International Journal of Pure and Applied Mathematics
Volume 115 No. 2 2017, 361-367
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v115i2.13

ijpam.eu

FIXED POINTS OF CHATTERJEE AND CIRIC CONTRACTIONS ON AN S-METRIC SPACE

T. Phaneendra[1 §], K. Kumara Swamy[2]

[1]Department of Mathematics
School of Advanced Sciences,
VIT University
Vellore, 632014, Tamil Nadu, INDIA

[2]Department of Mathematics
Hyderabad Institute of Technology & Management
Medchal District, Hyderabad, Telangana State, INDIA

Abstract: Unique fixed points are obtained for Chatterjee and Ciric contractions on an S-metric space, which are then shown to be S-contractive fixed points.

AMS Subject Classification: 54H25
Key Words: S-metric space, S-Cauchy sequence, Fixed point, S-contractive fixed point

1. Introduction

Let X be a nonempty set. Sedghi et al [7] introduced an S-metric $S : X \times X \times X \to [0, \infty)$ on X satisfying the following conditions:

(S1) $S(x, y, z) = 0$ if and only if $x, y, z \in X$ are such that $x = y = z$,

(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called an S-metric space. We obtain from Axiom (S2) that

$$S(x, x, y) = S(y, y, x) \text{ for all } x, y \in X. \tag{1.1}$$

Received: February 23, 2017
Revised: May 23, 2017
Published: July 14, 2017

§Correspondence author

Definition 1.1. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a S-metric space (X, S) is said to be S-convergent, if there exists a point x in X such that $S(x_n, x_n, x) \to 0$ as $n \to \infty$.

Definition 1.2. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a S-metric space (X, S) is said to be S-Cauchy if $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$.

Definition 1.3. The space (X, S) is said to be S-complete, if every S-Cauchy sequence in X converges in it.

The well-known infimum property of real numbers states that a nonempty and bounded set of real numbers has an infimum in $\mathbb{R}$. In particular,

Lemma 1.1. *If S is a nonempty subset of nonnegative real numbers, then $\alpha = \inf S \geq 0$ and $\lim_{n\to\infty} p_n = \alpha$ for some sequence $\langle p_n \rangle_{n=1}^{\infty}$ in S.*

As an elementary application of Lemma 1.1, unique fixed points of a Chatterjee and Ciric- contractions are obtained in S-metric space. Further, the unique fixed points for these two contractions are shown to be S-contractive fixed points.

2. Main Results

Our first result is

Theorem 2.1. *Let f be a Chatterjee contraction on a complete S-metric space (X, S) with the choice*

$$S(fx, fx, fy) \leq \alpha \max\{S(fx, fx, y), S(fy, fy, y)\} \text{ for all } x, y \in X, \tag{2.1}$$

where $0 \leq \alpha < 1/3$. Then f has a unique fixed point.

Proof. We divide the proof into various steps:

Step 1 – Existence of the infimum

Define $A = \{S(fx, fx, x) : x \in X\}$. Then by Lemma 1.1, the infimum infimum a of A exists and is nonnegative.

Step 2 – Vanishing infimum

If $a > 0$, writing $y = fx$ in (2.1) and using (1.1), we get

$$S(fx, fx, f^2x) \leq \alpha \max\left\{S(fx, fx, fx), S(f^2x, f^2x, x)\right\}$$

$$\leq \alpha[S(f^2x, f^2x, fx) + S(f^2x, f^2x, fx) + S(x, x, fx)]$$
$$= \alpha[2S(fx, fx, f^2x) + S(fx, fx, x)]$$

or

$$(1 - 2\alpha)S(fx, fx, f^2x) \leq \alpha S(fx, fx, x)$$
$$S(fx, fx, f^2x) \leq \left(\tfrac{\alpha}{1-2\alpha}\right) S(fx, fx, x)$$

This implies that $0 < a \leq \alpha a < a$, which is again a contradiction. Hence $a = 0$.

Step 3 – Existence of a sequence

Hence, there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X such that

$$S(fx_n, fx_n, x_n) \in A \text{ for all } n = 1, 2, 3, ... \text{ and } \lim_{n\to\infty} S(fx_n, fx_n, x_n) = 0. \quad (2.2)$$

Step 4 – $\langle x_n \rangle_{n=1}^{\infty}$ is S-Cauchy
In fact, by $(S2)$ and (1.1), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq S(x_n, x_n, fx_n) + S(x_n, x_n, fx_n) + S(x_m, x_m, fx_n) \\ &= 2S(x_n, x_n, fx_n) + S(x_m, x_m, fx_n) \\ &\leq 2S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m) + S(x_m, x_m, fx_m) \\ &\qquad + S(fx_n, fx_n, fx_m) \\ &= 2[S(fx_n, fx_n, x_n) + S(fx_m, fx_m, x_m)] \\ &\qquad + S(fx_n, fx_n, fx_m). \end{aligned} \quad (2.3)$$

Now, with $x = x_n$ and $y = x_m$, (2.1) gives,

$$\begin{aligned} S(fx_n, fx_n, fx_m) &\leq \alpha \max\{S(fx_n, fx_n, x_m), S(fx_m, fx_m, x_n)\} \\ &\leq \alpha \max\big\{[2S(fx_n, fx_n, x_n) + S(x_m, x_m, x_n)], \\ &\qquad [2S(fx_m, fx_m, x_m) + S(x_n, x_n, x_m)]\big\} \\ &= \alpha[2S(fx_n, fx_n, x_n) + 2S(fx_m, fx_m, x_m) + 2S(x_n, x_n, x_m)] \end{aligned}$$

Inserting this in (2.3), we get

$$\begin{aligned} S(x_n, x_n, x_m) &\leq (2\alpha + 1)[S(fx_n, fx_n, x_n) + S(fx_m, fx_m, x_m)] \\ &\qquad + 2\alpha S(x_n, x_n, x_m) \end{aligned}$$

or

$$(1-2\alpha)S(x_n,x_n,x_m) \le (2\alpha+1)[S(fx_n,fx_n,x_n)+S(fx_m,fx_m,x_m)]$$

so that

$$S(x_n,x_n,x_m) \le \left(\tfrac{2\alpha+1}{1-2\alpha}\right)[S(x_n,x_n,fx_n)+S(x_m,x_m,fx_m)].$$

Applying the limit as $m,n\to\infty$ in this and using (2.2) we obtain that $\langle x_n\rangle_{n=1}^{\infty}$ is a S-Cauchy sequence in X.

Step 5 – S-convergence

Since, X is S-complete, we find the point p in X such that

$$\lim_{n\to\infty} x_n = p. \tag{2.4}$$

Step 6 – S-convergent limit as a fixed point

Again repeatedly using ($S2$),

$$\begin{aligned} S(fp,fp,p) &\le S(fp,fp,fx_n)+S(fp,fp,fx_n)+S(p,p,fx_n).\\ &= 2S(fp,fp,fx_n)+S(p,p,fx_n)\\ &= 2S(fx_n,fx_n,fp)+S(fx_n,fx_n,p) \end{aligned} \tag{2.5}$$

Now, from (2.1) with $x=x_n$ and $y=p$, it follows that

$$\begin{aligned} S(fx_n,fx_n,fp) &\le \alpha \max\left\{S(fx_n,fx_n,p),S(fp,fp,x_n)\right\}\\ &\le \alpha[S(fx_n,fx_n,p)+S(fp,fp,x_n)] \end{aligned} \tag{2.6}$$

Substituting (2.6) in (2.5), we get

$$\begin{aligned} S(fp,fp,p) &\le 2\alpha[S(fx_n,fx_n,p)+S(fp,fp,x_n)]+S(fx_n,fx_n,p)\\ &= (2\alpha+1)S(fx_n,fx_n,p)+2\alpha S(fp,fp,x_n) \end{aligned}$$

In the limiting case as $n\to\infty$, this in view of (2.2) and (2.4) implies $S(fp,fp,p)=0$ or $fp=p$. Thus p is a fixed point.

Step 7 – Uniqueness of the fixed point

Let q be another fixed point of f. Then, (2.1) with $x=p$ and $y=q$ gives

$$\begin{aligned} S(p,p,q) &= S(fp,fp,fq)\\ &\le \alpha \max\left\{S(fp,fp,q),S(fq,fq,p)\right\} \end{aligned}$$

$$= \alpha \max \{S(p,p,q), S(q,q,p)\}$$
$$= \alpha S(p,p,q)$$

or

$$(1-\alpha)S(p,p,q) \leq 0$$

so that $p = q$. That is, p is the unique fixed point of f. □

Our next result is:

Theorem 2.2. *Let f be a Ciric-type contraction on a complete S-metric space (X,S) such that*

$$S(fx,fx,fy) \leq \alpha \max \{S(x,x,y), S(fx,fx,x), S(fx,fx,y), S(fy,fy,x), S(fy,fy,y)\} \tag{2.7}$$

for all $x, y \in X$, where $0 \leq \alpha < 1/3$. Then f has a unique fixed point.

A unique fixed point p for (2.7) is obtained, similar to the previous proof and is omitted here.

3. S-Contractive Fixed Point

The notion of a G-metric space was introduced by Mustafa and Sims in [1], as a generalization of a metric space. In this setting, contractive fixed points were introduced in [2]. For further study on this idea, one can refer to [3, 4, 5, 6].

Now, we have

Definition 3.1. Let f be a self-map on an S-metric space (X,S). A fixed point p of f is a contractive fixed point, if for every $x_0 \in X$, the f-orbit $O_f(x_0) = \langle x_0, fx_0, ..., f^n x_0, ...\rangle$ converges to p.

We now show that the unique fixed point p of Chatterjee contraction (2.1) is an S-contractive fixed point.

Proof. Writing $x = f^{n-1}x_0, y = p$ in (2.1), we get

$$\begin{aligned} S(f^n x, f^n x, p) &= S(f^n x, f^n x, fp) \\ &\leq \alpha \max \{S(f^n x, f^n x, p), S(fp, fp, f^{n-1}x)\} \\ &= \alpha \max \{S(f^n x, f^n x, p), S(f^{n-1}x, f^{n-1}x, fp)\} \end{aligned}$$

$$\begin{aligned}
&\le \max\left\{S(f^nx,f^nx,p),[2S(f^{n-1}x,f^{n-1}x,f^nx)+S(fp,fp,f^nx)]\right\}\\
&= \max\left\{S(f^nx,f^nx,p),[2S(f^{n-1}x,f^{n-1}x,f^nx)+S(f^nx,f^nx,p)]\right\}\\
&\le \alpha[2S(f^{n-1}x,f^{n-1}x,f^nx)+S(f^nx,f^nx,p)]
\end{aligned}$$

or

$$\begin{aligned}
(1-\alpha)S(f^nx,f^nx,p) &\le 2\alpha S(f^{n-1}x,f^{n-1}x,f^nx)\\
S(f^nx,f^nx,p) &\le \left(\tfrac{2\alpha}{1-\alpha}\right)S(f^{n-1}x,f^{n-1}x,f^nx). \qquad (3.1)
\end{aligned}$$

Proceeding the limit as $n\to\infty$ in (3.1) , we get $S(f^nx,f^nx,p)\to 0$. Thus $f^nx_0\to p$ for each $x_0\in X$. Thus p is a S-contractive fixed point of f. □

We finally show that the unique fixed point p of Ciric contraction (2.7) is an S-contractive fixed point as follows:

Proof. Writing $x=f^{n-1}x_0, y=p$ in (2.7), and then using (1.1), we get

$$\begin{aligned}
S(f^nx,f^nx,fp) &= S(f^nx,f^nx,p)\\
&\le \alpha\max\left\{S(f^{n-1}x,f^{n-1}x,p),S(f^nx,f^nx,f^{n-1}x),S(f^nx,f^nx,p),\right.\\
&\qquad\qquad \left.S(fp,fp,f^{n-1}x),S(fp,fp,p)\right\}\\
&\le \alpha\max\left\{S(f^{n-1}x,f^{n-1}x,p),S(f^nx,f^nx,f^{n-1}x),S(f^nx,f^nx,p)\right\}\\
&\le \alpha\max\left\{[2S(f^{n-1}x,f^{n-1}x,p)+S(p,p,f^nx)],\right.\\
&\qquad\qquad \left.S(f^nx,f^nx,f^{n-1}x),S(f^nx,f^nx,p)\right\}\\
&\le \alpha[2S(f^{n-1}x,f^{n-1}x,p)+S(f^nx,f^nx,p)]
\end{aligned}$$

or

$$\begin{aligned}
(1-\alpha)S(f^nx,f^nx,p) &\le 2\alpha S(f^{n-1}x,f^{n-1}x,f^nx)\\
S(f^nx,f^nx,p) &\le \left(\tfrac{2\alpha}{1-\alpha}\right)S(f^{n-1}x,f^{n-1}x,f^nx). \qquad (3.2)
\end{aligned}$$

As $n\to\infty$ in (3.1) , we see that

$$S(f^nx,f^nx,p)\to 0.$$

Thus $f^nx_0\to p$ for each $x_0\in X$. Thus p is a S-contractive fixed point of f. □

References

[1] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *Journal of Nonlinear and Convex Anal.*, **7**, No. 2 (2006), 289-297.

[2] T. Phaneendra, K. Kumara Swamy, Unique fixed point in G-metric space through greatest lower bound properties, *NoviSad J. Math.*, **43** (2013), 107-115.

[3] T. Phaneendra, S. Saravanan, On some misconceptions and chatterjee-type G-contraction, *Int. J. Pure Appl. Math.*, **109**, No. 4, 789-797.

[4] T. Phaneendra, S. Saravanan, On G-contractive fixed points, *Jnanabha*, **46**, 105-112.

[5] T. Phaneendra, S. Saravanan, Bounded orbits and G-contractive fixed points, *Comm. Appl. Anal.*, **20**, 441-457.

[6] S. Saravanan, T. Phaneendra, Fixed point as a G-contractive fixed point, *Int. J. Appl. Engg. Res.*, **11**, No. 1 (2016), 316-319.

[7] Shaban Sedghi, Nabi Shobe, Abdelkrim Aliouche, A generalization of fixed point theorems in S-metric spaces, *Matematiqki Vesnik*, **64**, No. 3 (2012), 258-266.