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GENERALIZED FIXED POINT THEOREMS IN G-METRIC SPACE UNDER AN IMPLICIT RELATION

T. Phaneendra

Department of Mathematics
School of Advanced Sciences
VIT University
Vellore, 632014, Tamil Nadu, INDIA

Abstract: Common fixed point theorems are proved in a G-metric space for four self-maps through the notions of weak compatibility, common limit range property and an implicit relation. These are generalized versions of the results of [3], [4] and the metrical fixed point theorem obtained in [5].

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Key Words: G-metric space, common limit range property, implicit relation, weakly compatible maps, coincidence point, common fixed point

1. Introduction

Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}$ such that

(G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y)$
$= G(y, z, x) = G(z, y, x)$ for all $x, y, z \in X$

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$

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Then G is called a G-metric on X and the pair (X, G) denotes a G-metric space. Axiom (G5) is known as the *rectangle inequality* (of G). This notion was introduced by Mustafa and Sims [1] as a generalization of metric space. From this definition, it immediately follows that

$$G(x, y, y) = 0 \quad \Rightarrow \quad x = y \quad \text{for all} \quad x, y \in X \tag{1.1}$$

and

$$G(x, y, y) \leq 2G(x, x, y) \text{ for all } x, y \in X. \tag{1.2}$$

A G-metric space (X, G) is said to be *symmetric*, if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Example 1.1. Let (X, d) be a metric space. Define

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x) \text{ for all } x, y, z \in X \tag{1.3}$$

Then (X, G) is a symmetric G-metric space.

Let x, y and z be the vertices of a triangle in a plane. Then $d(x, y)$ denotes the length of the side joining x and y and $G(x, y, z)$ represents the *perimeter* of the triangle.

The following terminology was developed by Mustafa et al in [1] and [2]:

Definition 1.1. A sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is said to be *G-convergent* with limit $p \in X$, if $\lim_{n,m \to \infty} G(p, x_n, x_m) = 0$, that is, if for any $\epsilon > 0$ there is a positive integer N such that $G(x_n, x_m, p) < \epsilon$ for all $n, m \geq N$.

Lemma 1.1. *The following statements are equivalent in a G-metric space* (X, G) :

(a) $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *is G-convergent with limit* $p \in X$,

(b) $\lim_{n \to \infty} G(x_n, x_n, p) = 0$,

(c) $\lim_{n \to \infty} G(x_n, p, p) = 0$.

First we give the following useful notation in a G-metric space (X, G) :

Definition 1.2. A point $p \in X$ is a *coincidence point* of self-maps f and r on X, if $fp = rp = u$, where u is a *point of coincidence* of f and r.

Definition 1.3. Self-maps f and r on X are *weakly compatible*, if they commute at their coincidence point.

Definition 1.4. Self-maps f and r on X satisfy the *property* (EA), if there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} rx_n = u \quad \text{for some} \quad u \in X. \tag{1.4}$$

Definition 1.5. Self-maps f and r on X satisfy the CLR_r*-property* if there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} rx_n = rp \quad \text{for some} \quad p \in X. \tag{1.5}$$

With these ideas, the following was proved in [3]:

Theorem 1.1. *Let f and r be self-maps on X such that*

$$\begin{aligned} G(fx, fy, fz) \le k \max\big\{&G(rx, ry, rz), G(rx, fx, fx), G(rx, fy, fy),\\ &G(rx, fz, fz), , G(ry, fy, fy)), G(ry, fx, fx),\\ &G(ry, fz, fz), G(rz, fz, fz), G(rz, fx, fx),\\ &G(rz, fy, fy)\big\} \text{ for all } x, y, z \in X, \end{aligned} \tag{1.6}$$

where $0 \le k < 1/4$. *If* (f, r) *satisfies the* CLR_r*-property and r is weakly compatible with f, then f and r will have a unique common fixed point.*

Writing $z = y$, we immediately get

Corollary 1.1. *Let f and r be self-maps on X such that*

$$\begin{aligned} G(fx, fy, fy) \le k \max\big\{&G(rx, ry, ry), G(rx, fx, fx), G(ry, fy, fy),\\ &G(rx, fy, fy), G(ry, fx, fx)\big\} \text{ for all } x, y \in X, \end{aligned} \tag{1.7}$$

where $0 \le k < 1/4$. *If* (f, r) *satisfies the* CLR_r*-property and r is weakly compatible with f, then f and r will have a unique common fixed point.*

Theorem 1.2 (Theorem 2.5, [4])**.** *Let f and r be self-maps on X and for all* $x, y \in X$ *either*

$$\begin{aligned} G(fx, fy, fy) \le q \max\big\{&G(ry, fy, fy),\\ &G(rx, fy, fy), G(ry, fx, fx)\big\} \end{aligned} \tag{1.8}$$

or

$$\begin{aligned} G(fx, fy, fy) \le q \max\big\{&G(ry, ry, fy),\\ &G(rx, rx, fy), G(ry, ry, fx)\big\} \end{aligned} \tag{1.9}$$

holds good, where $0 \le q < 1$. *If the range of f is contained in the range of r and* $r(X)$ *is a complete subspace of X, then f and r will have a unique common fixed point, provided r is weakly compatible with f.*

Remark 1.1. When the range of values of q is restricted to $[0, 1/4)$, the right hand side of (1.8) is less than or equal to the right hand side of (1.7). In other words, (1.7) will be weaker than (1.8), when $q \in [0, 1/4)$. Also, given $x_0 \in X$, if $f(X) \subset r(X)$, then we can define the sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X with the choice

$$fx_{n-1} = rx_n \text{ for } n = 1, 2, 3, ... \tag{1.10}$$

It can be shown that $\langle rx_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in $r(X)$ and hence converges in it, provided $r(X)$ is complete. Thus (f, r) satisfies the CLR_r-property whenever $f(X) \subset r(X)$ and $r(X)$ is complete. Hence Corollary 1.1 is a partial generalization of Theorem 1.2 when $q \in [0, 1/4)$.

In this paper, first we extend Corollary 1.1 to four self-maps using the notion of an implicit relation (*cf.* Section 2), which is a generalization of the metrical fixed point theorem proved in [5]. Then we derive a generalization of Theorem 1.2 under a certain condition by slightly altering the contraction conditions of the first result.

2. Main Results and Discussion

The notion of *implicit-type relations* were first introduced by Popa [6] to cover several contractive conditions and unify fixed point theorems in metric spaces (See [7], [8], [9] and so on). Recently Popa and Patriciu [10] inserted a continuous implicit relation $\phi : \mathbb{R}_+^4 \to \mathbb{R}$ to prove fixed point theorems in a G-metric spaces. In this paper, we employ a lower semicontinuous implicit function $\psi : \mathbb{R}_+^6 \to \mathbb{R}$, which is nondecreasing in each coordinate variable, such that

(P_a) $\psi(l, 0, 0, l, l, 0) > 0$ for all $l > 0$,

(P_b) $\psi(l, 0, l, 0, 0, l) > 0$ for all $l > 0$,

(P_c) $\psi(l, l, 0, 0, l, l) > 0$ for all $l > 0$.

Example 2.1. Let $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}$.

Example 2.2. Let $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\left\{t_2, \beta t_3 + \alpha t_4, \frac{t_5+t_6}{2}\right\}$, where $\beta \geq 0$ and $0 < \alpha < 1$.

Example 2.3. Let $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\left\{t_2, \alpha t_3, \alpha t_4, \frac{t_5+t_6}{2}\right\}$, where $0 < \alpha < 1$.

Example 2.4. Let $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}$.

Example 2.5. Let $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - [at_2 + bt_3 + ct_4 + e(t_5 + t_6)]$, where a, b, c and e are nonnegative numbers with $a + b + c + 2e < 1$.

Example 2.6. Let $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - [at_1^2 t_2 + bt_1 t_3 t_4 + ct_5^2 t_6 + dt_5 t_6^2]$, where $a > 0$, $b, c, e \geq 0$ such that $a + c + e < 1$ and $a + b < 1$.

Example 2.7. Let $\psi(l_1, l_2, l_3, l_4, l_5, l_6) = l_1^2 - al_2^2 - \frac{bl_5 l_6}{l_3^2 + l_4^2 + 1}$, where $a = 1/2$ and $b = 1/4$.

Our first main result is

Theorem 2.1. *Let f, g, h and r be self-maps on X such that for all $x, y \in X$ any two of the following inequalities hold good:*

$$\psi(G(fx, gy, gy), G(rx, ry, ry), G(rx, fx, fx), G(ry, gy, gy), G(rx, gy, gy), G(ry, fx, fx)) \leq 0, \tag{2.1}$$

$$\psi(G(gx, hy, hy), G(rx, ry, ry), G(rx, gx, gx), G(ry, hy, hy), G(rx, hy, hy), G(ry, gx, gx)) \leq 0, \tag{2.2}$$

$$\psi(G(hx, fy, fy), G(rx, ry, ry), G(rx, hx, hx), G(ry, fy, fy), G(rx, fy, fy), G(ry, hx, hx)) \leq 0. \tag{2.3}$$

Suppose that one of the pairs (f, r), (g, r) and (h, r) satisfies the CLR_r-property. If r is weakly compatible with any one of f, g and h, then all the four maps f, g, h and r will have a common coincidence point u, which will be their unique common fixed point.

Proof. Suppose f and r satisy the CLR_r-property. Then we can find a $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that (1.5) holds good.

First we prove that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} rx_n = rp. \tag{2.4}$$

Writing $x = y = x_n$ in (2.1), we get

$$\psi(G(fx_n, gx_n, gx_n), G(rx_n, rx_n, rx_n), G(rx_n, fx_n, fx_n), G(rx_n, gx_n, gx_n), G(rx_n, gx_n, gx_n), G(rx_n, fx_n, fx_n)) \leq 0.$$

Applying the limit as $n \to \infty$ and then using (1.5) and the lower semicontinuity of ψ, this yields

$$\psi(G(rp, q, q), 0, 0, G(rp, q, q), G(rp, q, q), 0) \leq 0, \tag{2.5}$$

where $q = \lim_{n\to\infty} gx_n$. If $G(rp, q, q) > 0$, then (2.5) gives a contradiction to (P_a). Thus $G(rp, q, q) = 0$ so that $q = rp$, in view of (1.1).

Again, taking $x = y = x_n$ in (2.2), we get

$$\psi(G(gx_n, hx_n, hx_n), G(rx_n, rx_n, rx_n), G(rx_n, gx_n, gx_n), G(rx_n, hx_n, hx_n),$$
$$G(rx_n, hx_n, hx_n), G(rx_n, gx_n, gx_n)) \le 0,$$

Applying the limit as $n \to \infty$ and the lower semicontinuity of ψ, this yields

$$\psi(G(rp, t, t), 0, 0, G(rp, t, t), G(rp, t, t), 0) \le 0,$$

where $\lim_{n\to\infty} hx_n = t$. This would also give a contradiction to (P_a) if $G(rp, t, t) > 0$. Thus $G(rp, t, t) = 0$ so that $t = rp$, in view of (1.1). This proves (2.4).

Similarly (2.4) can be established whenever (g, r) or (h, r) satisfies the CLR_r-property under (2.1) and (2.2). The other cases that one of (f, r), (g, r) and (h, r) satisfies the CLR_r-property under [(2.2) and (2.3)] or [(2.1) and (2.3)] will prove (2.4).
We shall prove that f, g, h and t have a common coincidence in the following three cases:

Case (a): (f, r) is weakly compatible,
Case (b): (g, r) is weakly compatible,
Case (c): (h, r) is weakly compatible

Case (a) (f, r) is weakly compatible and any one of the pairs [(2.1), (2.2)], [(2.1), (2.3)] and [(2.2), (2.3)] holds good:

$$fp = rp. \tag{2.6}$$

If possible we assume that $fp \ne rp$ so that $l = G(rp, fp, fp) > 0$ and $k = G(fp, fp, rp) > 0$ by (G4), and $l \ge k/2$, by (1.2).

Then writing $x = p$ and $y = x_n$ in (2.1), we get

$$\psi(G(fp, gx_n, gx_n), G(rp, rx_n, rx_n), G(rp, fp, fp),$$
$$G(rx_n, gx_n, gx_n), G(rp, gx_n, gx_n), G(rx_n, fp, fp)) \le 0.$$

Applying the limit as $n \to \infty$ and using (2.4) and the lower semicontinuity of ψ, this yields

$$\psi(k, 0, l, 0, 0, l) \le 0.$$

Since ψ is nondecreasing in each coordinate variable, the above imlpies that

$$\psi\left(\frac{k}{2}, 0, \frac{k}{2}, 0, 0, \frac{k}{2}\right) \leq \psi(k, 0, l, 0, 0, l) \leq 0.$$

which contradicts the choice (P_b). Therefore (2.6) must hold good.

Since f and r commute at the coincidence point p, it follows that $frp = rfp$ or

$$fu = ru, \text{ where } fp = rp = u. \tag{2.7}$$

Again, (2.1) with $x = y = u$ and (2.7) gives

$$\begin{aligned}\psi(G(fu, gu, gu), G(ru, ru, ru), G(ru, fu, fu),\\ G(ru, gu, gu), G(ru, gu, gu), G(ru, fu, fu)) \leq 0,\end{aligned}$$

or

$$\psi(G(fu, gu, gu), 0, 0, G(fu, gu, gu), G(fu, gu, gu), 0) \leq 0,$$

which will contradict with (P_a) if $G(fu, gu, gu) > 0$.
Hence $G(fu, gu, gu) = 0$ so that (1.1) gives $fu = gu = ru$.

Suppose that (2.2) holds good. With $x = u = y$, this gives

$$\begin{aligned}\psi(G(gu, hu, hu), G(ru, ru, ru), G(ru, gu, gu),\\ G(ru, hu, hu), G(ru, hu, hu), G(ru, gu, gu)) \leq 0\end{aligned}$$

or that $\psi(G(gu, hu, hu), 0, 0, G(gu, hu, hu), G(gu, hu, hu), 0) \leq 0$.
This again contradicts with (P_a) if $G(gu, hu, hu) > 0$ so that

$$G(gu, hu, hu) = 0 \text{ or } gu = hu.$$

In other words, u is a common coincidence point of f, g, h and r, that is

$$fu = gu = hu = ru, \text{ where } fp = rp = u. \tag{2.8}$$

On the other hand, if (2.3) holds good, then writing $x = y = u$ in this, followed by (2.7) and $fu = gu$, and proceeding as above, we get $gu = hu$ and hence (2.8).
In this way, (2.8) follows in either case [(2.1), (2.2)] and [(2.1), (2.3)].

Now consider the inequalities (2.2) and (2.3).

Writing $x = x_n$ and $y = p$ in (2.3), we get

$$\psi(G(hx_n, fp, fp), G(rx_n, rp, rp), G(rx_n, hx_n, hx_n), \\ G(rp, fp, fp), G(rx_n, fp, fp), G(rp, hx_n, hx_n)) \leq 0.$$

Applying the limit as $n \to \infty$ and using (2.4) and the lower semicontinuity of ψ, we get

$$\psi(G(rp, fp, fp), 0, 0, G(rp, fp, fp), G(rp, fp, fp), 0) \leq 0.$$

This gives a contradiction to (P_a) if $G(rp, fp, fp) > 0$. Thus

$$G(rp, fp, fp) = 0 \text{ or } rp = fp = u, \text{ by } (1.1)$$

and hence (2.7) follows from the weak compatibility of (f, r).

Again from (2.3) with $x = u = y$ and (2.7), we see that

$$\psi(G(hu, fu, fu), G(ru, ru, ru), G(ru, hu, hu), \\ G(ru, fu, fu), G(ru, fu, fu), G(ru, hu, hu)) \leq 0$$

or $\psi(G(hu, fu, fu), 0, G(fu, hu, hu), 0, 0, G(fu, hu, hu)) \leq 0$. This, in view of (1.2) and the nondecreasing nature of ψ, gives

$$\psi\left(\frac{G(hu,fu,fu)}{2}, 0, \frac{G(hu,fu,fu)}{2}, 0, 0, \frac{G(hu,fu,fu)}{2}\right) \leq 0.$$

This would be against (P_b) if $G(fu, fu, hu) > 0$. Therefore, $G(fu, fu, hu) = 0$ or $fu = hu$ and hence $fu = hu = ru$.

But then, (2.2) with $x = u = y$ and (2.7) imply that

$$\psi(G(gu, fu, fu), 0, G(fu, gu, gu), 0, 0, G(fu, gu, gu)) \leq 0,$$

This also, in view of (1.2) and the nondecreasing nature of ψ, gives

$$\psi\left(\frac{G(gu,fu,fu)}{2}, 0, \frac{G(gu,fu,fu)}{2}, 0, 0, \frac{G(gu,fu,fu)}{2}\right) \leq 0,$$

which again contradicts (P_b) if $G(fu, fu, gu) > 0$. Thus $G(fu, fu, gu) = 0$ so that $fu = gu$, from (1.1), and thus (2.8) follows.

Case (b): Let (g, r) be weakly compatible.

Subcase (i): Apply (2.1) with $x = x_n$ and $y = p$ to get p as a coincidence point of g and r and hence to get $u = gp = rp$ as their coincidence point. Then we again use (2.1) with $x = y = u$ to get u as a coincidence point of f and g. Further we use (2.2) with $x = y = u$ so that u will be a coincidence point of g and h. Or else we use (2.3) with $x = y = u$ to get u as a coincidence point of f and h.

Subcase (ii): Apply (2.2) with $x = p$ and $y = x_n$ to get p as a coincidence point of g and r and hence to get $v = gp = rp$ as their coincidence point. Then we again use (2.2) with $x = y = v$ to get v as a coincidence point of g and h. Further we use (2.3) with $x = y = v$ so that v will be a coincidence point of h and f. Or else we use (2.1) with $x = y = v$ to get v as a coincidence point of f and g.

Case (c): Let (h, r) be weakly compatible.

Subcase (i): Apply (2.2) with $x = x_n$ and $y = p$ to get p as a coincidence point of h and r, and to get $w = hp = rp$ also as their coincidence point. Then we again use (2.2) with $x = y = w$ to get w as a coincidence point of g and h. Further we use (2.3) with $x = y = w$ so that w will be a coincidence point of h and f. Or else we use (2.1) with $x = y = w$ to get w as a coincidence point of f and g.

Subcase (ii): Apply (2.3) with $x = p$ and $y = x_n$ to get p as a coincidence point of h and r, and hence $z = hp = rp$ also as their coincidence point. Then we again use (2.3) with $x = y = z$ to get z as a coincidence point of h and f. Further we use (2.1) with $x = y = z$ so that z is a coincidence point of f and g.

Thus we get (2.8) in all the cases.

Now we employ (2.1) with $x = u$ and $y = x_n$, which in the limit as $n \to \infty$ gives u as a fixed point of f and hence a common fixed point of f, g, h and r. In fact, (2.1) with $x = u$ and $y = x_n$ gives

$$\psi(G(fu, gx_n, gx_n), G(ru, rx_n, rx_n), G(ru, fu, fu), \\ G(rx_n, gx_n, gx_n), G(ru, gx_n, gx_n), G(rx_n, fu, fu)) \leq 0.$$

Proceeding the limit as $n \to \infty$, in this and using (2.8) and lower semicontinuity of ψ, we obtain

$$\psi(G(fu, u, u), G(fu, u, u), 0, 0, G(fu, u, u), G(u, fu, fu)) \leq 0$$

which from (1.2) gives

$$\psi\left(\frac{G(fu,u,u)}{2}, \frac{G(fu,u,u)}{2}, 0, 0, \frac{G(fu,u,u)}{2}, \frac{G(fu,u,u)}{2}\right) \leq 0,$$

since ψ is nondecreasing. This is a contradiction to (P_c) if $G(fu,u,u) > 0$, proving that $G(fu,u,u) = 0$ or $fu = u$, in view of (1.1). This together with (2.8) implies that u is a common fixed point of f, g, h and r.

To establish the uniqueness of the common fixed point, we assume that a and b are two distinct common fixed points of f, g, h and r so that $G(a,b,b) > 0$ and

$$fa = ga = ha = ra = a \quad \text{and} \quad fb = gb = hb = rb = b. \tag{2.9}$$

Then writing $x = a$ and $y = b$ in (2.1), we get

$$\psi(G(fa,gb,gb), G(ra,rb,rb), G(ra,fa,fa), \\ G(rb,gb,gb), G(ra,gb,gb), G(rb,fa,fa)) \leq 0.$$

Using (2.9) in this, we obtain

$$\psi(G(a,b,b), G(a,b,b), 0, 0, G(a,b,b), G(b,a,a)) \leq 0$$

or

$$\psi\left(\tfrac{G(a,b,b)}{2}, \tfrac{G(a,b,b)}{2}, 0, 0, \tfrac{G(a,b,b)}{2}, \tfrac{G(a,b,b)}{2}\right) \leq 0,$$

which leads to a contradiction to (P_c). Therefore $a = b$.

In other words, the common fixed point of f, g, h and r is unique. □

Taking $g = h = f$ in Theorem 2.1, we get

Corollary 2.1. *Let f and r be self-maps on X such that*

$$\psi(G(fx,fy,fy), G(rx,ry,ry), G(rx,fx,fx), G(ry,fy,fy), \\ G(rx,fy,fy), G(ry,fx,fx)) \leq 0 \text{ for all } x, y \in X, \tag{2.10}$$

Suppose that (f,r) satisfies the CLR_r-property. If r is weakly compatible with f, then f and r will have a coincidence point u, which will be their unique common fixed point.

To obtain the result of [5] as an important consequence of Theorem 2.1, we need the following notions in a metric space (X,d):

Definition 2.1. Given $x_0 \in X$ and f, g, h and r self-maps on X, if there exist points $x_1, x_2, x_3, \ldots$ in X such that

$$fx_{3n-3} = rx_{3n-2}, gx_{3n-2} = rx_{3n-1}, hx_{3n-1} = rx_{3n},\ n = 1, 2, 3, \ldots, \tag{2.11}$$

then $\langle rx_n\rangle_{n=1}^{\infty}$ is an (f,g,h)-*orbit* at x_0 *relative* to r.

The space X is (f,g,h)-*orbitally complete* at x_0 *relative* to r if every Cauchy sequence in an (f,g,h)-orbit at x_0 *relative* to r converges in X, and X is (f,g,h)-*orbitally complete* relative to r if it is (f,g,h)-orbitally complete at each $x_0 \in X$ relative to r.

Definition 2.2. Self-maps f,g,h and r satisfy the property (EA) if there exists a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} rx_n = u \text{ for some } u \in X. \tag{2.12}$$

The following was the main result proved in [5]]:

Corollary 2.2. *Let f, g, h and r be self-maps on a metric space (X,d) satisfying the property (EA). For all $x,y \in X$, suppose that any two of the following inequalities hold good:*

$$\psi(d(fx,gy), d(rx,ry), d(rx,fx), \\ d(ry,gy), d(rx,gy), d(ry,fx)) \le 0, \tag{2.13}$$
$$\psi(d(gx,hy), d(rx,ry), d(rx,gx), \\ d(ry,hy), d(rx,hy), d(ry,gx)) \le 0, \tag{2.14}$$
$$\psi(d(hx,fy), d(rx,ry), d(rx,hx), \\ d(ry,fy), d(rx,fy), d(ry,hx)) \le 0. \tag{2.15}$$

Suppose that $r(X)$ is (f,g,h)-orbitally complete relative to r and r is weakly compatible with any one of f, g and h, then all the four maps f, g, h and r will have a unique common coincidence point which will also be their unique common fixed point.

Remark 2.1. Let (X,d) be a metric space. Define the G metric as in Example 1.1. Then (X,G) is a symmetric G-metric space and

$$G(fx,gy,gy) = d(fx,gy),\ G(rx,ry,ry) = d(rx,ry), \\ G(rx,fx,fx) = d(rx,fx) \text{ etc. for all } x,y \in X.$$

Hence (2.13)-(2.15) are particular cases of (2.1)-(2.3) respectively. Since the property (EA) and the orbital completeness of $r(X)$ imply the CLR_r-property, we see that the conclusion of Theorem 2.2 follows from that of Theorem 2.1.

Next we see that Corollary 1.1 is a particular case of Corollary 2.1.

Remark 2.2. We write

$$\psi(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-k\max\left\{t_2,t_3,t_4,t_5,t_6\right\} \text{ for all } t_i\geq 0,\ i=1,2,...,6,$$

where $0 \leq k < 1/4$. Then (1.7) follows from (2.19) and hence Corollary 1.1 is a particular case of Corollary 2.1. In view of the choice of k, it may be noted from the proof of Corollary 1.1 given in [3], that the symmetry of X can be dropped when the inequality (2.10) is condensed as (1.7) .

Slightly altering the inequalities (2.1)-(2.3), one can obtain the following result:

Theorem 2.2. *Let f, g, h and r be self-maps on a symmetric G-metric space (X, G) such that for all $x, y \in X$ any two of the following inequalities hold good:*

$$\psi(G(fx,gy,gy),G(rx,rx,ry),G(rx,rx,fx),G(ry,ry,gy),$$
$$G(rx,rx,gy),G(ry,ry,fx))\leq 0, \tag{2.16}$$
$$\psi(G(gx,hy,hy),G(rx,rx,ry),G(rx,rx,gx),G(ry,ry,hy),$$
$$G(rx,rx,hy),G(ry,ry,gx))\leq 0, \tag{2.17}$$
$$\psi(G(hx,fy,fy),d(rx,rx,ry),G(rx,rx,hx),G(ry,ry,fy),$$
$$G(rx,rx,fy),G(ry,ry,hx))\leq 0. \tag{2.18}$$

Suppose that one of the pairs (f, r), (g, r) and (h, r) satisfies the CLR_r-property. If r is weakly compatible with any one of f, g and h, then all the four maps f, g, h and r will have a common coincidence point u, which will be their unique common fixed point.

Again with $g = h = f$ in Theorem 2.2, we get

Corollary 2.3. *Let f and r be self-maps on X such that*

$$\psi(G(fx,fy,fy),G(rx,rx,ry),G(rx,rx,fx),G(ry,ry,fy),$$
$$G(rx,rx,fy),G(ry,ry,fx))\leq 0 \text{ for all } x,y\in X. \tag{2.19}$$

If (f, r) satisfies the CLR_r-property and r is weakly compatible with f, then f and r will have a coincidence point u. Further if X is symmetric, then u will become their unique common fixed point.

Remark 2.3. The symmetry of X is not used in Corollary 2.1 and Corollary 2.3 to obtain the coincidence point of f and r, unlike in Theorem 2.2.

The following is a unification of Corollary 2.1 and Corollary 2.3, whose proof is simple:

Corollary 2.4. *Let f and r be self-maps on X satisfying* (2.10) *or* (2.19) *for all $x, y \in X$. If (f, r) satisfies the CLR_r-property and r is weakly compatible with f, then f and r will have a coincidence point u. Further u will become their unique common fixed point if X is symmetric.*

Remark 2.4. Again taking

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - q \max \left\{t_4, t_5, t_6\right\} \text{ for all } t_i \geq 0,\ i = 1, 2, ..., 6,$$

where $0 \leq q < 1$, we see that ψ is continuous and hence lower semicontinuous. Also (1.8) and (1.9) follow from (2.10) and (2.19) respectively.

Let $x_0 \in X$ be arbitrary. Then as in Remark 1.1, the sequence $\langle rx_n \rangle_{n=1}^{\infty}$ with the choice (1.10) is a Cauchy sequence in $r(X)$ and hence converges in it. That is

$$\lim_{n\to\infty} fx_{n-1} = \lim_{n\to\infty} rx_n = rp \text{ for some } p \in X. \tag{2.20}$$

It is not difficult to prove that $\lim_{n\to\infty} fx_n = rp$. In other words, f and r satisfy the CLR_r-property. Therefore it follows that $u = rp$ is a coindence point. From the proof given in [4] it follows that u is a common fixed point of f and r, wherein the symmetry of X is not needed. Thus Corollary 2.4 is a significant generalization of Theorem 1.2.

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