Form-Finding of Tensegrity Structures based on Force Density Method

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Abstract

Background/Objectives: Determining equilibrium configuration of a tensegrity structure is known as form-finding. This paper critically reviews various form-finding techniques based on force density method. **Methods/ Statistical Analysis:** The most appropriate method out of all the form-finding methods, to ascertain new formation of tensegrity structures is Force density method. The techniques based on Force density method such as Advanced form-finding method, Adaptive force density method, Form-finding via Genetic algorithm and Algebraic tensegrity form-finding are reviewed in detail along with scope, limitation and suitability of each method. **Findings:** This paper lists out the input and output parameters of each technique and comparing the effectiveness of each. Most of these methods require only type of the member (tension or compression) and topology of the tensegrity structure provided by connectivity matrix as initial parameters. Form-finding of regular shaped structures such as cable domes is best performed by Advanced form-finding method. Adaptive form-finding method is found to be efficient for irregular shaped structures and also to search novel configurations. Form-finding by means of genetic algorithm provides solutions for regular shaped tensegrity structures. Algebraic method is found to be highly efficient and automated general method among all the form-finding methods discussed in this paper. A structure's evolving geometry can be better regulated by this method. **Applications/ Improvements:** This paper presents the techniques which require minimum initial parameters for form-finding of tensegrity structures which also helps in choice of methods based on known parameters of the structure.

Keywords: Eigen values, Force Density, Form-Finding, New Configuration, Tensegrity

1. Introduction

One of the trailblazers in the field of tensegrity structures, Fuller[1] coined the term tensegrity by merging the term 'tensional' with 'integrity'. The integration of discontinuous elements in compression (struts/ bars) by continuous elements (strings/ cables) in tension resulting in a stable form is represented by the term 'tensegrity'. These structures are considered to be the most important turning point in the evolution of special structures[2]. Versatile characteristics of tensegrity structures like self-erection, deployability, light weight, efficiency and aesthetic elegance make them highly desirable for large span stadium roofs to aerospace structures[3-7]. Determining equilibrium configuration of a tensegrity structure, the principle stage in its design is known as form-finding.

Various methods are established for form-finding process, such as analytical method, Non-linear programming method, Energy method, Dynamic relaxation method, Reduced coordinate method, Force density method, etc[8]. The most appropriate method out of all the form-finding methods, to ascertain new formation of tensegrity structures with minimum parameters is Force density method. Form-finding process requires some prior knowledge like topology of structure, lengths, cross sectional areas, Young's moduli and types of elements (tension or compression). The choice of form-finding method depends on the availability of initial parameters and complexity of the structure. Hence form-finding techniques which require minimum parameters are highly sought after. This paper critically reviews various form-finding techniques which are based on force density method and require very few

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initial parameters. The scope, limitation and suitability of each of those methods are also discussed along with comparison of the same.

2. Force Density Method

Scheck[9] introduced this method in 1974. All the elements of a tensegrity structure are pin jointed. In the process of form-finding it is assumed that, external loads acting on the structure are zero and loads due to self-weight are insignificant. The topology of structure, force density and types of elements are the parameters required for this method.

For a d-dimensional structure with b elements, n_f fixed nodes and n free nodes, the equations of equilibrium in x direction can be written as[9]

$$C_S^T Q C_S x_s = f_x \qquad (1)$$

The column vector contains external forces acting on nodes in x direction is represented by f_x. Incidence matrix or connectivity matrix is denoted by C_s ($\in R^{b \times (n+n_f)}$). Q ($\in R^{b \times b}$) is a diagonal square matrix, and x_s represents a column vector contains x coordinates. Similarly, equilibrium equations in y- and z- direction can be written in terms of respective coordinates. For an element k with initial node i and final node j (i < j), C_s and Q written as discussed by Tran and Lee[10-12], Schek[9] and Zhang and Ohsaki[13] as follows,

$$C_{S(k,p)} = \begin{cases} 1 \text{ for } p = i \\ -1 \text{ for } p = j \\ 0 \text{ otherwise} \end{cases} \qquad (2)$$

$$Q = diag(q) \qquad (3)$$

q ($\in R^b$) is the force density vector[9-12], defined by

$$q = \{q_1 \; q_2 \; ... \; q_b\}^T \qquad (4)$$

where each element of the above vector $q_i = p_i / l_i$ is the ratio of force p_i to length l_i, known as tension coefficient or force density. If free nodes numbered first, C_s can be separated as

$$C_S = [C \; C_f] \qquad (5)$$

Equation 1 takes the form as follows,

$$C^T Q C x = f_x - C^T Q C_f x_f \qquad (6)$$

The column vector containing x coordinates of free nodes and fixed nodes respectively are denoted by x and x_f. Self-stressed systems do not need any fixed nodes. So equation 6 can now be written in x, y and z direction as follows

$$Dx = 0 \qquad (7a)$$

$$Dy = 0 \qquad (7b)$$

$$Dz = 0 \qquad (7c)$$

where D ($\in R^{n \times n}$) is known as stress matrix[14] or force density matrix[8-13] which is given as

$$D = C^T Q C \qquad (8)$$

$$D = C^T diag(q) C \qquad (9)$$

Vassart and Motro[15] give D without using equation 8 or equation 9 as

$$D_{(i,j)} = \begin{cases} -q_k & \text{if member k connects nodes } i \& j \\ \sum_{k \in \omega} q_k & \text{for } i = j \\ 0 & \text{other cases} \end{cases} \qquad (10)$$

where ω denotes set of all the elements connected to node i. Equation 7 can be expressed as

$$D[x \; y \; z] = [0 \; 0 \; 0] \qquad (11)$$

Substituting equation 9 into equation 7, the equilibrium equations now be written as

$$C^T diag(q) C x = 0 \qquad (12a)$$

$$C^T diag(q) C y = 0 \qquad (12b)$$

$$C^T diag(q) C z = 0 \qquad (12c)$$

Equation 11 expressed in the simplified form as,

$$Aq = 0 \qquad (13)$$

Equilibrium matrix is denoted by A ($\in R^{dn \times b}$) and given by following equation,

$$A = \begin{bmatrix} C^T diag(Cx) \\ C^T diag(Cy) \\ C^T diag(Cz) \end{bmatrix} \qquad (14)$$

Hence force density method provides the equilibrium equations which have to be solved to find the nodal coor-

dinates. This method requires no lengths of members. But choosing right set of tension coefficients in the beginning stage of form-finding is difficult. The following sections introduce methods of form-finding which require minimum number of initial parameters, based on force density method.

3. Advanced Form-Finding Method

Advanced form-finding is a numerical method put forward by Tran and Lee[10-12]. This method requires only the topology by connectivity matrix in addition to member types, as initial information. Possible nodal coordinates sets and corresponding force density values, which ensure conditions related to rank deficiency of D and A are determined by iterative performance of eigenvalue decomposition of D followed by singular value decomposition (SVD) of A.

The rank deficiency of D and A matrices respectively are

$$n_D = n - rank\ (D) \qquad (15a)$$

$$n_A = b - rank\ (A) \qquad (15b)$$

Tran and Lee[10-12] give the most significant condition of rank deficiency associated with semi-definite stress matrix D as

$$n_D \geq d + 1 \qquad (16)$$

The condition of rank deficiency to have at least one self-stressed state is given by

$$s = n_A \geq 1 \qquad (17)$$

Number of independent self-stressed states is denoted by s.

Initial values of tension coefficients provided based on the element type. For strings +1 and for bars -1 are given as follows,

$$q^0 = [+1 + 1 \dots + 1 \dots - 1 \quad - 1 \dots - 1]^T \qquad (18)$$

Then, q^0 is used to compute stress matrix D by equation 9. Subsequently, eigenvalue decomposition of D is performed to determine nodal coordinates[10-12]. Substitution of these values in equation 13 to obtain tension coefficient vector q as a result of singular value decomposition of A as described by Pellegrino[16] is the next step. After that, the D matrix is updated by equation 9. Finally suitable sets of

nodal coordinates and force density values are found out by repeating the above mentioned steps till conditions as given by 16 & 17 are satisfied. Let the minimum values of rank deficiencies required for D and A be denoted by n^*_D and n^*_A respectively.

$$n^*_D = d + 1 \qquad (19)$$

$$n^*_A = 1 \qquad (20)$$

The factorised form of force density matrix D using eigenvalue decomposition[10-12] is as follows

$$D = \Phi\Delta\Phi^T \qquad (21)$$

Orthogonal matrix with eigenvector basis $\emptyset_i (\in R^n)$ of D as i[th] column is denoted by $\Phi(\in R^{n \times n})$. Diagonal matrix with the corresponding eigenvalues λ_i as diagonal elements is denoted by $\Delta(\in R^{n \times n})$. The eigenvalues are in the ascending order.

Number of eigenvalues of D with zero value is equal to the dimension of its null space. Let e be the non positive eigenvalues of D. When $e \leq n^*_D$, possible nodal coordinates are selected by taking the first orthonormal eigenvectors of Φ[10-12]. The form-finding process then iteratively updates q to make it a very small value till a possible extend to force initial n^*_D eigenvalues of D to zero. Thus D becomes positive semi-definite that in turn makes corresponding self-stressed structure super-stable[12]. When $e > n^*_D$, D matrix is not necessarily positive semi-definite and this technique will additionally calculate tangent stiffness matrix as given in Tran and Lee and Zhang and Ohsaki[12,17]. Constrained rigid body motions of a self-stressed structure ensure its stability, if tangent stiffness matrix of the structure is positive definite. Subsequently, singular value decomposition is performed on A, calculated from the updated coordinates obtained from previous step to obtain q.

This method is found to be suitable for regular tensegrity structures including cable domes in two dimensional and three dimensional spaces. However, the possibility of this method for structures involving large number of members and complex constraints need further investigation.

4. Adaptive Force Density Method

This numerical method proposed by Zhang and Ohsaki[13] also requires only topology and types of members as input parameters. It involves repetitive performance of eigen-

value analysis followed by spectral decomposition on D to determine possible set of tension coefficient values and nodal coordinates which satisfy required rank deficiency of D. A set of nodal coordinates which are independent in nature has to be assigned then to obtain exclusive and stable formation of the structure.

Let ω denote set of all elements connect with node i. i^{th} column D_i of D can be written as follows from equation 10,

$$B^i q = D_i \qquad (22)$$

The (j, k) component of B^i can be obtained from following expression where value of k ranges from 1 to b (number of elements).

$$B^i{}_{(j,k)} = \begin{cases} 1 & \text{if } i = j \text{ and } k \in \omega \\ -1 & \text{if member k connects nodes } i \ \& \ j \\ 0 & \text{otherwise} \end{cases} \qquad (23)$$

Let $B^T = (B^{1T}, \dots, B^{iT}, \dots, B^{nT})$ and $g^T = (D^T{}_1, \dots, D^T{}_i, \dots, D^T{}_n)$,

$$Bq = g \qquad (24)$$

The following expression is used to specify linear constraints related to some specific tension coefficient values using vector g^e and constant matrix B^e.

$$B^e q = g^e \qquad (25)$$

Let $\overline{B}^T = (B^T, B^{eT})$ and $\overline{g}^T = (g^T, g^{eT})$, From equation 24 and equation 25

$$\overline{B}q = \overline{g} \qquad (26)$$

Then least square solution of equation 26 is obtained as per the below expression[13].

$$q = \overline{B}^- \overline{g} \qquad (27)$$

Generalized inverse of $\overline{B}$ is denoted by $\overline{B}^-$ in the above expression.

Assign 0 to initial n^*_D eigenvalues of D to get $\overline{\Delta}$ with modified eigenvalues when non-positive eigenvalues of D, $e \le n^*_D$. Then equation 21 is updated as follows

$$\overline{D} = \Phi \overline{\Delta} \Phi^T \qquad (28)$$

Now $\overline{D}$ is having no negative eigenvalues and also with required rank deficiency n^*_D. The condition in which $e > n^*_D$ is not considered in this method. But, Advanced form-finding method addresses both of these situations[10-12].

In the beginning of form-finding process, topology of required tensegrity structure, linear constraints along with primary set of tension coefficient values are specified. Then possible tension coefficient values corresponding to $\overline{D}$ with rank deficiency n^*_D is iteratively found out using an algorithm based on equations 8, 27 and 28. Finally, a set of nodal coordinates which are independent in nature has to be assigned to obtain exclusive and stable formation of the structure.

The method described in this section explores new non-degenerate and asymmetric formations effectively. This can be achieved by altering values of primary set of tension coefficients provided at initial stage along with independent nodal coordinates specified at final stage of form-finding process. Since force or length of the members is not specified in the form-finding, this method cannot directly control the mechanical and geometrical properties of the structure.

5. Formfinding Via Genetic Algorithm

Genetic algorithm is one of the extensively utilized techniques for form-finding[18-20]. This technique proposed by Koohestani[18] requires connectivity of the structure and types of elements for the form-finding procedure. A constraint minimization problem is formulated as follows based on the fact that, D matrix must have at least t zero eigenvalues for structure with dimension d to be in self-stressed state. Value of t is 3 when d is 2 and 4 when d is 3.

$$\text{Minimize} \quad \alpha\beta \qquad (29)$$

Subject to the constraints given below

$$-1 \le q_i \le 1, \qquad q_i \ne 0, \qquad i = 1, 2, \dots, m \qquad (30)$$

where α and β are defined as

$$\alpha = \sum_{j=1}^{t} |\lambda_j| \qquad (31)$$

$$\beta = \sum_{i=1}^{m} \beta_i = \sum_{i=1}^{m} \frac{1}{|q_i|} \qquad (32)$$

Tension coefficient of ith member is denoted by q_i and i^{th} eigenvalue of D is denoted by λ_j.

Since eigenvalues of D are arranged in the ascending order, α is the sum of the least t eigenvalues of D. The constrained minimization problem is solved using genetic algorithm as the optimization tool, taking the

objective function given in 29 as the fitness function. The genetic algorithm parameters used are selected after considerable analysis and experiments and are available in Koohestani[18]. This method produces a set of tension coefficient values which can formulate a force density matrix that satisfies the conditions of rank deficiency. Hence, the nodal coordinates of the structure are acquired from the group of eigenvectors obtained from eigenvalues with value zero. Since the method is having force density as variable, the obtained coordinates need not result in a symmetric structure. So, Koohestani[18] propose two techniques incorporating symmetry conditions of the structure which forms linear combination of the selected eigenvalues from the first phase.

This is a robust technique for regular shaped structures. The main challenge for force density based methods is to ensure the symmetry of the structure. This method provides the appropriate solution for that. It is also suitable for small irregular shaped structures. Viability of this technique for large and irregular tensegrity structures needs additional research.

6. Algebraic Tensegrity Form-Finding

Masic et al[21] introduced Algebraic tensegrity form finding method in which the scope of force density method is extended. Requirements of the shape of the structure including linear shape constraints, constraints based on length of the member, constraints based on the symmetry of the structure (symmetry related to nodes and elements) are incorporated in the Force density method which in turn found to be reducing the force density variables. This method is compatible for various contemporary numerical optimization tools and gives quick solutions for complex tensegrity structures. Algebraic method is suitable for general, large and symmetric structures such as tensegrity towers, plates, shells etc. Compared to other methods, this method is having better control over the geometry of the structure.

7. Other Form-Finding Methods

There are some more form-finding methods other than the methods discussed above, which require only types of the elements and topology in the form of connectivity matrix as input parameters for the form-finding pro-

cess[19-22]. Numerical method proposed by Estrada et al.[22] is capable of finding new configurations of tensegrity structures without even considering length of strings or ratios of string to bar. Koohestani and Guest[23] suggested a new approach based on force density method with modification in variables and structure with Cartesian components of length of members as main variables. This method is found to be efficient for symmetric as well as non-regular and large tensegrity structures. Various studies are going on in the field of form-finding techniques which enhance design of tensegrity structures used in complex scenarios [3-7, 24].

8. Discussion and Conclusions

Force density method is one of the fundamental methods apposite to find new configurations of a tensegrity structure. Lengths of the members are not required for this method and this contains only linear equilibrium equations involving force densities. Selecting right force density values in the initial stage of form-finding is not easy. Four methods of form-finding, requiring minimum number of parameters and based on force density method are discussed in detail. Most of the methods require only type of the member (tension or compression) and topology of the tensegrity structure provided by connectivity matrix. These parameters do not require any material properties and lengths of the elements and hence effortless to formulate.

Advanced form-finding method is suitable for regular tensegrity structures such as cable domes. This method provides the solution for all cases of conditions related to eigenvalues unlike Adaptive method. But Adaptive form-finding method is found to be efficient for irregular shaped structures and also to search novel configurations with minimum alterations of values in initial and final steps. However this method is not able to directly control the mechanical and geometrical properties of the structure.

Form-finding by means of genetic algorithm provides solutions for regular tensegrity structures. Symmetry constrains are separately incorporated in latter step of process unlike Algebraic method. Algebraic method is a highly efficient and automated general method among all the mentioned form-finding methods. Constraints on shape and structural symmetry are included in conventional Force density method. Along with better control

over the geometry of the structure, this method is capable of providing rapid and reliable results for complex form-finding problems.

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