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Laplacian Energy of an Intuitionistic Fuzzy Graph

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Abstract

Background/Objectives: The concept of Laplacian energy of fuzzy graph is extended to the Laplacian energy of an intuitionistic fuzzy graph. **Methods/Statistical Analysis:** In this paper, we defined the adjacency matrix of an intuitionistic fuzzy graph and the Laplacian energy of an intuitionistic fuzzy graph is defined in terms of its adjacency matrix. **Findings:** The lower and upper bound for the energy of an intuitionistic fuzzy graph are also derived and verified with suitable intuitionistic fuzzy graphs. **Application/Improvements:** Laplacian spectra of intutionistic fuzzy graphs may reveal more analogous results in the chemical molecules.

Keywords: Fuzzy Graph, Intuitionistic Fuzzy Graph, Intuitionistic Fuzzy Set

1. Introduction

Fuzzy set[6] has emerged as a potential area of interdisciplinary research and fuzzy graph theory is of recent interest. The concept of a fuzzy graph relation was defined by Zadeh[13] and it has found applications in the analysis of cluster patterns. Rosenfield[11] considered the fuzzy relations on fuzzy relations on fuzzy sets and developed the structure of fuzzy graphs.

Atanassov[3] introduced the concept of intuitionistic fuzzy relation and Intuitionistic Fuzzy Graphs (IFG). Recent on the theory of Intuitionistic Fuzzy Sets (IFS) has been witnessing an exponential growth of mathematics and its applications. This ranges from normal mathematics to computer sciences, information sciences and communication systems. Graph spectrum appears in problems in statistical physics and in combinatorial optimization problems in mathematics. Spectrum of a graph also plays an important role in pattern recognition, modelling virus propagation in computer networks and in securing personal data in databases. A concept related to the spectrum of a graph is that of energy.

In this paper we are concerned with simple graphs. Let G be a graph with n vertices and m edges and we say this G is a (n,m) graph.

Let d_i be the degree of i^{th} vertex of G,i=1,2,...,n. The spectrum of the graph G, consisting of the numbers λ_1, λ_2, ..., λ_n is the spectrum of its adjacency matrix[5]. The Laplacian spectrum of the graph G, consisting of the numbers μ_1, μ_2,...,μ_n is the spectrum of its Laplacian matrix[12].

The ordinary and laplacian graph Eigen values obey the following well-known relations

$$\sum_{i=1}^{n}\lambda_i = 0, \sum_{i=1}^{n}\lambda_i^2 = 2m, \sum_{i=1}^{n}\mu_i = 2m, \sum_{i=1}^{n}\mu_i^2 = 2m + \sum_{i=1}^{n} d_i^2$$

Furthermore, if the graph G has n components (n ≥ 1) and if the Laplacian Eigen values are labelled so that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ then $\mu_{n-i} = 0$ for i=0,1,2, ...p-1 and and $\mu_{n-p} > 0$.

Laplacian energy of a graph G is equal to the sum of distances of the Laplacian Eigen values of G and the average degree d(G) of G.

In this paper we introduce the concept of Laplacian energy of intuitionistic fuzzy graphs. Section 2 consists of

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preliminaries and definition of laplacian energy of fuzzy graph and in section 3, we present the Laplacian energy of an intuitionistic fuzzy graph. The lower and upper bounds for the Laplacian energy of an intuitionistic fuzzy graph are also derived in Section 4. We give the conclusion in the last section.

2. Preliminaries

2.1 Laplacian Energy LE(G)

Definition 2.1: Let G be a simple graph possessing n vertices and m edges. The Laplacian energy LE(G) of a graph G is equal to the sum of absolute values of the Eigen values $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ of G and average degree d(G) of G. This concept has been investigated in12 by L(G) = $\sum_{i=1}^{n}|r_i|$.

The auxiliary Eigen values r_i, i=1,2,...,n defined as $r_i = \mu_i - \frac{2m}{n}$.

We have $\sum_{i=1}^{n} r_{i_p} = 0, \sum_{i=1}^{n} r_i^2 = 2M$,

where $M = m + n + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2$, of G is a (n ,m)-graph and its Laplacian Eigen values are $\mu_1, \mu_2, \ldots, \mu_n$ then the Laplacian energy of G denoted by LE(G) is equal to $\sum_{i=1}^{n}|r_i|$

i.e $LE(G) = \sum_{I=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$

Thus the above definition is well chosen is seen from the following bounds that should be $LE(G) = \sqrt{2M_n}$

$$LE(G) = \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]}$$

$$2\sqrt{M} \leq LE(G) \leq 2M$$

Definition 2.2: An edge whose end points are the same is called a loop. A graph without loops and parallel edges is called a simple graph. Two vertices that are connected by an edge is called adjacent. The adjacency matrix $A=[a_{ij}]$ for a graph G = (V, E) is a matrix with n rows and n columns, n = |V| and its entries defined by

$$a_{ij} = \begin{cases} 1 & if\ (v_i, v_j) \in E \\ 0 & otherwise \end{cases}$$

2.2 Fuzzy Set

Definition 2.3: Fuzzy set: [See 12] Let X be a nonempty set. A fuzzy set A in X is defined as A = { (x, μ_A(x))/x ∈ X} which is characterized by a membership function $\mu_A(x):X \to [0,1]$ and a fuzzy set satisfying the following condition, $\mu_A(x)+\gamma_A(x) = 1$ where $\gamma_A(x) = 1-\mu_A(x)$ is the non membership function .

2.3 Intuitionistic Fuzzy Set

Definition 2.4: [See 3] Let X be a non empty set. An intuitionistic fuzzy set A in X is defined as A= { (x, μ_A(x), γ_A(x) /x ∈ X} which is characterized by a membership function $\mu_A(x):X \to [0,1]$ and the non membership function $\gamma_A(x):X \to [0,1]$ and satisfying the following condition

1. $0 \leq \mu_A(x)+\gamma_A(x) \leq 1, \forall\ x \in X$
2. $0 \leq \mu_A(x)+\gamma_A(x), \Pi_A(x) \leq 1, \forall\ x \in X$
3. $\Pi_A(x) = 1 - \mu_A(x) - \gamma_A(x)$

Where Π_A (x) is called intuitionistic fuzzy index of element x in A, the value denotes a measure of non determinity. Obviously if $\Pi_A(x) = 0$, then the intuitionistic fuzzy set A is just a Zadeh's fuzzy set.

2.4 Fuzzy Graph

Definition 2.5: [See 7] A fuzzy graph with V as the underlying set is a pair of functions G = (σ, μ) where σ : V→[0,1] is a fuzzy subset and μ : V x V→[0,1] is a symmetric fuzzy relation on the fuzzy subset σ for all u,v ∈ V such that μ(u,v)≤ σ(u) Λ σ(v). The underlying crisp graph of $\tilde{G} = (\sigma,\mu)$ is denoted by G = (V, E) where E ⊆ V x V.

A fuzzy relation can also be expressed by a matrix called fuzzy relation matrix M = $[m_{ij}]$ where $m_{ij} = \mu(u_i,u_j)$

Throughout this paper, we suppose $\tilde{G}$ is undirected without loops and σ(u)=1 for each u∈V.

Definition 2.6: Let $\tilde{G} = (\sigma,\mu)$ be a fuzzy graph with p vertices and q edges. The adjacency matrix of $\tilde{G} = (\sigma,\mu)$ is a square matrix of order p whose $\tilde{A}(\tilde{G}) = [\tilde{a}_{ij}]$ where $\tilde{a}_{ij} = \mu(u_i, u_j)$ entry is as the strength of relation between the vertices u_i and u_j .

Example 2.1: Suppose $\tilde{G}_1$ is a fuzzy graph depicted in Figure 1. Adjacency matrix of the fuzzy graph $\tilde{G}_1$ is

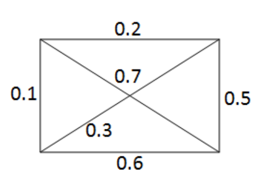

Figure 1. A fuzzy graph $\tilde{G}_1$.

$$\tilde{A}=\begin{bmatrix} 0 & 0.2 & 0.7 & 0.1 \\ 0.2 & 0 & 0.5 & 0.3 \\ 0.7 & 0.5 & 0 & 0.6 \\ 0.1 & 0.3 & 0.6 & 0 \end{bmatrix}$$

Definition 2.7: Let u be a vertex of the fuzzy graph $\tilde{G}=(\sigma,\mu)$. The degree of u is defined as

$$d_{\tilde{G}}(u)=\sum_{uv\in E(\tilde{G})}\mu(uv)$$

Definition 2.8: Let $\tilde{G}=(\sigma,\mu)$ be a fuzzy graph on G= (V, E) of $d_{\tilde{G}}(v)=k$ for all v∈V, then $\tilde{G}$ is said to be a regular fuzzy graph of degree k or a k-regular fuzzy graph.

Definition 2.9: Let $\tilde{G}=(\sigma,\mu)$ be a fuzzy graph with n vertices and m edges. The degree matrix of $\tilde{G}=(\sigma,\mu)$ is a square matrix of order p whose $D(\tilde{G})=[\tilde{d}_{ij}]$ and we have

$$\tilde{d}_{i,j}=\begin{cases} d_{\tilde{G}}(v_i) & if\ i=j \\ 0 & otherwise \end{cases}$$

Theorem 2.1: Suppose $\{\tilde{\lambda}_1,\tilde{\lambda}_2,...,\tilde{\lambda}_n\}$ are the Eigen values of the fuzzy adjacency matrix $\tilde{A}(\tilde{G})\ with\ \tilde{\lambda}_1\geq\tilde{\lambda}_1\geq...\geq\tilde{\lambda}_n$, then i)$\sum_{i=1}^{n}\tilde{\lambda}_i=0$ ii)$\sum_{i=1}^{n}\tilde{\lambda}_i^2=2\sum_{1\leq i\leq j\leq n}m_{ij}^2$

Example 2.2: Let $\tilde{G}_2$ be a fuzzy graph with |V| = n vertices. The adjacency matrix of $\tilde{G}_2$ in Figure 2 is

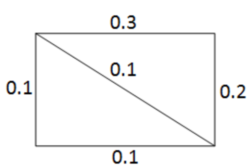

Figure 2. A fuzzy graph $\tilde{G}_2$.

$$\tilde{A}=\begin{bmatrix} 0 & 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.2 & 0.1 \\ 0 & 0.2 & 0 & 0.3 \\ 0.1 & 0.1 & 0.3 & 0 \end{bmatrix}$$

The spectrum of $\tilde{G}_1$ is {-0.34,-0.1,0,0.44} we have also

$$\sum_{i=1}^{4}\tilde{\lambda}_i=-0.34+(-0.1)+0+0.44=0$$

$$\sum_{i=1}^{4}\tilde{\lambda}_i=0.1156+0.01+0+0.1936=2\sum_{1\leq i\leq j\leq n}m_{ij}^2=0.32$$

Definition 2.10: Let $\tilde{A}(\tilde{G})$ be an adjacency matrix and $D(\tilde{G})=[\tilde{d}_{ij}]$ be a degree matrix of $\tilde{G}=(\sigma,\mu)$. The matrix $L(\tilde{G})=D(\tilde{G})-A(\tilde{G})$ is defined as fuzzy Laplacian matrix of $\tilde{G}$.

Example 2.3: The Laplacian matrix of the fuzzy graph $\tilde{G}_2$ is

$$\tilde{L}=\begin{bmatrix} 0.2 & -0.1 & 0 & -0.1 \\ -0.1 & 0.4 & -0.2 & -0.1 \\ 0 & -0.2 & 0.5 & -0.3 \\ -0.1 & -0.1 & -0.3 & 0.5 \end{bmatrix}$$

Definition 2.11: Let L ($\tilde{G}$) be a fuzzy Laplacian matrix of t $\tilde{G}=(\sigma,\mu)$. The Laplacian polynomial of G is the characteristic polynomial of its laplacian matrix, $\phi(\tilde{G},\tilde{\mu})=\det(\tilde{\mu}I_n-L(\tilde{G}))$. The roots of $\phi(\tilde{G},\tilde{\mu})$ is the fuzzy Laplacian Eigen values of $\tilde{G}$.

Theorem 2.2: Let $\tilde{G}=(\sigma,\mu)$ be a fuzzy graph with |V|=n vertices and $\tilde{\mu}_1\geq\tilde{\mu}_2\geq...\geq\tilde{\mu}_n$ is the Laplacian Eigen values of the fuzzy graph $\tilde{G}=(\sigma,\mu)$ then

$$\sum_{i=1}^{n}\tilde{\mu}_i=2\sum_{1\leq i\leq j\leq n}m_{ij}$$

$$\sum_{i=1}^{n}\tilde{\mu}_i^2=2\sum_{1\leq i\leq j\leq n}m_{ij}^2+\sum_{i=1}^{n}d_{\tilde{G}}^2(u_i)$$

Corollary 2.1: Let $\tilde{G}=(\sigma,\mu)$ be a fuzzy graph with $|V|=p$ vertices and $\tilde{\mu}_1\geq\tilde{\mu}_2\geq...\geq\tilde{\mu}_n$ is the Laplacian Eigen values of fuzzy graph $\tilde{G}=(\sigma,\mu)$ where

$$\tilde{\gamma}_i=\tilde{\mu}_i-\frac{2\sum_{1\leq i\leq j\leq n}m_{ij}}{n}$$ then we obtain

$$1.\sum_{i=1}^{n}\tilde{\gamma}_i=0\quad 2.\sum_{i=1}^{n}\tilde{\gamma}_i^2=2M$$

Where $$M=\sum_{1\leq i\leq j\leq n}m_{ij}^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\tilde{G}}(u_i)-\frac{2\sum_{1\leq i\leq j\leq n}m_{ij}}{n}\right)^2$$

Definition 2.12: Let $\tilde{G}=(\sigma,\mu)$ be a fuzzy graph with $|V|=n$ vertices and $\tilde{\mu}_1\geq\tilde{\mu}_2\geq...\geq\tilde{\mu}_n$ be the Laplacian Eigen values of fuzzy graph $\tilde{\mu}_1\geq\tilde{\mu}_2\geq...\geq\tilde{\mu}_n$. The Laplacian energy of fuzzy graph $\tilde{G}=(\sigma,\mu)$ is defined as

$$LE(\tilde{G})=\left|\tilde{\mu}_i-\frac{2\sum_{1\leq i\leq j\leq n}\mu(u_i,u_j)}{n}\right|$$

Example 2.4: Suppose $\tilde{G}$ is a fuzzy graph depicted in Figure 3a, the Laplacian spectrum of $\tilde{G}$ is

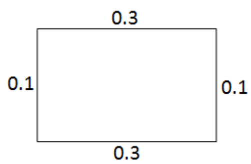

Figure 3a. Graph G

$\left(\tilde{\mu}_1 = 0.2, \tilde{\mu}_2 = 0.6, \tilde{\mu}_3 = 0.2, \tilde{\mu}_4 = 0\right)$ we also have

3. Laplacian Energy of Intuitionistic Fuzzy Graph

In this section, we define the Laplacian energy of an intuitionistic fuzzy graph without loops.

Definition 3.1: [See 8] An intuitionistic fuzzy graph is defined as G=(V, E, μ, γ) where V is the set of vertices and E is the set of edges, μ is a fuzzy membership function defined on VxV and γ is a fuzzy non membership function we define μ (v_i, v_j) by μ_{ij} and γ (v_i, v_j) by γ_{ij} such that
1. $0 \le \mu_{ij} + \gamma_{ij} \le 1$

2. $0 \le \mu_{ij,} \gamma_{ij}, \pi_{ij} \le 1$ where $\pi_{ij} = 1 - \mu_{ij} - \gamma_{ij}$. Hence (VxV, μ, γ) is an
Intuitionistic fuzzy graph.
Example 3.1:

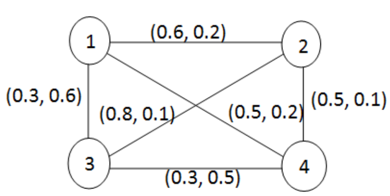

Figure 3b. $\tilde{G}_3$ Cloud environment.

Definition 3.2: [See 9] An intuitionistic fuzzy adjacency matrix of an intuitionistic fuzzy graph is defined as the adjacency matrix of the corresponding intuitionistic fuzzy graph. That is for an intuitionistic fuzzy graph G=(V,E, μ, γ) an intuitionistic fuzzy adjacency matrix is defined by A(IG)=[a_{ij}] where a_{ij}(μ_{ij}, γ_{ij}) . Note that μ_{ij} represents the strength of the relationship between v_i and v_j and γ_{ij} the strength of non relationship between v_i and v_j.

For an intuitionistic fuzzy graph in Figure 3b the adjacency matrix is

$$A(IG) = \begin{bmatrix} 0 & (0.6,0.2) & (0.5,0.2) & (0.3,0.6) \\ (0.6,0.2) & 0 & (0.5,0.1) & (0.8,0.1) \\ (0.5,0.2) & (0.5,0.1) & 0 & (0.3,0.5) \\ (0.3,0.6) & (0.8,0.1) & (0.3,0.5) & 0 \end{bmatrix}$$

Definition 3.3: The adjacency matrix of an intuitionistic fuzzy graph can be written as two matrices one containing the entries as membership values and other containing membership values i.e. A (I (G)) = [(μ_{ij}), (γ_{ij})] where

$$A(\mu_{ij}) \begin{bmatrix} 0 & 0.6 & 0.5 & 0.3 \\ 0.6 & 0 & 0.5 & 0.8 \\ 0.5 & 0.5 & 0 & 0.3 \\ 0.3 & 0.8 & 0.3 & 0 \end{bmatrix} and\ A(\gamma_{ij}) = \begin{bmatrix} 0 & 0.2 & 0.2 & 0.6 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0 & 0..5 \\ 0.6 & 0.1 & 0.5 & 0 \end{bmatrix}$$

Definition 3.4: The Eigen values of an intuitionistic fuzzy adjacency matrix A (IG) is defined as (X, Y) Where X is the set of Eigen values of A (μ_{ij}) and Y is the set of Eigen values of A (γ_{ij}).

Definition 3.5: Let A (IG) be a adjacency matrix and D (IG) =[d_{ij}] be a degree matrix of IG=(V, E, μ, γ). The matrix L (IG) =D (IG)-A (IG) is defined as intuitionistic fuzzy Laplacian matrix of IG.

Here Laplacian matrix of an intuitionistic fuzzy graph can be written as two matrices one containing the entries as membership values and other containing non membership values
i.e. L (IG) =$\left[\left(L\left(\mu_{ij}\right)\right),\left(L\left(\gamma_{ij}\right)\right)\right]$

Example 3.2: The Laplacian matrix of the membership values of an intuitionistic fuzzy graph $G_3$is

$$L[\mu_{ij}] = \begin{bmatrix} 1.4 & -0.6 & -0.5 & -0.3 \\ -0.6 & 1.9 & -0.5 & -0.8 \\ -0.5 & -0.5 & 1.3 & -0.3 \\ -0.3 & -0.8 & -0.3 & 1.4 \end{bmatrix} and\ L[\gamma_{ij}] = \begin{bmatrix} 1.0 & -0.2 & -0.2 & -0.6 \\ -0.2 & 0.4 & -0.1 & -0.1 \\ -0.2 & -0.1 & 0.8 & -0.5 \\ -0.1 & -0.1 & -0.5 & 1.2 \end{bmatrix}$$

Definition 3.6: Let L[μ_{ij}] be a intuitionistic fuzzy Laplacian matrix of IG = (V, E, μ, γ)). The laplacian polynomial of intuitionistic fuzzy graph is the characteristic polynomial of its Laplacian matrix ϕ (IG, μ) = det$\left(\tilde{\mu} I_n - L(IG)\right)$. The roots of ϕ (IG, μ) is the intuitionistic fuzzy Laplacian Eigen values of IG.

For this purpose, we define the Laplacian energy of intuitionistic fuzzy graph IG is defined as
Theorem 3.1: Let $\tilde{G} = (V, E, \sigma, \mu)$ be an intuitionistic fuzzy

graph graph with $|V| = n$ vertices and $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq ... \geq \tilde{\mu}_n$ is the Laplacian Eigen values of membership values of intuitionistic fuzzy graph then

$$1.\sum_{i=1}^{n}\lambda_i^2 = 2\sum_{1\leq i\leq j\leq n}\mu_{ij} \quad 2.\sum_{i=1}^{n}\lambda_i^2 = 2\sum_{1\leq i\leq j\leq n}\mu_{ij}^2 + \sum_{i=1}^{n}d^2_{\mu_{ij}(G)}(u_i)$$

Proof: 1. Since $\tilde{L}\left[\mu_{ij}(G)\right]$ is a symmetric matrix and those Laplacian Eigen values are non negative such that tr=$\left(L\left(\mu_{ij}(G)\right)\right) = \sum_{i=1}^{n} d_{u_{ij}(G)}(u_i) = 2\sum_{1\leq i\leq j\leq n}\mu_{ij}$

2. According to definition of Laplacian matrix, we have

$$\begin{bmatrix} d_{\mu_{ij}(G)}(u_1) & -\mu(u_1,u_2) & .... & -\mu(u_1,u_n) \\ -\mu(u_2,u_1) & d_{\mu_{ij}(G)}(u_2) & .... & -\mu(u_2,u_n) \\ . & . & . & . \\ -\mu(u_n,u_1) & -\mu(u_n,u_2) & .... & d_{\mu_{ij}(G)}(u_n) \end{bmatrix}$$

Then obtain tr $\left(L_{i,i}^2\right) = \sum_{i=1}^{n}\mu_i^2$

Where

$$tr(L_{ii}^2) = [d^2_{\mu_{ij}(G)}(u_1) + \mu^2(u_1,u_2) + ... + \mu^2(u_1,u_n)] +$$
$$[\mu^2(u_2,u_1) + d^2_{\mu_{ij}(G)}(u_2) + ... + \mu^2(u_2,u_n)] + ... + [\mu^2(u_n,u_1) + \mu^2(u_n,u_2) + ... + d^2_{\mu_{ij}(G)}(u_n)]$$
$$= 2\sum_{1\leq i\leq j\leq n}\mu_{ij}^2 + \sum_{I=1}^{N} d^2_{\mu_{ij}(G)}(u_i)$$

Similarly we can prove that for the non membership function intuitionistic fuzzy Laplacian matrix as

$$1.\sum_{i=1}^{n}\delta_i = 2\sum_{1\leq i\leq j\leq n}\gamma_{ij}$$

$$2.\ \sum_{i=1}^{n}\delta_i^2 = 2\sum_{1\leq i\leq j\leq n}\gamma_{ij}^2 + \sum_{i=1}^{n} d^2_{r_{ij}(G)}(u_i)$$

Where $\delta_1 \geq \delta_2 \geq ... \geq \delta_n$ is the Laplacian Eigen values of membership values of intuitionistic fuzzy graph.

Corollary 3.1: Let IG = (V, E, σ, μ) be an intuitionistic fuzzy graph of membership function with $|V|$ = n vertices and $\lambda_1 \geq \lambda_2 \geq .. \geq \lambda_n$ is the membership Laplacian Eigen values of intuitionistic fuzzy graph ,where $\xi_i = \lambda_i - \dfrac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}$ then we obtain

$$1.\sum_{i=1}^{n}\xi_i = 0\ (b)\sum_{i=1}^{n}\xi_I^2 = 2M$$

Where M = $\sum_{1\leq i\leq j\leq n}\mu_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(G)}(u_i) - \dfrac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2$

Similarly for membership Laplacian Eigen values of intuitionistic fuzzy graph where, $\chi_i = \delta_i - \dfrac{2\sum_{1\leq i\leq j\leq n} r_{ij}}{n}$ then we have

$$1.\sum_{i=1}^{n}\chi_i = 0\ (b)\sum_{i=1}^{n}\chi_i^2 = 2N$$

Where N = $\sum_{1\leq i\leq j\leq n} r_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{r_{ij}(G)}(u_i) - \dfrac{2\sum_{1\leq i\leq j\leq n} r_{ij}}{n}\right)^2$

Definition 3.7: Let $\tilde{G} = (V, E, \sigma, \mu)$ be an intuitionistic fuzzy graph with $|V|$ = n Vertices and $\lambda_1 \geq \lambda_2 \geq .. \geq \lambda_n$ is the membership Laplacian Eigen values of IG. The Laplacian energy intuitionistic fuzzy graph is defined as,

$$LE\left(\mu_{ij}(G)\right) = \left|\lambda_i - \frac{2\sum_{1\leq i\leq j\leq n}\mu\left(u_i,u_j\right)}{n}\right|$$

The Laplacian energy of intuitionistic fuzzy graph G = (V, E, μ, γ) is defined as $\left[LE\left(\mu_{ij}(G)\right), LE\left(\gamma_{ij}(G)\right)\right]$

Example 3.3: For an intuitionistic fuzzy graph in Figure 3

spec(L(μ_{ij}(G_3)))={0.0,1.5520,1.8506,2.5975}

spec(L(γ_{ij}(G_3)))={0.0,0.5210,1.0900,1.7890}

$$LE\left(\mu_{ij}(G_3)\right) = \left|\left(0 - \frac{2(3)}{4}\right)\right| + \left|\left(1.5520 - \frac{2(3)}{4}\right)\right| + \left|\left(1.8506 - \frac{2(3)}{4}\right)\right| + \left|\left(2.5975 - \frac{2(3)}{4}\right)\right|$$

= 1.5+0.052+0.3506+1.0975 = 3.0001

$$LE\left(\gamma_{ij}(G_3)\right) = \left|\left(0 - \frac{2(1.7)}{4}\right)\right| + \left|\left(0.5210 - \frac{2(1.7)}{4}\right)\right| + \left|\left(1.0900 - \frac{2(1.7)}{4}\right)\right| + \left|\left(1.7890 - \frac{2(1.7)}{4}\right)\right|$$

=0.85 +0.329+0.24+0.939 =2.358

Here $\sum_{1\leq i\leq j\leq n}\mu\left(u_i,u_j\right)$=0.6+0.5+0.3+0.3+0.8+0.5=3 and

$\sum_{1\leq i\leq j\leq n}\gamma(u_i,u_j) =$ 0.2+0.2+0.6+0.1+0.1+0.5 = 1.7

∴The Laplacian energy of an intuitionistic fuzzy graph G_3 is [3.0001,2.358]

4. Results

Theorem 4.1: Let G be an intuitionistic fuzzy graph (with out loops) with |V| = n and |E| = m and (IG) = ((μ_{ij}), (γ_{ij})) be an intuitionistic fuzzy adjacency matrix and L(IG) = (L(μ_{ij}), (γ_{ij})) be an intuitionistic fuzzy laplacian matrix of G then

1.

$$LE\left(\mu_{ij}(G)\right) \leq \sqrt{\left(2\sum_{1\leq i\leq j\leq n}\mu_{ij}^2 + \sum_{i=1}^{n}\left(d_{\mu_{ij}(G)}(u_i) - \frac{2\sum_{1\leq i\leq j\leq n}u_{ij}}{n}\right)^2\right)n}$$

2. $$LE\left(\gamma_{ij}(G)\right) \le \sqrt{\left(2\sum_{1\le i\le j\le n}\gamma_{ij}^2 + \sum_{i=1}^{n}\left(d_{\gamma_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}\gamma_{ij}}{n}\right)^2\right)n}$$

Proof: Apply Cauchy –Schwarz inequality to (1,..., 1) and $(|\xi_1|,|\xi_2|,\dots,|\xi_n|)$, we get

$$\left|\sum_{i=1}^{n}\tilde{\xi}_i\right|^2 \le n\sum_{i=1}^{n}\left|\tilde{\xi}_i\right|^2$$

Where $LE\left(\mu_{ij}(G)\right) \le \sqrt{n\sum_{i=1}^{n}\left|\xi_i\right|^2} = \sqrt{2Mn}$

Since M = $\sum_{1\le i\le j\le n}\mu_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}\mu_{ij}}{n}\right)^2$ we have

$$LE\left(\mu_{ij}(G)\right) \le \sqrt{\left(2\sum_{1\le i\le j\le n}\mu_{ij}^2 + \sum_{i=1}^{n}\left(d_{\mu_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}u_{ij}}{n}\right)^2\right)n}$$

Similarly, we can prove

$$LE\left(\gamma_{ij}(G)\right) \le \sqrt{\left(2\sum_{1\le i\le j\le n}\gamma_{ij}^2 + \sum_{i=1}^{n}\left(d_{\gamma_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}\gamma_{ij}}{n}\right)^2\right)n}$$

Theorem 4.2: Let G be an intuitionistic fuzzy graph with $|V| = n$ and $L(IG) = ((\mu_{ij}), (\gamma_{ij}))$ be an intuitionistic fuzzy Laplacian matrix of G then

1. $$LE\left(\mu_{ij}(G)\right) \ge 2\sqrt{\left(\sum_{1\le i\le j\le n}\mu_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}u_{ij}}{n}\right)^2\right)}$$

2. $$LE\left(\gamma_{ij}(G)\right) \ge 2\sqrt{\left(\sum_{1\le i\le j\le n}\gamma_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\gamma_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}\gamma_{ij}}{n}\right)^2\right)}$$

Proof: From the definition 3.7, we have

$$\left(LE\left(\mu_{ij}(G)\right)\right)^2 = \left(\sum_{i=1}^{n}\left|\xi_i\right|\right)^2 = \sum_{i=1}^{n}\left|\xi_i\right|^2 + 2\sum_{1\le i\le j\prec n}\left|\xi_i\right|\left|\xi_j\right| \ge 4M$$

We get
$$LE\left(\mu_{ij}(G)\right) \le 2\sqrt{\left(\sum_{1\le i\le j\le n}\mu_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}u_{ij}}{n}\right)^2\right)}$$

Hence the result.
Similarly

$$LE\left(\gamma_{ij}(G)\right) \le 2\sqrt{\left(\sum_{1\le i\le j\le n}\gamma_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\gamma_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}\gamma_{ij}}{n}\right)^2\right)}$$

Theorem 4.3: Let G be an intuitionistic fuzzy graph with $|V| = n$ and $L(IG) = ((\mu_{ij}), (\gamma_{ij}))$ be an intuitionistic fuzzy Laplacian matrix of G then

1. $$LE(\mu(G)) \le \tilde{\xi}_1 + \sqrt{(n-1)\left(2\sum_{1\le i\le j\le n}\mu_{ij}^2 + \sum_{i=1}^{n}\left(d_{\mu_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}u_{ij}}{n}\right)^2 - \xi_1^2\right)}$$

2. $$LE(\gamma(G)) \le \tilde{\chi}_1 + \sqrt{(n-1)\left(2\sum_{1\le i\le j\le n}\gamma_{ij}^2 + \sum_{i=1}^{n}\left(d_{\gamma_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}\gamma_{ij}}{n}\right)^2 - \tilde{\chi}_1\right)}$$

Proof: Apply Cauchy –Schwarz inequality to (1,...,1) and $((|\xi_1|,|\xi_2|,\dots,|\xi_n|)$,we get

$$\left|\sum_{i=2}^{n}\xi_i\right|^2 \le (n-1)\left(\left|\tilde{\xi}_2\right|^2 + \dots + \left|\tilde{\xi}_n\right|^2\right)$$

Where $LE(\mu(G)) - \tilde{\xi}_1 \le \sqrt{(n-1)\left(2M - \xi_1^2\right)}$,

Since M = $\sum_{1\le i\le j\le n}\mu_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(G)}(u_i) - 2\frac{\sum_{1\le i\le j\le n}\mu_{ij}}{n}\right)^2$, thus

$$LE(\mu(G)) \le \tilde{\xi}_1 + \sqrt{(n-1)\left(2\sum_{1\le i\le j\le n}\mu_{ij}^2 + \sum_{i=1}^{n}\left(d_{\mu_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}u_{ij}}{n}\right)^2 - \xi_1^2\right)}$$

This complete the proof

$$LE(\gamma(G)) \le \tilde{\chi}_1 + \sqrt{(n-1)\left(2\sum_{1\le i\le j\le n}\gamma_{ij}^2 + \sum_{i=1}^{n}\left(d_{\gamma_{ij}(G)}(u_i) - \frac{2\sum_{1\le i\le j\le n}\gamma_{ij}}{n}\right)^2 - \tilde{\chi}_1\right)}$$

Corollary 4.1: Let G be a k-regular intuitionistic fuzzy graph with $|V| = n$ and $L(IG) = (L(\mu(G))) = (L(\gamma(G)))$ be an k-regular intuitionistic fuzzy graph laplacian matrix of G then

1. $LE(\mu(G)) \le \tilde{\xi}_1 + \sqrt{(n-1)\left(2\sum_{1\le i\le j\le n}\mu_{ij}^2 - \tilde{\xi}_1^2\right)}$
2. $LE(\gamma(G)) \le \tilde{\chi}_1 + \sqrt{(n-1)\left(2\sum_{1\le i\le j\le n}\gamma_{ij}^2 - \tilde{\chi}_1^2\right)}$

Proof: Since G is a k-regular intuitionistic fuzzy graph and
$$k = d_{\mu(G)}(u_i) = \frac{2\sum_{1\le i\le j\le n}\mu_{ij}}{n}$$

Substituting this in the previous theorem we have

1. $LE(\mu(G)) \le \tilde{\xi}_1 + \sqrt{(n-1)\left(2\sum_{1\le i\le j\le n}\mu_{ij}^2 - \tilde{\xi}_1^2\right)}$

Similarly we have

$$2.\ LE(\gamma(G)) \le \tilde{\chi}_1 + \sqrt{(n-1)\left(2\sum_{1\le i\le j\le n}\gamma_{ij}^2 - \tilde{\chi}_1^2\right)}.$$

Example 4.1: (Illustration to Theorem 4.1 and 4.2) for intuitinistic fuzzy graph in Figure 3b.

LE ($\mu(G_3)$)= 3.0001 its lower bound = 2.6758 and upper bound = 3.78.

LE ($\gamma(G_3)$) = 2.358 its lower bound = 1.881 and upper bound = 2.6608

5. Conclusion

The Laplacian matrix and energy for an intutionistic fuzzy graph are defined. Some results on Laplacian spectra of intutionistic fuzzy graphs may reveal more analogous results of these kinds and will be discussed in the forth coming papers.

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