2 y) + p_\alpha(y, T_1 x)\big]\right\}.$$

Also suppose that
   (i) *if $y \in (T_1 x_0)_\alpha$, then $y, x_0 \in X$ are comparable,*
   (ii) *if $x, y \in X$ are comparable, then every $u \in (T_1 x)_\alpha$ and every $v \in (T_2 y)_\alpha$ are comparable,*
   (iii) *if a sequence $\{x_n\}$ in X converges to $x \in X$ and its consecutive terms are comparable, then x_n and x are comparable for all n.*
Then there exists a point $x \in X$ such that $x_\alpha \subset T_1 x$ and $x_\alpha \subset T_2 x$.

Proof Let x_0 in X. Since $(T_1 x_0)_\alpha \neq \emptyset$, then there exists $x_1 \in X$ such that $x_1 \in (T_1 x_0)_\alpha$. By assumption (i), x_0 and x_1 are comparable. Since $(T_2 x_1)_\alpha$ is a nonempty compact subset of

X, there exists $x_2 \in (T_2 x_1)_\alpha$ such that

$$d(x_1, x_2) = p_\alpha(x_1, T_2 x_1) \leq D_\alpha(T_1 x_0, T_2 x_1).$$

Moreover, x_1 and x_2 are comparable. Continuing this process, one obtains a sequence $\{x_n\}$ in X such that $x_{2n+1} \in (T_1 x_{2n})_\alpha$ and $x_{2n+2} \in (T_2 x_{2n+1})_\alpha$ for all $n \geq 0$, x_{2n} and x_{2n+1} are comparable and

$$d(x_{2n+1}, x_{2n+2}) \leq D_\alpha(T_1 x_{2n}, T_2 x_{2n+1}).$$

Since x_{2n} and x_{2n+1} are comparable, by taking x_{2n} for x and x_{2n+1} for y in the inequality (1), it follows that

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &\leq D_\alpha(T_1 x_{2n}, T_2 x_{2n+1}) \\
&\leq \varphi\big(M(x_{2n}, x_{2n+1})\big) + \mathbf{L} \min\big\{ p_\alpha(x_{2n}, T_1 x_{2n}), \\
&\quad p_\alpha(x_{2n+1}, T_2 x_{2n+1}), p_\alpha(x_{2n}, T_2 x_{2n+1}), p_\alpha(x_{2n+1}, T_1 x_{2n}) \big\},
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
M(x_{2n}, x_{2n+1}) &= \max\bigg\{ d(x_{2n}, x_{2n+1}), p_\alpha(x_{2n}, T_1 x_{2n}), p_\alpha(x_{2n+1}, T_2 x_{2n+1}), \\
&\qquad \frac{1}{2}\big[p_\alpha(x_{2n}, T_2 x_{2n+1}) + p_\alpha(x_{2n+1}, T_1 x_{2n}) \big] \bigg\} \\
&= \max\bigg\{ d(x_{2n}, x_{2n+1}), p_\alpha(x_{2n}, T_1 x_{2n}), p_\alpha(x_{2n+1}, T_2 x_{2n+1}), \\
&\qquad \frac{1}{2} p_\alpha(x_{2n}, T_2 x_{2n+1}) \bigg\} \\
&\leq \max\bigg\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{1}{2} d(x_{2n}, x_{2n+2}) \bigg\} \\
&= \max\big\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \big\}.
\end{aligned}$$

Therefore from (2), we have

$$d(x_{2n+1}, x_{2n+2}) \leq \varphi\big(\max\big\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \big\}\big).$$

If $d(x_{2n}, x_{2n+1}) = 0$, it follows that $d(x_{2n+1}, x_{2n+2}) = 0$. Now, $x_{2n} = x_{2n+1} = x_{2n+2}$ implies $x_{2n+1} \in (T_1 x_{2n})_\alpha = (T_1 x_{2n+1})_\alpha$ and $x_{2n+1} = x_{2n+2} \in (T_2 x_{2n+1})_\alpha$, then the proof is finished. Therefore, we assume $d(x_{2n}, x_{2n+1}) > 0$. By Lemma 3.1, we get $\varphi(t) < t$ for each $t > 0$.

Consequently, if $d(x_{2n+1}, x_{2n+2}) > d(x_{2n}, x_{2n+1})$, for some n, then we have

$$d(x_{2n+1}, x_{2n+2}) \leq \varphi\big(d(x_{2n+1}, x_{2n+2})\big) < d(x_{2n+1}, x_{2n+2}),$$

which is a contradiction. Therefore

$$d(x_{2n+1}, x_{2n+2}) \leq \varphi\big(d(x_{2n}, x_{2n+1})\big) < d(x_{2n}, x_{2n+1}),$$

that is,

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}).$$ (3)

Similarly it can be shown that

$$d(x_{2n+3}, x_{2n+2}) \leq \varphi\big(d(x_{2n+2}, x_{2n+1})\big) < d(x_{2n+2}, x_{2n+1}),$$

that is,

$$d(x_{2n+3}, x_{2n+2}) < d(x_{2n+2}, x_{2n+1}).$$ (4)

Therefore, for all n, we get

$$d(x_n, x_{n+1}) \leq \varphi\big(d(x_{n-1}, x_n)\big)$$

$$\cdots$$

$$\leq \varphi^n\big(d(x_0, x_1)\big).$$

Hence

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m})$$

$$\leq \varphi^n\big(d(x_0, x_1)\big) + \cdots + \varphi^{n+m-1}\big(d(x_0, x_1)\big)$$

$$= \sum_{k=n}^{n+m-1} \varphi^k\big(d(x_0, x_1)\big).$$

Since $\sum_{n=1}^{\infty} \varphi^n(d(x_0, x_1)) < \infty$, then $\{x_n\}$ is a Cauchy sequence in X. Now, from the completeness of X, there exists $x \in X$ such that $x_n \to x$ as $n \to +\infty$ and since consecutive terms of $\{x_n\}$ are comparable, by hypothesis also x_n and x are comparable for all n. Now, we claim that $p_\alpha(x, T_2 x) = 0$ for each $\alpha \in [0, 1]$. If not, then for some $\alpha \in [0, 1]$, we have $p_\alpha(x, T_2 x) > 0$. Consider

$$p_\alpha(x, T_2 x) \leq d(x, x_{2n+1}) + p_\alpha(x_{2n+1}, T_2 x)$$

$$\leq d(x, x_{2n+1}) + D_\alpha(T_1 x_{2n}, T_2 x)$$

$$\leq d(x, x_{2n+1}) + \varphi\bigg(\max\bigg\{d(x_{2n}, x), p_\alpha(x_{2n}, T_1 x_{2n}), p_\alpha(x, T_2 x),$$

$$\frac{1}{2}\big[p_\alpha(x_{2n}, T_2 x) + p_\alpha(x, T_1 x_{2n})\big]\bigg\}\bigg)$$

$$+ \mathbf{L} \min\big\{p_\alpha(x_{2n}, T_1 x_{2n}), p_\alpha(x, T_2 x), p_\alpha(x_{2n}, T_2 x), p_\alpha(x, T_1 x_{2n})\big\}$$

$$= d(x, x_{2n+1}) + \varphi\bigg(\max\bigg\{d(x_{2n}, x), d(x_{2n}, x_{2n+1}), p_\alpha(x, T_2 x),$$

$$\frac{1}{2}\big[p_\alpha(x_{2n}, T_2 x) + d(x, x_{2n+1})\big]\bigg\}\bigg)$$

$$+ \mathbf{L} \min\big\{d(x_{2n}, x_{2n+1}), p_\alpha(x, T_2 x), p_\alpha(x_{2n}, T_2 x), p_\alpha(x, T_1 x_{2n})\big\}.$$

We note that $d(x_{2n}, x) \to 0$, $d(x_{2n}, x_{2n+1}) \to 0$, and $p_\alpha(x_{2n}, T_2x) \to p_\alpha(x, T_2x)$ as $n \to +\infty$. This implies that there exists $n_0 \in \mathbb{N}$ such that

$$\max\left\{ d(x_{2n}, x), d(x_{2n}, x_{2n+1}), \frac{1}{2}\left[p_\alpha(x_{2n}, T_2x) + d(x, x_{2n+1}) \right] \right\} \leq p_\alpha(x, T_2x)$$

for all $n \geq n_0$. Consequently, we have

$$p_\alpha(x, T_2x) \leq \varphi\big(p_\alpha(x, T_2x)\big) + \mathbf{L}\min\big\{ d(x_{2n}, x_{2n+1}), p_\alpha(x, T_2x), p_\alpha(x_{2n}, T_2x), p_\alpha(x, T_1x_{2n}) \big\}$$

for all $n \geq n_0$, which on taking the limit as $n \to +\infty$ gives

$$p_\alpha(x, T_2x) \leq \varphi\big(p_\alpha(x, T_2x)\big) < p_\alpha(x, T_2x),$$

a contradiction. Hence $p_\alpha(x, T_2x) = 0$ and so $x_\alpha \subset T_2x$. Similarly we deduce that $x_\alpha \subset T_1x$. □

From Theorem 3.2, assuming $\varphi(t) = qt$ with $0 < q < 1$ and $\mathbf{L} = 0$, we deduce the following result.

Corollary 3.3 *Let $(X, d, \preceq)$ be a complete ordered metric space and $T_1, T_2 : X \to W_\alpha(X)$ be two fuzzy mappings satisfying*

$$D_\alpha(T_1x, T_2y) \leq q\max\left\{ d(x, y), p_\alpha(x, T_1x), p_\alpha(y, T_2y), \frac{1}{2}\left[p_\alpha(x, T_2y) + p_\alpha(y, T_1x) \right] \right\} \qquad (5)$$

for all comparable elements $x, y \in X$. Also suppose that
  (i) *if $y \in (T_1x_0)_\alpha$, then $y, x_0 \in X$ are comparable,*
  (ii) *if $x, y \in X$ are comparable, then every $u \in (T_1x)_\alpha$ and every $v \in (T_2y)_\alpha$ are comparable,*
  (iii) *if a sequence $\{x_n\}$ in X converges to $x \in X$ and its consecutive terms are comparable, then x_n and x are comparable for all n.*
Then there exists a point $x \in X$ such that $x_\alpha \subset T_1x$ and $x_\alpha \subset T_2x$.

Now, we give an illustrative example, by adapting Example 6 in [46]; also we refer to the same paper for a better understanding of the situation.

Example 3.4 Let $X = [0, 1]$ endowed with the usual order of real numbers and the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Clearly (X, d) is a complete (ordered) metric space. Let $\alpha \in (0, 1/2)$ and define $\varphi : [0, +\infty) \to [0, +\infty)$ and $T_1, T_2 : X \to W_\alpha(X)$ by

$$\varphi(t) = \begin{cases} t^3/(1+t) & \text{if } x \in [0, 1], \\ \frac{1}{2} & \text{if } x \in (1, +\infty), \end{cases}$$

$$(T_10)(x) = (T_21)(x) = \begin{cases} 1 & \text{if } x = 0, \\ \alpha & \text{if } x \in (0, 1/2], \\ \alpha/2 & \text{if } x \in (1/2, 1], \end{cases}$$

$$(T_1 1)(x) = (T_2 0)(x) = \begin{cases} 1 & \text{if } x = 0, \\ 2\alpha & \text{if } x \in (0, 1/2], \\ \alpha/2 & \text{if } x \in (1/2, 1], \end{cases}$$

$$(T_1 z)(x) = (T_2 z)(x) = \begin{cases} 1 & \text{if } x = 0, \\ \alpha & \text{if } x \in (0, 1/2], \\ 0 & \text{if } x \in (1/2, 1], \end{cases} \quad \text{where } z \in (0, 1).$$

Then we discuss the existence of fixed fuzzy points of mappings T_1 and T_2. To this aim, we note that $(T_i 0)_\alpha = (T_i z)_\alpha = (T_i 1)_\alpha = [0, 1/2]$, $(T_i 0)_{\alpha/2} = (T_i 1)_{\alpha/2} = [0, 1]$, and $(T_i z)_{\alpha/2} = [0, 1/2]$, where $i = 1, 2$. Consequently, it is easy to show (see also [46]) that all the hypotheses of Theorem 3.2 are satisfied. In particular, condition (1) holds trivially since $D_\alpha(T_1 x, T_2 y) = 0$ for all $x, y \in X$. We conclude that each $x \in [0, 1/2]$ is such that $x_\alpha \subset T_1 x$ and $x_\alpha \subset T_2 x$.

On the other hand, in view of Definition 2.4, we can apply our Theorem 3.2 to establish the existence of a common fixed point of T_1 and T_2. In this case, we note that $(T_i 0)_1 = (T_i z)_1 = (T_i 1)_1 = \{0\}$, and hence $x = 0$ is a common fixed point of T_1 and T_2.

Now, we briefly discuss the validity of our theorem. In fact, a question that arises naturally is: 'Is it possible to prove this kind of result without assuming that $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$?'. In the sequel we provide a positive answer to the above question. Precisely, Theorem 3.2 still holds if the condition:

(a) $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$,

is replaced by

(b) $\varphi : [0, +\infty) \to [0, +\infty)$ is a right-continuous function such that $\varphi(t) < t$ for all $t > 0$.

Next, we give the proof of Theorem 3.2 under condition (b). To this aim, we recall the following lemma.

Lemma 3.5 *Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = 0.$$

If $\{x_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences converge to ε when $k \to +\infty$:

$$\begin{aligned} & \{d(x_{2m_k}, x_{2n_k})\}, \qquad \{d(x_{2m_k}, x_{2n_k+1})\}, \\ & \{d(x_{2m_k-1}, x_{2n_k})\}, \qquad \{d(x_{2m_k-1}, x_{2n_k+1})\}. \end{aligned} \tag{6}$$

Remark 3.6 Note that assertions similar to the above lemma (see, for example, [10]) were proved and used to obtain several fixed point results in many papers.

Finally, we state and prove the following result.

Theorem 3.7 *Let $(X, d, \preceq)$ be a complete ordered metric space and $T_1, T_2 : X \to W_\alpha(X)$ be two fuzzy mappings satisfying*

$$D_\alpha(T_1 x, T_2 y) \le \varphi(M(x, y)) + \mathbf{L} \min\{p_\alpha(x, T_1 x), p_\alpha(y, T_2 y), p_\alpha(x, T_2 y), p_\alpha(y, T_1 x)\} \tag{7}$$

for all comparable elements $x, y \in X$, where $\mathbf{L} \geq 0$,

$$M(x, y) = \max\left\{d(x, y), p_\alpha(x, T_1 x), p_\alpha(y, T_2 y), \frac{1}{2}\left[p_\alpha(x, T_2 y) + p_\alpha(y, T_1 x)\right]\right\}$$

and $\varphi : [0, +\infty) \to [0, +\infty)$ *is a right-continuous function such that* $\varphi(t) < t$ *for all* $t > 0$.
Suppose that

(i) *if* $y \in (T_1 x_0)_\alpha$, *then* $y, x_0 \in X$ *are comparable*,
(ii) *if* $x, y \in X$ *are comparable, then every* $u \in (T_1 x)_\alpha$ *and every* $v \in (T_2 y)_\alpha$ *are comparable*,
(iii) *if a sequence* $\{x_n\}$ *in X converges to* $x \in X$ *and its consecutive terms are comparable, then* x_n *and* x *are comparable for all* n.

Then there exists a point $x \in X$ *such that* $x_\alpha \subset T_1 x$ *and* $x_\alpha \subset T_2 x$.

Proof Following the proof of Theorem 3.2, we can construct a sequence $\{x_n\}$ such that (3) and (4) hold. It follows that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \quad \text{for all } n. \tag{8}$$

Thus, in this case $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of positive numbers and so there exists $r \geq 0$ such that $\lim_{n \to +\infty} d(x_n, x_{n+1}) = r$. Now, if $r > 0$, then passing to the limit when $n \to +\infty$ in $d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n))$, and using the properties of φ, we get

$$r \leq \varphi(r) < r,$$

a contradiction and so we have proved that $\lim_{n \to +\infty} d(x_n, x_{n+1}) = r = 0$.

Now, suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then Lemma 3.5 implies that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the sequences (6) converge to ε (from above) when $k \to +\infty$. Therefore, using (7) with $x = x_{2m_k}$ and $y = x_{2n_k+1}$, we get

$$D_\alpha(T_1 x_{2m_k}, T_2 x_{2n_k+1}) \leq \varphi\big(M(x_{2m_k}, x_{2n_k+1})\big) + \mathbf{L}\min\big\{p_\alpha(x_{2m_k}, T_1 x_{2m_k}),$$
$$p_\alpha(x_{2n_k+1}, T_2 x_{2n_k+1}), p_\alpha(x_{2m_k}, T_2 x_{2n_k+1}),$$
$$p_\alpha(x_{2n_k+1}, T_1 x_{2m_k})\big\}, \tag{9}$$

where

$$M(x_{2m_k}, x_{2n_k+1}) = \max\Big\{d(x_{2m_k}, x_{2n_k+1}), p_\alpha(x_{2m_k}, T_1 x_{2m_k}), p_\alpha(x_{2n_k+1}, T_2 x_{2n_k+1}),$$
$$\frac{1}{2}\big[p_\alpha(x_{2m_k}, T_2 x_{2n_k+1}) + p_\alpha(x_{2n_k+1}, T_1 x_{2m_k})\big]\Big\}$$
$$\to \varepsilon, \quad \text{as } k \to +\infty.$$

Using the properties of φ, we obtain the contradiction $\varepsilon \leq \varphi(\varepsilon) < \varepsilon$, since $\varepsilon > 0$. Thus $\{x_{2n}\}$ is a Cauchy sequence and hence also $\{x_n\}$ is a Cauchy sequence. The rest of the proof is the same as the proof of Theorem 3.2 and so to avoid repetition we omit the details. □

To conclude this section, we give an example to illustrate Theorem 3.7, in the case of a single mapping.

Example 3.8 Let a, b, and c be three real numbers such that $a < b < c$ and consider $X = \{a, b, c\}$ endowed with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let $\alpha \in (0, 1/3)$ and define $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \begin{cases} t^2/2 & \text{if } t \leq 1, \\ \sqrt{t}/2 & \text{if } t > 1, \end{cases}$$

and $T : X \to W_\alpha(X)$ by

$$(Ta)(x) = (Tb)(x) = \begin{cases} 1 & \text{if } x = a, \\ \alpha/2 & \text{if } x = b, \\ \alpha & \text{if } x = c, \end{cases} \qquad (Tc)(x) = \begin{cases} 1 & \text{if } x = a, \\ \alpha/2 & \text{if } x = b, \\ 3\alpha & \text{if } x = c. \end{cases}$$

Firstly, searching for fixed fuzzy points, we notice that $(Ta)_{\alpha/2} = (Tb)_{\alpha/2} = (Tc)_{\alpha/2} = \{a, b, c\}$ and $(Ta)_\alpha = (Tb)_\alpha = (Tc)_\alpha = \{a, c\}$. Also, it is easy to show that all the hypotheses of Theorem 3.7 with $T_1 = T_2 = T$ are satisfied and hence c_α is a fixed fuzzy point of X. Secondly, searching for fixed points, from $(Ta)_1 = (Tb)_1 = (Tc)_1 = \{a\}$ we deduce that a is a fixed point of T.

4 Application to ordinary fuzzy differential equation

In this section, we present a situation where our obtained results can be applied. Precisely, we study the existence of solution for the second order nonlinear boundary value problem:

$$\begin{cases} x''(t) = k(t, x(t), x'(t)), & t \in [0, \Lambda], \Lambda > 0, \\ x(t_1) = x_1, \\ x(t_2) = x_2, & t_1, t_2 \in [0, \Lambda], \end{cases} \tag{10}$$

where $k : [0, \Lambda] \times W(X) \times W(X) \to W(X)$ is a continuous function. This problem is equivalent to the integral equation

$$x(t) = \int_{t_1}^{t_2} G(t, s) k(s, x(s), x'(s)) \, ds + \beta(t), \quad t \in [0, \Lambda], \tag{11}$$

where the Green's function G is given by

$$G(t, s) = \begin{cases} \frac{(t_2 - t)(s - t_1)}{t_2 - t_1} & \text{if } t_1 \leq s \leq t \leq t_2, \\ \frac{(t_2 - s)(t - t_1)}{t_2 - t_1} & \text{if } t_1 \leq t \leq s \leq t_2, \end{cases}$$

and $\beta(t)$ satisfies $\beta'' = 0$, $\beta(t_1) = x_1$, $\beta(t_2) = x_2$. Let us recall some properties of $G(t, s)$, precisely we have

$$\int_{t_1}^{t_2} |G(t, s)| \, ds \leq \frac{(t_2 - t_1)^2}{8}$$

and

$$\int_{t_1}^{t_2} \left| G_t(t,s) \right| ds \leq \frac{(t_2 - t_1)}{2}.$$

If necessary, the reader can refer to [47, 48] for a more detailed explanation of the background of the problem. Here, we shall prove our result, by establishing the existence of a common fixed point for a pair of integral operators defined as

$$T_i(x)(t) = \int_{t_1}^{t_2} G(t,s) k_i \big(s, x(s), x'(s)\big) ds + \beta(t), \quad t \in [0, \Lambda], i \in \{1, 2\}, \tag{12}$$

where $k_1, k_2 \in C([0, \Lambda] \times W(X) \times W(X), W(X))$, $x \in C^1([0, \Lambda], W(X))$, and $\beta \in C([0, \Lambda], W(X))$.

Theorem 4.1 *Assume that the following conditions are satisfied*:

(a) $k_1, k_2 : [0, \Lambda] \times W(X) \times W(X) \to W(X)$ *are increasing in its second and third variables,*

(b) *there exists $x_0 \in C^1([0, \Lambda], W(X))$ such that, for all $t \in [0, \Lambda]$, we have*

$$x_0(t) \leq \int_{t_1}^{t_2} G(t,s) k_1 \big(t, x_0(s), x_0'(s)\big) ds + \beta(t),$$

where $t_1, t_2 \in [0, \Lambda]$,

(c) *there exist $\gamma, \delta > 0$ such that, for all $t \in [0, \Lambda]$, we have*

$$\left| k_1 \big(t, x(t), x'(t)\big) - k_2 \big(t, y(t), y'(t)\big) \right| \leq \gamma \left| x(t) - y(t) \right| + \delta \left| x'(t) - y'(t) \right|$$

for all comparable $x, y \in C^1([0, \Lambda], W(X))$,

(d) *for $\gamma, \delta > 0$ and $t_1, t_2 \in [0, \Lambda]$ we have*

$$\gamma \frac{(t_2 - t_1)^2}{8} + \delta \frac{(t_2 - t_1)}{2} < 1,$$

(e) *if $x, y \in C^1([0, \Lambda], W(X))$ are comparable, then every $u \in (T_1 x)_1$ and every $v \in (T_2 y)_1$ are comparable.*

Then the pair of nonlinear integral equations

$$x(t) = \int_{t_1}^{t_2} G(t,s) k_i \big(s, x(s), x'(s)\big) ds + \beta(t), \quad t \in [0, \Lambda], i \in \{1, 2\} \tag{13}$$

has a common solution in $C^1([t_1, t_2], W(X))$.

Proof Consider $\mathcal{C} = C^1([t_1, t_2], W(X))$ with the metric

$$D(x, y) = \max_{t_1 \leq t \leq t_2} \big(\gamma \left| x(t) - y(t) \right| + \delta \left| x'(t) - y'(t) \right| \big).$$

The space $(\mathcal{C}, D)$ is a complete metric space, which can also be equipped with the partial ordering given by

$$x, y \in \mathcal{C}, \quad x \preceq y \quad \Leftrightarrow x(t) \quad \leq y(t) \quad \text{for all } t \in [0, \Lambda].$$

In [12], it is proved that $(\mathcal{C}, \preceq)$ satisfies the following condition:

(r) for every nondecreasing sequence $\{x_n\}$ in $\mathcal{C}$ convergent to some $x \in \mathcal{C}$, we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Let $T_1, T_2 : \mathcal{C} \to \mathcal{C}$ be two integral operators defined by (12); clearly, T_1, T_2 are well defined since k_1, k_2, and β are continuous functions. Now, x^* is a solution of (13) if and only if x^* is a common fixed point of T_1 and T_2.

By hypothesis (a), T_1, T_2 are increasing and, by hypothesis (b), $x_0 \preceq T_1(x_0)$. Consequently, in view of condition (r), hypotheses (i)-(iii) of Corollary 3.3 hold true.

Next, for all comparable $x, y \in \mathcal{C}$, by hypothesis (c) we have successively

$$\left| T_1(x)(t) - T_2(y)(t) \right| \leq \int_{t_1}^{t_2} \left| G(t,s) \right| \left| k_1\left(s, x(s), x'(s)\right) - k_2\left(s, y(s), y'(s)\right) \right| ds$$

$$\leq D(x,y) \int_{t_1}^{t_2} \left| G(t,s) \right| ds \leq \frac{(t_2 - t_1)^2}{8} D(x,y) \tag{14}$$

and

$$\left| \left(T_1(x)\right)'(t) - \left(T_2(y)\right)'(t) \right| \leq \int_{t_1}^{t_2} \left| G_t(t,s) \right| \left| k_1\left(s, x(s), x'(s)\right) - k_2\left(s, y(s), y'(s)\right) \right| ds$$

$$\leq D(x,y) \int_{t_1}^{t_2} \left| G_t(t,s) \right| ds \leq \frac{(t_2 - t_1)}{2} D(x,y). \tag{15}$$

From (14) and (15), we obtain easily

$$D(T_1 x, T_2 y) \leq \left(\gamma \frac{(t_2 - t_1)^2}{8} + \delta \frac{(t_2 - t_1)}{2} \right) D(x,y).$$

Consequently, in view of hypothesis (d), the contractive condition (5) is satisfied with

$$q = \gamma \frac{(t_2 - t_1)^2}{8} + \delta \frac{(t_2 - t_1)}{2} < 1.$$

Therefore, Corollary 3.3 applies to T_1 and T_2, which have a common fixed point $x^* \in \mathcal{C}$, that is, x^* is a common solution of (13). $\qquad \square$

As an immediate consequence of Theorem 4.1, in the case $T_1 = T_2 = T$, we find that the integral equation (11) has a solution in $\mathcal{C}$, and hence the second order nonlinear boundary value problem (10) has a solution.

5 Conclusions

Our Theorem 3.2 gives a contribution to the 'fixed point arena' in the sense of generalization by using fuzziness under ordered metric spaces and by assuming the validity of the contractive condition only on elements that are comparable in respect to partial ordering. Moreover, using recent ideas in the literature [13, 23, 24, 31], it is possible to extend our result to non-Archimedean fuzzy metric spaces and probabilistic metric spaces endowed with a partial ordering induced by an appropriate function.

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Acknowledgements

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