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Induced Matching Partition of Petersen and Circulant Graphs

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Abstract

Distributing the overall workload evenly among a set of processors in VLSI designs to achieve high speed-ups in computation has been widely studied as a *graph partitioning problem*. Determining induced matching *k*-partition number even when $k = 2$ is an *NP*-complete problem. In this paper we deal with the induced matching partition for Petersen graphs and circulant graphs and determine their induced matching partition numbers.

Keywords: Matching, Circulant graph, Petersen graph, Induced matching partition.

Nomenclature	
G	Graph
$\Delta(G)$	maximum degree of Graph G
imp	induced matching partition number
$P(n, 2)$	Petersen graph
$G(n; \pm\{1, 2, ..., j\})$	Circulant graph

1. Introduction

Identifying parallelism in a problem by partitioning its data and tasks among the processors of a parallel computer is a fundamental issue in parallel computing. This problem can be modeled as a *graph partitioning problem* in which the vertices of a graph are divided into a specified number of subsets such that few edges join two vertices in different subsets [1]. Graph partition problems are some of the most well-studied problems both in graph theory and in computer-science. Standard examples of partition problems include *k*-colorability. Most of these problems are computationally hard even to approximate, but it was observed in the 90's [2-3] that many of these partition problems have good approximations when the input graph is dense.

A matching in a graph $G = (V, E)$ is a subset M of edges, no two of which have a vertex in common. A matching is called *induced* if the subgraph of G induced by the endpoints of edges in M is 1-regular. A matching is called *M perfect* if every vertex in G is an endpoint of one of the edges in M. A *near-perfect* matching covers all but exactly one vertex. Let G be a graph with a perfect matching. An induced matching *k*-partition of a graph G which has a perfect matching is a *k*-partition $(V_1, V_2, ..., V_k)$ of $V(G)$ such that, for each i $(1 \leq i \leq k)$, $E(V_i)$ is an induced matching of G that covers V_i, or equivalently, the subgraph $G[V_i]$ of G induced by V_i is 1-regular. The induced matching partition number of a graph G, denoted by *imp* (G), is the minimum integer k such that G has an induced matching *k*-partition. The induced matching *k*-partition problem asks whether a given graph G has an induced *k*-partition or not.

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Historically, the induced matching k-partition problem was first studied as a combinatorial optimization problem [4]. The induced matching k-partition problem is *NP*-complete. Further the problem is *NP*-complete even for $k = 2$ and for 3-regular planar graphs, respectively [4-5]. Aifeng et al. [6] studied the computational complexity of the induced matching k-partition problem for graphs with small diameters. They proved that the induced matching 2-partition problem of graphs with diameter 6 and induced matching 3-partition problem of graphs with diameter 2 are *NP*-complete. Further they showed that the induced matching 2-partition problem of graphs with diameter 2 is polynomially solvable. Yuan and Wang [7] have characterized graphs G with $imp\ (G) = 2\ \Delta\ (G) - 1$ where $\Delta\ (G)$ is the maximum degree of G. The induced matching partition problem has been studied for certain interconnection networks such as butterfly networks, hypercubes, cube-connected cycles and grids [8].

The induced matching partition number has been determined for augmented and wrapped butterfly networks [9], honeycomb networks, honeycomb torus and sierpinski gasket graphs of even dimension. Further the induced matching partition numbers of sierpinski gasket graphs of odd dimension and sierpinski graphs have been determined for which there exists a near perfect matching [10]. This motivates us to introduce the concept of induced matching partition number for graphs with near-perfect matching. Further, in this paper, we find algorithms to identify induced matching partition of Petersen graphs and circulant graphs thereby determining their induced matching partition numbers.

Definition 1 *Let G be a graph with a near-perfect matching. An induced matching k-partition of a graph $G_v = G - \{v\}$ which has a perfect matching is a k-partition $(V_1, V_2, ..., V_k)$ of $V\ (G_v)$ such that, for each i $(1 \le i \le k)$, $E\ (V_i)$ is an induced matching of G_v that covers V_i, or equivalently, the subgraph $G_v\,[V_i]$ of G_v induced by V_i is 1-regular. The induced matching partition number of G_v denoted by $imp\ (G_v)$, is the minimum integer k such that G_v has an induced matching k-partition.*

If G_v does not have perfect matching, $imp\ (G_v)$ is taken as infinity. The induced matching partition number of G denoted by $imp\ (G)$ is defined as $imp\ (G) = min_{\ v \in V}\ imp\ (G_v)$.

Remark 1 If G is vertex-transitive, $imp\ (G)$ is equal to $imp\ (G_v)$ for any $v \in V$ and hence it is enough to study $imp\ (G_v)$ for an arbitrarily chosen vertex v in G.

Theorem 1 *Let G be a cycle of length $4k$, $k \ge 1$. Then $imp\ (G) = 2$.*

Proof. Let $V = \{1, 2, 3, ..., 4k\}$ be the vertex set of G. Let $V_1 = \{1, 2, 5, 6, 9, 10, ..., 4k - 3, 4k - 2\}$ and $V_2 = \{3, 4, 7, 8, 11, 12, ..., 4k - 1, 4k\}$ be a partition of V. For any $x \in V$, we have either $x \in V_1$ or $x \in V_2$. Without loss of generality let $x \in V_1$. Now the open neighborhood of x denoted by $N\ (x)$ is given by $N\ (x) = \{x - 1, x + 1\}$. Therefore either $x - 1 \in V_1$ and $x + 1 \in V_2$ or $x + 1 \in V_1$ and $x - 1 \in V_2$. Thus $G\ [V_1]$ and $G\ [V_2]$ are 1-regular. Hence $imp\ (G) = 2$. □

Theorem 2 [7] *If a graph G has a perfect matching then $imp\ (G) \le 2\ \Delta\ (G) - 1$ and $imp\ (G) = 2\ \Delta\ (G) - 1$ if and only if G is isomorphic to either K_2 or C_{4k+2} or the Petersen graph, where C_n is the cycle of length n.*

Corollary 1 $imp\ (C_{4k+2}) = 3$.

As in Theorem 1, when n is odd, we obtain the induced matching partition number of C_n.

Theorem 3 *For a cycle C_n of odd length, $imp\ (C_n) = 2$.*

2. Petersen Graph

The Petersen graph has fascinated many graph theorists over the years because of its appearance as a counterexample in many places. Because of its ambiguity, it seemed a natural graph to be used in many places. The graph is named after Julius Petersen, who in 1898 constructed it to be the smallest bridgeless cubic graph with no three-edge-coloring. In 1950 Coxeter [11] introduced a family of graphs generalizing the Petersen graph. The Petersen graph is the most efficient small network in terms of node degree, diameter, and network size. Due to its unique and optimal properties, several network topologies based on the Petersen graph have been proposed and investigated in the literature [12].

A generalized Petersen graph $P\ (n, m)$, $n \ge 3$, $1 \le m \le \lceil (n - 1) / 2 \rceil$, consists of an outer n-cycle $u_1, u_2, ..., u_n$, a set of n spokes (u_i, v_i), $1 \le i \le n$ and n inner cycle edges (v_i, v_{i+m}) with indices taken modulo n. It is a 3-regular graph and contains $2n$ vertices and $3n$ edges. For $1 \le i \le n$, we call the vertices u_i and v_i of $P\ (n, m)$ as *outer rim* and *inner rim* vertices respectively. In this section we consider Petersen graphs with $m = 2$ and call a generalized Petersen graph $P\ (n, 2)$ simply a Petersen graph. See Fig. 1(a).

Theorem 4 *Let G be the Petersen graph P* $(n, 2)$. *Then imp* $(G) \geq 3$.

Proof. Suppose on the contrary that V_1, V_2 form an induced matching 2-partition of G. It is clear that every 5-cycle in G has exactly one edge in $E(V_1)$ or $E(V_2)$. All other edges are in $E(V_1) \cup E(V_2)$.

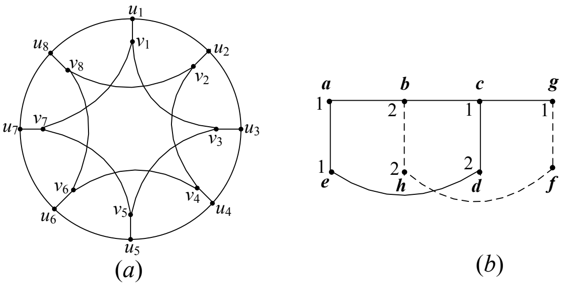

Fig. 1. (*a*) $P(8, 2)$ (*b*) The 5-cycle a, b, c, d, e and b, h, f, g, c

The girth of G is 5 and we consider a 5-cycle in G say a, b, c, d, e in which the edge (a, e) is in $E(V_1)$ or $E(V_2)$. See Fig. 1 (*b*). Without loss of generality let us assume that (a, e) is in $E(V_1)$. Now $N(b) = \{a, c, h\}$ and $N(c) = \{b, d, g\}$. Then there exists a 5-cycle say b, h, f, g, c in which the edge $(b, h) \in E(V_2)$ and $(c, g) \in E(V_1)$, which is a contradiction. □

Procedure INDUCED MATCHING PARTITION *P* (*n*, 2)

Input: A generalized Petersen graph $P(n, 2)$, $n > 10$.

Algorithm:

(i) Label the inner rim vertex v_i as 1, 2, or 3 according as i $(mod\ 12)$ lies in the interval [1, 4], [5, 8] or [9, 12] respectively, whenever (*a*) $i \leq n - 4$ if $n \equiv 0\ mod\ 4$ (*b*) $i \leq n - 5$ if $n \equiv 1\ mod\ 4$ (*c*) $i \leq n - 10$ if $n \equiv 2\ mod\ 4$ and (*d*) $i \leq n - 7$ if $n \equiv 3\ mod\ 4$. Go to (ii) if $n \equiv 0\ mod\ 4$, to (iii) if $n \equiv 1\ mod\ 4$, to (iv) if $n \equiv 2\ mod\ 4$ and to (v) if $n \equiv 3\ mod\ 4$.

(ii) Label the four vertices v_{n-3}, v_{n-2}, v_{n-1} and v_n as 3 if label of v_{n-4} is 2 and as 2 otherwise. Go to (vi). See Fig. 2 (*a*).

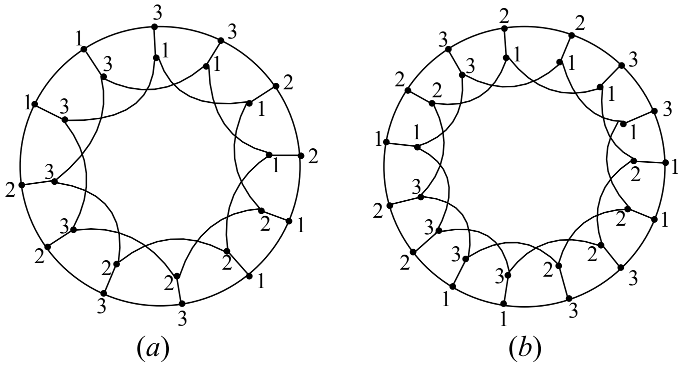

Fig. 2. (*a*) $P(12, 2)$ (*b*) $P(15, 2)$

(iii) Label the four vertices v_{n-4}, v_{n-3}, v_{n-2} and v_{n-1} as 3 if label of v_{n-5} is 2 and as 2 otherwise. Label v_n as 2 or 3 according as the label of v_{n-1} is 3 or 2 respectively. Go to (vi).

(iv) Label v_{n-1} and v_n as 2 and 3 respectively and the four vertices v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2} as 1. Label the remaining vertices v_{n-9}, v_{n-8}, v_{n-7} and v_{n-6} as 2 or 3 according as the label of v_{n-10} is 3 or 2 respectively. Go to (vi).

(v) Label the five vertices v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3} and v_n as 2 or 3 according as the label of v_{n-7} is not equal to 2 or 3 respectively. Label v_{n-2} as 1. Label v_{n-1} as 2 or 3 according as the label of v_n is 3 or 2 respectively. Go to (vi). See Fig. 2 (*b*).

(vi) Label u_n as 1 when $n \equiv 0$ *mod* 4 and as the label of v_n otherwise. Label vertex u_{n-1} as that of v_{n-1} when $n \equiv 2, 3$ *mod* 4 and the vertex u_{n-2} as that of v_{n-2} when $n \equiv 3$ *mod* 4.

(vii) On the outer cycle, move simultaneously from u_1 in the clockwise direction and from u_{n-i} in the anticlockwise direction according as $n \equiv i$ *mod* 4, $i = 0, 1, 2$ or 3 respectively. If v_i and u_{i-1} receive the same label in the clockwise direction then label u_i as that of v_{i-1} and if v_i and u_{i+1} receive the same label in the anticlockwise direction then label u_i as that of v_{i+1}. Otherwise u_i, v_i, u_{i-1} receive distinct labels and u_i, v_i, u_{i+1} receive distinct labels.

(viii) Having labeled u_i let its adjacent vertex u_{i+1} in the clockwise direction and u_{i-1} in the anticlockwise direction receive the same label till we arrive at u_k where $k = 2\lfloor n/4 \rfloor - 1$ in the clockwise direction and $k = 2\lfloor n/4 \rfloor + 2$ in the anticlockwise direction.

End INDUCED MATCHING PARTITION $P(n, 2)$

Output: *imp* $(P(n, 2)) = 3$.

Proof of Correctness: The vertices that receive labels 1, 2 or 3 are in V_1, V_2 and V_3 respectively. For any $u \in V_i$, $i = 1, 2, 3$, exactly one vertex in $N(u) \in V_i$. Thus $G[V_1]$, $G[V_2]$ and $G[V_3]$ are 1-regular. Therefore *imp* $(G) = 3$. □

3. Circulant Networks

Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [13]. The undirected circulant networks arise in the context of Mesh Connected Computer suited for parallel processing of data, such as the well-known ILLIAC type computers [14]. Generally, the ILLIAC network with n^2 processors can be represented as a circulant graph $G(n^2; \pm\{1, n\})$. Circulant graphs are intensively researched in computer science, graph theory and discrete mathematics [15].

The circulant network is a natural generalization of double loop network, which was first considered by Wong and Coppersmith [16]. A circulant undirected graph, denoted by $G(n; \pm\{1, 2, ..., j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$, $n \geq 3$ is defined as an undirected graph consisting of the vertex set $V = \{0, 1, ..., n-1\}$ and the edge set $E = \{(i, j): |j - i| \equiv s \ (mod\ n), s \in \{1, 2, ..., j\}\}$. The graph in Fig. 3 is the circulant graph $G(8; \pm\{1, 2\})$. It is clear that $G(n; \pm 1)$ is an undirected cycle C_n and $G(n; \pm\{1, 2, ..., \lfloor n/2 \rfloor\})$ is the complete graph K_n. We observe that $C_n = G(n, \pm 1)$ is a subgraph of $G(n; \pm\{1, 2, ..., j\})$ for every j, $1 \leq j \leq \lfloor n/2 \rfloor$. The circulant graph is a vertex-transitive graph.

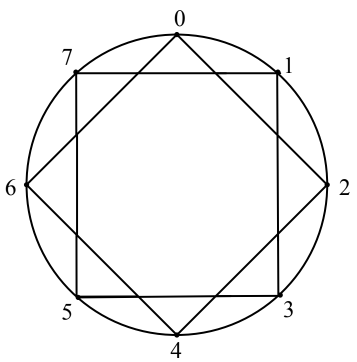

Fig. 3. $G(8, \pm\{1, 2\})$

Lemma 1 *For the complete graph* K_n, *imp* $(K_n) = \lfloor n/2 \rfloor$.

Theorem 5 *For the circulant graph* $G(n, \pm\{1, 2, ..., j\})$, *imp* $(G) \geq \lfloor (j+1)/2 \rfloor + 1$.

Proof. Let $V_1, V_2, ..., V_k$ be a k-partition of $V(G)$ such that the subgraph $G[V_i]$ of G induced by V_i is 1-regular. Let x be any vertex in $G(n, \pm\{1, 2, ..., j\})$ and $N(x) = \{x_1, x_2, ..., x_j, y_1, y_2, ..., y_j\}$. Suppose $x \in V_1$, then exactly one of x_i or $y_i \in V_1$, $1 \leq i \leq j$. Without loss of generality let $x_i \in V_1$. Since the graph induced by the vertices $x, y_1, y_2, ..., y_j$ is a complete graph on $j + 1$ vertices, in view of Lemma 1, *imp* $(G) \geq \lfloor (j+1)/2) \rfloor + 1$. □

Theorem 1 together with Corollary 1 implies the following theorem.

Theorem 6 *For the circulant graph* $G(n, \pm\{1, 2\})$, $imp(G)$ equals 2 or 3 according as $n = 4k$ *or* $n = 4k + 2$ respectively.

Proof. Let $n = 4k$ or $4k + 2$. Then $G(n, \pm\{1, 2\})$ is comprised of the outer cycle of length n and two disjoint inner cycles of length $n / 2$. Without loss of generality let $(0, 1) \in E(V_1)$. Then either $(2, 4) \in E(V_2)$ or $(2, 3) \in E(V_2)$. If $(2, 4) \in E(V_2)$, then 3 must be in V_3. On the other hand if $(2, 3) \in E(V_2)$, proceeding in the clockwise direction, we get $V_1 = \{0, 1, 4, 5, 8, 9, ..., n-4, n-3\}$ and $V_2 = \{2, 3, 6, 7, 10, 11, ..., n-2, n-1\}$ if $n = 4k$. Again $V_1 = \{0, 1, 4, 5, 8, 9, ..., n-6, n-5\}$, $V_2 = \{2, 3, 6, 7, 10, 11, ..., n-4, n-3\}$ and $V_3 = \{n-2, n-1\}$ if $n = 4k + 2$. □

$G(n, \pm\{1, 2\})$ when $n = 4k + 1$ or $4k + 3$ is a near-perfect graph. Since G is vertex-transitive it is enough to consider $G \setminus \{v\}$ for any $v \in V$. Without loss of generality let v be the vertex labeled 0. Proceeding as in Theorem 6, we have the following theorem.

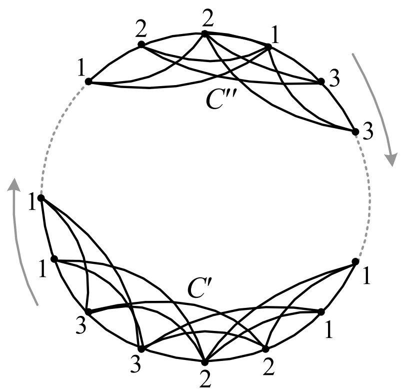

Fig. 4. $G(6k + 2, \pm\{1, 2, 3\})$

Theorem 7 *For the circulant graph* $G(n, \pm\{1, 2\})$, $imp(G)$ equals 2 or 3 according as $n = 4k + 1$ or $n = 4k + 3$ respectively.

Theorem 8 *For the circulant graph* $G(n, \pm\{1, 2, 3\})$, $n > 15$, $imp(G) = 3$.

Proof. The cycle C_n is a subgraph of $G(n; \pm\{1, 2, 3\})$. Divide the cycle C_n into two arcs such that number of vertices in arc C' is a multiple of 6 and the number of vertices in arc C'' is $n \pmod 6 + 12$, $n > 15$. Without loss of generality let the vertices on C' be labeled 1, 1, 2, 2, 3, 3 for every consecutive six vertices in the clockwise sense.

If $n \bmod 6 = 0$, the remaining 12 vertices in C'' are labeled as 1, 1, 2, 2, 3, 3, 1, 1, 2, 2, 3, 3 in the clockwise sense. If $n \bmod 6 = 2$, the remaining 14 vertices in C'' are labeled as 1, 2, 2, 1, 3, 3, 2, 2, 1, 1, 3, 2, 2, 3 in the clockwise sense. If $n \bmod 6 = 4$, the remaining 16 vertices in C'' are labeled as 1, 1, 2, 2, 3, 1, 1, 3, 2, 2, 1, 1, 3, 2, 2, 3 in the clockwise sense. It can be manually checked that V_i containing vertices labeled i, $1 \le i \le 3$ yield an induced matching partition of G. The case when $n \bmod 6 = 1, 3, 5$ leads to a near perfect matching which can be obtained in a similar manner. □

Conjecture *For the circulant graph* $G(n, \pm\{1, 2, ..., j\})$, $imp(G) = \lfloor (j+1)/2 \rfloor + 1$, $j \ge 4$.

4. Conclusion

In this paper, the induced matching partition numbers of Petersen graphs and Circulant graphs have been determined. As the induced matching k-partition problem is NP-complete even for $k = 2$, it is worth investigating the same for interconnection networks for all k. We have also identified classes of Petersen and Ciruculant graphs with $k = 3$.

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