LEVEL SETS OF CONDITION SPECTRUM

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ABSTRACT. For $0 < \epsilon \leq 1$ and an element a of a complex unital Banach algebra $\mathcal{A}$, we prove the following topological properties about the level sets of condition spectrum:

(1) If $\epsilon = 1$ then the $1-$level set of condition spectrum of a has an empty interior unless a is a scalar multiple of the unity.
(2) If $0 < \epsilon < 1$ then the $\epsilon-$level set of condition spectrum of a has an empty interior in the unbounded component of the resolvent set of a. Further, we show that, if the Banach space X is complex uniformly convex or X^* is complex uniformly convex then for any operator T acting on X, the level set of $\epsilon-$condition spectrum of T has an empty interior.

1. INTRODUCTION

Let $\mathcal{A}$ be a complex Banach algebra with unity e and Ω be an open subset of $\mathbb{C}$. We shall identify $\lambda.e = \lambda$ for any $\lambda \in \mathbb{C}$. As most of our results are trivial for the elements which are scalar multiple of the unity, we denote the set of all elements in $\mathcal{A}$ which are not scalar multiple of the unity by $\mathcal{A} \setminus \mathbb{C}e$. A function $f : \Omega \to \mathcal{A}$ is said to be differentiable at the point $\mu \in \Omega$ (see [13], Definition 3.3) if there exists an element $f'(\mu) \in \mathcal{A}$ such that

$$\lim_{\lambda \to \mu} \left\| \frac{f(\lambda) - f(\mu)}{\lambda - \mu} - f'(\mu) \right\| = 0.$$

If f is differentiable at every point in Ω then f is said to be analytic in Ω.

Consider a non constant analytic function $f : \Omega \to \mathcal{A}$. For $M > 0$, we ask the following question,

$$\text{does the level set} := \{\lambda \in \Omega : \|f(\lambda)\| = M\} \text{ have nonempty interior?} \tag{1.1}$$

Answer to the above question depends on the topology of Ω and the Banach algebra $\mathcal{A}$. The following two examples shows that for some $M > 0$ and for a general non constant analytic Banach algebra valued function, interior of the level set may be empty or may not be empty.

Example 1.1. If Ω be a connected open subset of $\mathbb{C}, \mathcal{A} = \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a non constant analytic map then by maximum modulus theorem, for any $M > 0$ the interior of the level set defined in (1.1) is empty.

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Example 1.2. Consider $\Omega = \mathbb{C}$, $\mathcal{A} = \mathbb{M}_2(\mathbb{C}) := \left\{ A : A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ where } a_{ij} \in \mathbb{C} \right\}$ with norm $\|A\|_\infty = \max_{1 \leq i \leq 2} \left\{ \sum_{j=1}^{2} |a_{ij}| \right\}$. Define $\psi : \mathbb{C} \to \mathbb{M}_2(\mathbb{C})$ by $\psi(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. For any $\mu \in \mathbb{C}$, it is easy to see

$$\lim_{\lambda \to \mu} \left\| \frac{\psi(\lambda) - \psi(\mu)}{\lambda - \mu} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\|_\infty = 0.$$

Thus ψ is analytic. Moreover

$$\|\psi(\lambda)\|_\infty = \begin{cases} 1 & \text{if } |\lambda| \leq 1 \\ |\lambda| & \text{if } |\lambda| > 1. \end{cases}$$

The level set of ψ for $M = 1$ is

$$\{\lambda \in \mathbb{C} : \|\psi(\lambda)\|_\infty = 1\} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

Clearly 0 is an interior point to the above set.

For $a \in \mathcal{A}$ the resolvent set of a is defined as $\{\lambda \in \mathbb{C} : (a - \lambda) \in \mathcal{A}^{-1}\}$ where $\mathcal{A}^{-1}$ denotes set of all invertible elements of $\mathcal{A}$. Resolvent set is denoted as $\rho(a)$ and it is known that $\rho(a)$ is an open subset of $\mathbb{C}$. Complement of $\rho(a)$ is called the spectrum of a, it is denoted by $\sigma(a)$. It is well known that $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$. The spectral radius of a is defined as

$$r(a) := \sup \{|\lambda| : \lambda \in \sigma(a)\}.$$

The map $R : \rho(a) \to \mathcal{A}$ defined by $R(\lambda) = (a - \lambda)^{-1}$ is called resolvent map and we know that the resolvent map is an analytic Banach algebra valued map.

For $\epsilon > 0$, Globevnik in [10] raised the following question,

does the level set $\{\lambda \in \rho(a) : \|(a - \lambda)^{-1}\| = \epsilon\}$ have nonempty interior?

He was unable to answer this question, he showed that (a) the resolvent norm of an element of a unital Banach algebra cannot be constant on an open subset of the unbounded component of the resolvent set and (b) the resolvent norm of a bounded linear operator on a Banach space cannot be constant on an open set if the underlying space is complex uniformly convex (see definition(4.4)). One can find, some more answers related to this question in [2], [3] and [4]. In [14] (Theorem 3.1), Shargorodsky proved, there exists an invertible bounded operator T acting on the Banach space

$$\ell_\infty(\mathbb{Z}) := \left\{ x = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) \,\middle|\, \sup_{-\infty \leq i \leq \infty} |x_i| < \infty \text{ and } x_i \in \mathbb{C} \right\}$$

with norm $\|x\|_* = \sup_{k \neq 0} |x_k| + |x_0|$ such that $\|(T - \lambda)^{-1}\|$ is constant in a neighborhood of $\lambda = 0$, which is an affirmative answer to the question of Globevnik. Refer [6], for the results related to the level sets of resolvent norm of a linear operator.

The concept of condition spectrum was first introduced by Kulkarni and Sukumar in [11] and because of the inequality in the definition, it is evident that in order to understand the condition spectrum geometrically one has to know more about its boundary set. Since the boundary set is the subset of the level sets of condition spectrum, one has to concentrate on the level sets. The definition of the level sets of condition spectrum is the following,

Definition 1.3. Let $0 < \epsilon \leq 1$. The $\epsilon-$level set of condition spectrum of $a \in \mathcal{A}$ is defined as

$$L_\epsilon(a) := \left\{ \lambda \in \mathbb{C} : \|(a-\lambda)\| \|(a-\lambda)^{-1}\| = \frac{1}{\epsilon} \right\}$$

In computational point of view, if we are sure that the level sets of condition spectrum do not contain any interior point then it can help us to trace out the boundary set of condition spectrum. Because of the reasons discussed so for, in this paper we focus on the following question: is the interior of $L_\epsilon(a)$ non empty?

For $0 < \epsilon < 1$ and $a \in \mathcal{A} \setminus \mathbb{C}e$, the preliminary section of this paper, discusses the basic facts about $L_\epsilon(a)$. In section 3, we construct some example to show the contrast between the topological property of $L_1(a)$ and $L_\epsilon(a)$ and we prove that $L_1(a)$ has empty interior. For $0 < \epsilon < 1$, in section 4, we study about the interior property of $L_\epsilon(a)$.

Throughout this paper $B(a, r)$ denotes the open ball in $\mathbb{C}$ with center a and radius r and $B(X)$ denotes set of all bounded linear operators defined on the complex Banach space X.

2. PRELIMINARIES

In this section we introduce some definition and terminology used in this paper. We also prove some basic properties of the level sets of condition spectrum.

Definition 2.1. ([11], Definition 2.5) Let $0 < \epsilon < 1$. The ϵ-condition spectrum of $a \in \mathcal{A}$ is defined as

$$\sigma_\epsilon(a) = \left\{ \lambda \in \mathbb{C} : \|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| \geq \frac{1}{\epsilon} \right\},$$

with the convention that $\|(a-\lambda)\| \, \|(a-\lambda)^{-1}\| = \infty$ if $(a-\lambda)$ is not invertible.

Note 2.2. For $0 < \epsilon < 1$, it is clear that $L_\epsilon(a) \subset \sigma_\epsilon(a)$. If $a = \lambda$ for some $\lambda \in \mathbb{C}$ then $L_\epsilon(a) = \emptyset$ and so interior of $L_\epsilon(a) = \emptyset$, further $L_1(a) = \mathbb{C} \setminus \{\lambda\}$ and so $L_1(a) \neq \emptyset$.

Consider the Banach algebra $\mathbb{M}_2(\mathbb{C})$. For every $A \in \mathbb{M}_2(\mathbb{C})$, the 2-norm of A is defined as $\|A\| = s_{\max}(A)$ where $s_{\max}(A)$ denotes the maximum singular value of A. For any $0 < \epsilon < 1$, we find out explicitly $\epsilon-$ level set of condition spectrum of an upper triangular 2×2 matrix.

Proposition 2.3. *Let* $0 < \epsilon < 1$ *and* $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$. *Then*

$$L_\epsilon(A) = \left\{ \mu \in \mathbb{C} : \frac{\left(\sqrt{(|\mu-a|+|\mu-c|)^2 + |b|^2} + \sqrt{(|\mu-a|-|\mu-c|)^2 + |b|^2} \right)^2}{4|\mu-a||\mu-c|} = \frac{1}{\epsilon} \right\} \tag{2.1}$$

Proof. For $A \in \mathbb{M}_2(\mathbb{C})$ with 2-norm, we have the following

$$\|A-\mu\|= s_{\max}(A-\mu) \text{ and } \left\|(A-\mu)^{-1}\right\| = \frac{1}{s_{\min}(A-\mu)}.$$

where $s_{\min}(A-\mu)$ denotes the minimum singular value of $A-\mu$. Hence

$$L_\epsilon(A) = \left\{\mu : \frac{s_{\max}(A-\mu)}{s_{\min}(A-\mu)} = \frac{1}{\epsilon}\right\}.$$

Now

$$s_{\max}(A-\mu)s_{\min}(A-\mu) = |\det(A-\mu)|= |a-\mu||c-\mu|. \tag{2.2}$$

$$\begin{aligned}[s_{\max}(A-\mu)]^2 + [s_{\min}(A-\mu)]^2 &= \operatorname{trace}[(A-\mu)^*(A-\mu)] \\ &= |\mu-a|^2+|\mu-c|^2+|b|^2 \end{aligned} \tag{2.3}$$

From the above two equations, we get

$$[s_{\max}(A-\mu) \pm s_{\min}(A-\mu)]^2 = (|\mu-a|\pm|\mu-c|)^2 + |b|^2 \tag{2.4}$$

After simplification, we see $L_\epsilon(a)$ as given in equation 2.1 □

Note 2.4. We know that any complex matrix is unitarily similar to an upper triangular matrix and hence the level set of any matrix $A \in \mathbb{M}_2(\mathbb{C})$ is of the form given in proposition 2.3.

For $0 < \epsilon < 1$. If $a \in \mathcal{A} \setminus \mathbb{C}e$ then boundary of $\sigma_\epsilon(a)$ is a subset of $L_\epsilon(a)$. Since $\sigma_\epsilon(a)$ is a non empty compact set, so $L_\epsilon(a)$ is also non empty set. Following example shows that every element of $L_\epsilon(a)$ need not come from boundary of $\sigma_\epsilon(a)$.

Example 2.5. Consider the Banach space $\ell_\infty(\mathbb{Z})$ with norm

$$\|x\|_*= |x_0|+\sup_{n\neq 0}|x_n| \text{ where } x = \left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right),$$

where the box represents the zeroth coordinate of an element in $\ell_\infty(\mathbb{Z})$. Take an operator $A \in B\left(\ell_\infty(\mathbb{Z})\right)$ such that

$$A\left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right) = \left(\cdots, x_{-2}, x_{-1}, x_0, \boxed{\frac{x_1}{4}}, x_2, x_3, \cdots\right).$$

For $\epsilon = \dfrac{1}{5}$, we prove that the scalar 0 belongs to $L_\epsilon(A)$ but not in the boundary of $\sigma_\epsilon(A)$. By theorem 3.1 in [14],

$$\left\|(A-\lambda)^{-1}\right\| = 4 \text{ for } \lambda \in B\left(0, \frac{1}{4}\right). \tag{2.5}$$

For any $x \in \ell_\infty(\mathbb{Z})$,

$$\|Ax\|_*= \left(\left|\frac{x_1}{4}\right| + \sup_{n\neq 1}|x_n|\right) \leq \frac{5}{4}\|x\|_*. \tag{2.6}$$

Take the unit norm element $y = (y_k)_{k=-\infty}^{\infty}$ such that $y_k = \begin{cases} 1, & \text{for } k = 1,2 \\ 0, & \text{otherwise} \end{cases}$.

It is easy to see $\|Ay\|_*= \dfrac{5}{4}$, thus $\|A\|= \dfrac{5}{4}$. Equation(2.5) and the fact $\|A\|= \dfrac{5}{4}$

together implies $\|A\|\|A^{-1}\|= 5$ and hence $0 \in L_\epsilon(A)$. Consider the unit norm element $y = (y_k)_{k=-\infty}^{\infty}$ such that

$$y_k = \begin{cases} 1, \text{ for } k = 1, 4 \\ -\overline{\lambda} \text{ for } k = 3 \\ 0, \text{ otherwise.} \end{cases}$$

where $\lambda \in B\left(0, \frac{1}{4}\right) \setminus \{0\}$. Then

$$\begin{aligned} \|(A-\lambda)y\|_* &= \left\|\left(\cdots, y_{-1} - \lambda y_{-2}, y_0 - \lambda y_{-1}, \boxed{\frac{y_1}{4} - \lambda y_0}, y_2 - \lambda y_1, y_3 - \lambda y_2, \cdots\right)\right\|_* \\ &= \left|\frac{y_1}{4} - \lambda y_0\right| + \sup_{n\neq 0} |y_{n+1} - \lambda y_n| > \frac{5}{4}. \end{aligned}$$

Hence

$$\|A-\lambda\|> \frac{5}{4}, \text{ for } \lambda \in B\left(0, \frac{1}{4}\right) \setminus \{0\}. \tag{2.7}$$

From equation (2.5) and equation (2.7), we get

$$\|A-\lambda\|\|(A-\lambda)^{-1}\|> 5, \text{ for } \lambda \in B\left(0, \frac{1}{4}\right) \setminus \{0\}.$$

Thus $B\left(0, \frac{1}{4}\right) \subset \sigma_\epsilon(A)$, this clearly tells us 0 is not a boundary point of $\sigma_\epsilon(A)$.

Note 2.6. From theorem 3.1 in [11], we know that $\sigma_\epsilon(a)$ is a perfect set, for any $a \in \mathcal{A}\setminus\mathbb{C}e$. But from the last example, we observe $L_\epsilon(a)$ need not to be a perfect set. Whereas the the following proposition shows that $L_\epsilon(a)$ is a compact set with uncountable cardinality.

Proposition 2.7. *Let $0 < \epsilon < 1$. If $a \in \mathcal{A} \setminus \mathbb{C}e$ then $L_\epsilon(a)$ is a compact subset of $\mathbb{C}$ with uncountable number of elements.*

Proof. For $a \in \mathcal{A}\setminus\mathbb{C}e$, we know $L_\epsilon(a)$ is a closed subset of $\sigma_\epsilon(a)$ and hence $L_\epsilon(a)$ is compact. Suppose $L_\epsilon(a)$ has countable number of elements then we choose an isolated point λ_0 from the boundary of $\sigma_\epsilon(a)$. There exist an $r > 0$ such that

$$B(\lambda_0, r) \cap \sigma(a) = \emptyset, B(\lambda_0, r) \cap \sigma_\epsilon(a) \neq \emptyset, B(\lambda_0, r) \cap \sigma_\epsilon(a)^c \neq \emptyset.$$

Take $E := B(\lambda_0, r) \setminus L_\epsilon(a)$ and define the following function

$$\phi : E \to \mathbb{C} \text{ by } \phi(\lambda) = \|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\|.$$

Clearly ϕ is continuous and

$$E = \left\{\lambda \in \rho(a) : \phi(\lambda) > \frac{1}{\epsilon}\right\} \bigcup \left\{\lambda \in \rho(a) : \phi(\lambda) < \frac{1}{\epsilon}\right\}.$$

This is a contradiction to the fact that E is connected. Thus $L_\epsilon(a)$ has uncountable number of points. $\square$

3. 1-LEVEL SET OF CONDITION SPECTRUM

This section deals with $1-$level set of condition spectrum. We mainly prove interior of $1-$level set of condition spectrum is empty and they also give a better geometric picture of $1-$level set of condition spectrum (see lemma 3.4). In fact excluding the case when the number of elements in $\sigma(a)$ is two, we prove that $L_1(a)$ contains at most one element (see Theorem 3.5 and Theorem 3.8).

The following examples shows that the nature $L_1(a)$ is different from $L_\epsilon(a)$, particularly $L_1(a)$ may be empty, may be unbounded and may have countable number of points.

Example 3.1. The set $\mathcal{D} = \{f \in C([a,b]) \mid f' \in C([a,b])\}$ forms a complex unital Banach algebra with respect to pointwise addition, pointwise multiplication and with the norm $\|f\|_d = \|f\|_\infty + \|f'\|_\infty$. Since $\|f'\|_\infty \neq 0$ for every non scalar invertible element $f \in \mathcal{D}$, we have

$$\|f\|_d \|f^{-1}\|_d \geq \|f\|_\infty \|f^{-1}\|_\infty + \|f^{-1}\|_\infty \|f'\|_\infty > 1.$$

Thus $L_1(f) = \emptyset$ for every non scalar invertible element $f \in \mathcal{D}$.

Example 3.2. Consider the complex Hilbert space

$$\ell^2(\mathbb{N}) := \left\{ x = (x_1, x_2, x_3, x_4, \dots) \middle| \sum_{i=1}^{\infty} |x_i|^2 < \infty \text{ and } x_i \in \mathbb{C} \right\}$$

with norm

$$\|x\|_2 = \sum_{i=1}^{\infty} |x_i|^2 .$$

For some fixed $n \in \mathbb{N}$ with $n \geq 2$, consider an operator T in $B(\ell^2(\mathbb{N}))$ defined as

$$T(e_i) = \begin{cases} e_{(n+1)-i}, & \text{for } 1 \leq i \leq n \\ e_i, & \text{for all } i \geq n+1. \end{cases}$$

where the $e_i's$ form the standard orthonormal basis for $\ell^2(\mathbb{N})$. It is easy to see that $T = T^* = T^{-1}$ and $\sigma(T) = \{-1, 1\}$. For any $\lambda \in \rho(T)$ the operators $T - \lambda$ and $(T-\lambda)^{-1}$ are normal and so their norms are equal to its spectral radius. We have

$$\|T - \lambda\| = \max\{|\lambda - 1|, |\lambda + 1|\} \text{ and } \left\|(T-\lambda)^{-1}\right\| = \max\left\{\frac{1}{|\lambda - 1|}, \frac{1}{|\lambda + 1|}\right\}.$$

Hence

$$L_1(T) = \left\{\lambda : \frac{|\lambda - 1|}{|\lambda + 1|} = 1\right\} \bigcup \left\{\lambda : \frac{|\lambda + 1|}{|\lambda - 1|} = 1\right\} = \{\lambda : |\lambda - 1| = |\lambda + 1|\}.$$

This shows that $L_1(T)$ is unbounded.

Example 3.3. For $n \geq 2$, consider Banach space $\mathbb{C}^n$ with infinity norm. Take an operator $S \in B(\mathbb{C}^n)$ such that $S(e_i) = e_{(n+1)-i}$ where e_i is the standard basis of $\mathbb{C}^n$. It is clear that $S = S^{-1}$, $\|S\| = 1$ and $\sigma(S) = \{-1, 1\}$. For any $\lambda \in \rho(S)$, we observe the following

$$(S - \lambda)(e_i) = -\lambda e_i + e_{(n+1)-i} \text{ with } \|(S - \lambda)\| = 1 + |\lambda|.$$

and

$$(S-\lambda)^{-1}(e_i) = \frac{-1}{\lambda^2-1}\left(\lambda e_i + e_{(n+1)-i}\right) \text{ with } \|(S-\lambda)^{-1}\| = \frac{1+|\lambda|}{|\lambda^2-1|}.$$

It is easy to verify that $L_1(S) = \{0\}$.

Lemma 3.4. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $L_1(a)$ is nonempty then for each $\mu \in L_1(a)$*

$$\|a-\mu\| = |\mu-\lambda| \text{ and } \|(a-\mu)^{-1}\| = \frac{1}{|\mu-\lambda|} \text{ for all } \lambda \in \sigma(a).$$

Proof. Let $\mu \in L_1(a)$. Then

$$\begin{aligned}\left\|(a-\mu)^{-1}\right\| &\geq \frac{1}{\inf\{|\mu-\lambda| : \lambda \in \sigma(a)\}} \\ &\geq \frac{1}{\sup\{|\mu-\lambda| : \lambda \in \sigma(a)\}} \\ &= \frac{1}{r(a-\mu)} \geq \frac{1}{\|a-\mu\|} = \left\|(a-\mu)^{-1}\right\|\end{aligned}$$

Hence, $|\mu-\lambda| = \|a-\mu\|$ and $\dfrac{1}{|\mu-\lambda|} = \|(a-\mu)^{-1}\|$ for all $\lambda \in \sigma(a)$. □

Theorem 3.5. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $\sigma(a)$ has more than two element then $L_1(a)$ has atmost one element.*

Proof. Let $\lambda_1, \lambda_2, \lambda_3 \in \sigma(a)$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Suppose $L_1(a)$ has two distinct elements z_1 and z_2 then by lemma 3.4,

$$|z_1-\lambda_1| = |z_1-\lambda_2| = |z_1-\lambda_3| = \|a-z_1\|,$$

and

$$|z_2-\lambda_1| = |z_2-\lambda_2| = |z_2-\lambda_3| = \|a-z_2\|.$$

The above two equations imply us that two circles with distinct centers intersect in three distinct points. This is a contradiction. □

Theorem 3.6. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\sigma(a)$ has more than one element. Then interior of $L_1(a)$ is empty.*

Proof. If $\sigma(a)$ has more than two element then by Theorem 3.5, interior of $L_1(a)$ is empty. Next, we assume that $\sigma(a) = \{\lambda_1, \lambda_2\}$ with $\lambda_1 \neq \lambda_2$. Suppose there exists an $r > 0$ such that $B(\eta_0, r) \subseteq L_1(a)$ for some $\eta_0 \in \mathbb{C}$. By lemma 3.4, we have $|\lambda_1-\mu| = |\lambda_2-\mu|$ for all $\mu \in B(\eta_0, r)$. This is a contradiction. □

A well known problem in operator theory is that if $T \in B(X)$ with $\sigma(T) = \{1\}$ then under what additional conditions can we conclude $T = I$? In connection with this problem, Theorem 3.7, gives a sufficient condition. A survey article [15] contains details of many classical results related to this problem. Another sufficient condition is also given in [[11], Corollary 3.5] in terms of condition spectrum.

Theorem 3.7 ([1], Theorem 1.1)**.** *Let $a \in \mathcal{A}$. If $\sigma(a) = \{1\}$ and a is doubly power bound element of $\mathcal{A}$ which means* $\sup\{\|a^n\| : n \in \mathbb{Z}\} < \infty$ *then $a = e$.*

We prove the following result with the help of Theorem 3.7.

Theorem 3.8. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $\sigma(a) = \{\lambda\}$ then $L_1(a)$ is empty and in particular the interior of $L_1(a)$ is also empty.*

Proof. Suppose $L_1(a) \neq \emptyset$ and $\mu \in L_1(a)$ then by lemma 3.4, $\|a - \mu\| = |\mu - \lambda|$. Consider the element $b := \dfrac{(a-\mu)}{\lambda - \mu}$. It is clear that $\sigma(b) = \{1\}$. Since $\|b\| = 1$, we have $\|b^n\| \leq 1$ for all positive integers n. By Lemma 3.4, $\|b^{-1}\| = 1$ and hence $\|b^n\| \leq 1$ for all negative integers n and hence b is doubly power bound. By Theorem 3.7, we conclude that $b = e$, this implies $a = \lambda$, which is a contradiction. □

From Theorem 3.8, we observe, if $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $L_1(a)$ is non empty then $\sigma(a)$ contains more than one element and if $a \in \mathcal{A}$ with $L_1(a) = \mathbb{C} \setminus \{\mu\}$ for some $\mu \in \mathbb{C}$ then $a = \mu$. From example 3.1, we understand $L_1(a)$ may be empty for $a \in \mathcal{A} \setminus \mathbb{C}e$ with $\sigma(a)$ contains more than one element.

The following Theorem and Example 3.1 proves that $L_1(a) = \emptyset$ for some elements of every Banach algebra and every element of some Banach algebra

Theorem 3.9. *For any complex unital Banach algebra $\mathcal{A}$, there always exists an $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $L_1(a) = \emptyset$.*

Proof. Suppose there exists $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\sigma(a) = \{\lambda\}$ then by Theorem 3.6, $L_1(a) = \emptyset$. If there exists $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\sigma(a) = \{\lambda_1, \lambda_2\}$ with $\lambda_1 \neq \lambda_2$, then by proposition 9 in §7 of [8], there exists idempotents e_1 and e_2 such that $\sigma(ae_1) = \{\lambda_1\}, \sigma(ae_2) = \{\lambda_2\}$ and $a = ae_1 + ae_2$. We must have either $ae_1 \in \mathcal{A} \setminus \mathbb{C}e$ or $ae_2 \in \mathcal{A} \setminus \mathbb{C}e$, otherwise $a \notin \mathcal{A} \setminus \mathbb{C}e$. Hence by Theorem 3.6 we get $L_1(ae_1) = \emptyset$ or $L_1(ae_2) = \emptyset$. If there exists $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\{\lambda_1, \lambda_2, \lambda_3\} \subseteq \sigma(a)$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3$ then consider the following polynomial

$$p(z) = \frac{(z-\lambda_2)(z-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} - \frac{(z-\lambda_1)(z-\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}.$$

Clearly $\{-1, 0, 1\} \subseteq \sigma(p(a))$ and so by lemma 3.4, $L_1(p(a)) = \emptyset$. □

To get Theorem 3.8, we need a doubly power bound element $a \in \mathcal{A} \setminus \mathbb{C}e$. We now ask the following question, for $a \in \mathcal{A} \setminus \mathbb{C}e$ with $L_1(a)$ empty, is it necessary that a to be doubly power bound? We get a negative answer from the following example.

Example 3.10. Consider the Banach algebra $C[0,2]$ and element $f \in C[0,2]$ such that $f(x) = x$. By lemma 3.4, $L_1(f) = \emptyset$ but $\|f^n\|_\infty \to \infty$ as $n \to \infty$. Hence f is not a doubly power bound element.

4. $\epsilon-$LEVEL SET OF CONDITION SPECTRUM

For $0 < \epsilon < 1$ and $a \in \mathcal{A} \setminus \mathbb{C}e$, our first main result in this section proves $L_\epsilon(a)$ has empty interior in the unbounded component of the resolvent set of a. For that we prove a version of maximum modulus Theorem for product of n analytic vector valued functions (where $n \in \mathbb{N}$.) This proof is similar to the proof of Theorem 2.1 in [14].

Lemma 4.1. *Let Ω_0 be a connected open subset of $\mathbb{C}$, Ω is an open subset of Ω_0 and X be a complex Banach space. For $i = 1, \cdots, n$, suppose we have the following*

(1) $\psi_i : \Omega_0 \to X$ *are analytic vector valued functions.*

(2) $\prod_{i=1}^{n} \|\psi_i(\lambda)\| \leq M$ *for all* $\lambda \in \Omega$.

(3) $\prod_{i=1}^{n} \|\psi_i(\mu)\| < M$ *for some* $\mu \in \Omega_0$

Then $\prod_{i=1}^{n} \|\psi_i(\lambda)\| < M$ *for all* $\lambda \in \Omega$.

Proof. Suppose there exists $\lambda_0 \in \Omega$ such that

$$\prod_{i=1}^{n} \|\psi_i(\lambda_0)\| = M$$

then by Hahn-Banach Theorem for each $\psi_i(\lambda_0)$ there exists $g_i \in X^*$ such that $\|g_i\| = 1$ and

$$g_i\left(\psi_i(\lambda_0)\right) = \|\psi_i(\lambda_0)\|. \tag{4.1}$$

Consider the function

$$\phi : \Omega_0 \to \mathbb{C} \text{ defined by } \phi(\lambda) = \prod_{i=1}^{n} g_i(\psi_i(\lambda)).$$

ϕ is analytic because $g_i(\psi_i)$ is analytic on Ω_0 for each i. By assumption (2)

$$|\phi(\lambda)| = \left|\prod_{i=1}^{n} g_i(\psi_i(\lambda))\right| \leq \prod_{i=1}^{n} \|g_i\| \|\psi_i(\lambda)\| \leq M \text{ for all } \lambda \in \Omega.$$

Particularly for $\lambda_0 \in \Omega$ and from equation 4.1, we get

$$|\phi(\lambda_0)| = \left|\prod_{i=1}^{n} g_i(\psi_i(\lambda_0))\right| = \prod_{i=1}^{n} |g_i(\psi_i(\lambda_0))| = \prod_{i=1}^{n} \|\psi_i(\lambda_0)\| = M,$$

Thus $|\phi|$ attains local maximum at Ω_0. Since Ω_0 is connected by Maximum modulus Theorem ϕ is constant and $\phi \equiv M$. On the other hand, by assumption (3) and by the definition of all g_i, we have

$$M = |\phi(\mu)| = \left|\prod_{i=1}^{n} g_i(\psi_i(\mu))\right| \leq \prod_{i=1}^{n} \|g_i\| \|\psi_i(\mu)\| = \prod_{i=1}^{n} \|\psi_i(\mu)\| < M.$$

This is a contradiction. □

Theorem 4.2. *Let $M > 1, a \in \mathcal{A} \setminus \mathbb{C}e$ and Ω be an open subset in the unbounded component of $\rho(a)$. If*

$$\|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| \leq M \text{ for all } \lambda \in \Omega,$$

then

$$\|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| < M \text{ for all } \lambda \in \Omega.$$

Proof. Let Ω_0 be the unbounded component of $\rho(a)$. By our assumption, $\Omega \subset \Omega_0$ and

$$\|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| \leq M, \text{ for all } \lambda \in \Omega.$$

Since $\sigma_{\frac{1}{M}}(a)$ is compact, we must have

$$\left\{\lambda \in \mathbb{C} : \|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| < M\right\} \cap \Omega_0 \neq \emptyset.$$

Take $\mu \in \{\lambda \in \mathbb{C} : \|(a-\lambda)\| \|(a-\lambda)^{-1}\| < M\} \cap \Omega_0$. Apply lemma 4.1 to the analytic functions $\lambda \mapsto (a-\lambda)$ and $\lambda \mapsto (a-\lambda)^{-1}$ which are defined from Ω_0 to $\mathcal{A}$ and to the scalar $\mu \in \Omega_0$, we get $\|(a-\lambda)\| \|(a-\lambda)^{-1}\| < M$ for all $\lambda \in \Omega$. ☐

Corollary 4.3. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ and $0 < \epsilon < 1$. Then $L_\epsilon(a)$ has empty interior in the unbounded component of $\rho(a)$. In particular interior of $L_\epsilon(a)$ is empty if $\rho(a)$ is connected.*

Proof. Immediate from Theorem 4.2. ☐

Our next result shows if $T \in B(X)$ where X is a complex uniformly convex Banach space then interior of $L_\epsilon(T)$ is empty for $0 < \epsilon < 1$. We first see the notion of complex uniformly convex Banach space and some important remarks related to them.

Definition 4.4 ([9], Definition 2.4 (ii))**.** A complex Banach space X is said to be complex uniformly convex (uniformly convex) if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in X,\ \|y\| \geq \epsilon \text{ and } \|x+\zeta y\| \leq 1,\ \forall \zeta \in \mathbb{C}\,(\zeta \in \mathbb{R}),\ \text{with } |\zeta| \leq 1 \Rightarrow \|x\| \leq 1-\delta.$$

It is so obvious that every uniformly convex Banach space is complex uniformly convex space. It is proved in [5] that Hilbert spaces and L_p (with $1 < p < \infty$) spaces are uniformly convex Banach spaces and hence they are all complex uniformly convex Banach spaces. In [9], Theorem 1, Globevnik showed L_1 space is complex uniformly convex. The Banach space L_∞ is not complex uniformly convex Banach space. The dual space L_∞^* is isometrically isomorphic to a space of bounded finitely additive set functions (see [7], Chapter IV, section 8, Theorem 16 and Chapter III, section 1, Lemma 5). The space of bounded finitely additive set functions are complex uniformly convex space is proved in proposition 1.1 in [12] and so L_∞^* is complex uniformly convex.

Definition 4.5 ([9], Remark)**.** Consider a complex Banach space X and $\delta > 0$. We define $\omega_c(\delta)$ as follows

$$\omega_c(\delta) = \sup\{\|y\| : x, y \in X \text{ with } \|x\| = 1, \|x+\zeta y\| \leq 1+\delta, (\zeta \in B(0,1))\}$$

Remark 4.6. ([9], Remark) Let X be a complex Banach space. Then X is complex uniformly convex if and only if $\lim\limits_{\delta \to 0} \omega_c(\delta) = 0$.

Proof of the following Theorem is similar to the proof of proposition 2 in [10].

Theorem 4.7. *Let X be a complex uniformly convex Banach space and $M > 1$. If $T \in B(X)$ with*

$$\|T-\lambda\|\left\|(T-\lambda)^{-1}\right\| \leq M \text{ for all } \lambda \in B(0,1)$$

then

$$\|T-\lambda\|\left\|(T-\lambda)^{-1}\right\| < M \text{ for all } \lambda \in B(0,1).$$

Proof. We claim that there exists $\mu \in B(0,1)$ such that

$$\|T-\mu\|\left\|(T-\mu)^{-1}\right\| < M.$$

Suppose

$$\|(T-\lambda)\|\left\|(T-\lambda)^{-1}\right\| = M \text{ for all } \lambda \in B(0,1) \tag{4.2}$$

We arrive contradiction in 3 steps,
Step 1: In this step, we define sequence of function ψ_n from $B(0,1)$ to X for each $n \in \mathbb{N}$ and, we prove that each ψ_n is a bounded analytic vector valued function. We know that there exists a sequence $\{x_n\}$ with $\|x_n\|= 1$ such that

$$\lim_{n\to\infty} \left\|T^{-1}(x_n)\right\| = \left\|T^{-1}\right\|.$$

By Hahn-Banach Theorem, there exists $g \in B(X)^*$ such that $g(T) = \|T\|$ with $\|g\|= 1$. For each x_n, we define the following function

$$\psi_n : B(0,1) \to X \text{ by } \psi_n(\lambda) = \frac{[g(T-\lambda)]\,(T-\lambda)^{-1}\,x_n}{\|T\|\|T^{-1}\|}.$$

Now,

$$\begin{aligned}\|\psi_n(\lambda)\| &= \left\|\frac{[g(T-\lambda)]\,(T-\lambda)^{-1}\,x_n}{\|T\|\|T^{-1}\|}\right\| \\ &\leq \frac{\|g\|\|T-\lambda\|\|(T-\lambda)^{-1}\|\,\|x_n\|}{\|T\|\|T^{-1}\|}\end{aligned} \tag{4.3}$$

Since $\|g\|= 1$ and by equations (4.2) and (4.3), we get

$$\|\psi_n(\lambda)\|\leq 1. \tag{4.4}$$

Each ψ_n is analytic because the maps $\lambda \mapsto g(T-\lambda)$ and $\lambda \mapsto (T-\lambda)^{-1}$ are analytic.
Step 2 : In this step we apply Theorem 2 in [9] to the functions ψ_n and we see the consequence.
Applying, Theorem 2 in [9] to the function ψ_n, we get

$$\|\psi_n(\lambda) - \psi_n(0)\|\leq \left(\frac{2|\lambda|}{1-|\lambda|}\right) w_c\,(1 - \|\psi_n(0)\|) \text{ for all } \lambda \in B(0,1).$$

Substituting the corresponding values of ψ_n in the above equation

$$\left\|\frac{[g(T-\lambda)]\,(T-\lambda)^{-1}\,x_n}{\|T\|\|T^{-1}\|} - \frac{g(T)T^{-1}x_n}{\|T\|\|T^{-1}\|}\right\| \leq \left(\frac{2|\lambda|}{1-|\lambda|}\right) w_c \left(1 - \left\|\frac{g(T)T^{-1}x_n}{\|T\|\|T^{-1}\|}\right\|\right)$$

Apply $g(T) = \|T\|$ to the right side of the above inequality,

$$\left\| \frac{[g(T-\lambda)]\,(T-\lambda)^{-1}x_n}{\|T\|\|T^{-1}\|} - \frac{g(T)T^{-1}x_n}{\|T\|\|T^{-1}\|} \right\| \leq \left(\frac{2|\lambda|}{1-|\lambda|}\right) w_c \left(1 - \frac{\|T^{-1}x_n\|}{\|T^{-1}\|}\right)$$

Using Remark (4.6) and the fact $1 - \frac{\|T^{-1}x_n\|}{\|T^{-1}\|} \to 0$, we observe that

$$\lim_{n\to\infty} \left\| [g(T-\lambda)]\,(T-\lambda)^{-1}x_n - g(T)T^{-1}x_n \right\| = 0. \tag{4.5}$$

Substitute $(T-\lambda)^{-1} = T^{-1} + \lambda T^{-1}(T-\lambda)^{-1}$. We get,
$[g(T-\lambda)]\,(T-\lambda)^{-1} - g(T)T^{-1}$

$$\begin{aligned} &= g(T)\left[T^{-1} + \lambda T^{-1}(T-\lambda)^{-1}\right] - \lambda g(I)\,(T-\lambda)^{-1} - g(T)T^{-1} \\ &= \lambda\left[g(T)T^{-1} - g(I)\right](T-\lambda)^{-1}. \end{aligned} \tag{4.6}$$

Equation (4.5) and equation (4.6) yields

$$\lim_{n\to\infty} \left\| \left[g(T)T^{-1} - g(I)\right](T-\lambda)^{-1}x_n \right\| = 0. \tag{4.7}$$

and

$$\lim_{n\to\infty} \left\| [g(T) - g(I)T]\,(T-\lambda)^{-1}x_n \right\| = 0. \tag{4.8}$$

for all $\lambda \in B(0,1)$.

Step 3 : In this step, we get the required contradiction by applying the appropriate value for $g(I)$ to the equation (4.7) and equation (4.8).

Case 1 : If $g(I) = 0$ then equation (4.8) becomes,

$$\lim_{n\to\infty} \left\| g(T)\,(T-\lambda)^{-1}x_n \right\| = 0.$$

Since the operator $T-\lambda$ is continuous for any $\lambda \in B(0,1)$, and so

$$\lim_{n\to\infty} \|x_n\| = 0$$

which is a contradiction to $\|x_n\| = 1$.

Case 2 : If $|g(I)| \leq 1$ then from equation (4.7), we get

$$\lim_{n\to\infty} \left\| (T-\lambda)\left[g(T)T^{-1} - g(I)\right](T-\lambda)^{-1}x_n \right\| = 0. \tag{4.9}$$

Since the operators $(T-\lambda)$ and T^{-1} commutes, the above equation becomes

$$\lim_{n\to\infty} \left\| \left(g(T)T^{-1} - g(I)\right)x_n \right\| = 0.$$

By the triangle inequality,

$$\lim_{n\to\infty} \left(\|g(T)T^{-1}x_n\| - \|g(I)x_n\|\right) = 0.$$

The above equation implies,

$$\lim_{n\to\infty} \|T\| \left\|T^{-1}x_n\right\| = 1.$$

Thus

$$\lim_{n\to\infty} \left\|T^{-1}x_n\right\| = \frac{1}{\|T\|}.$$

We also know that $\lim_{n\to\infty} \left\|T^{-1}x_n\right\| = \left\|T^{-1}\right\|$. Hence $\|T\|\|T^{-1}\| = 1$. But we assumed that $\|T\|\|T^{-1}\| = M$. This is a contradiction to $M > 1$. Hence there exists $\mu \in B(0,1)$ such that $\|T-\mu\|\|(T-\mu)^{-1}\| < M$. Apply lemma 4.1, to the function $\lambda \mapsto (T-\lambda)$ and $\lambda \mapsto (T-\lambda)^{-1}$ defined from $B(0,1)$ to $B(X)$ and to the point μ, to get the required conclusion. □

Note 4.8. The above result holds for any open ball in the resolvent set of T. Suppose $\|T-\lambda\|\|(T-\lambda)^{-1}\| \leq M$ for all $\lambda \in B(\mu, r)$ and $M > 1$. If we define an operator $S := \dfrac{T-\mu}{r} \in B(X)$ then $S \in B(X)$ and $\|S-\lambda\|\,\|(S-\lambda)^{-1}\| \leq M$ for all $\lambda \in B(0,1)$. In order to prove $\|T-\lambda\|\|(T-\lambda)^{-1}\| < M$ for all $\lambda \in B(\mu, r)$, we apply Theorem (4.7) to the operator S.

Corollary 4.9. *Let X be a complex Banach space such that the dual space X^* is complex uniformly convex and $M > 1$. Suppose $T \in B(X)$ with*

$$\|T-\lambda\|\left\|(T-\lambda)^{-1}\right\| \leq M \text{ for all } \lambda \in B(0,1),$$

then

$$\|T-\lambda\|\left\|(T-\lambda)^{-1}\right\| < M \text{ for all } \lambda \in B(0,1).$$

Proof. Consider the transpose linear map $T^* \in B(X^*)$. For any $\lambda \in B(0,1)$, we have

$$\|(T-\lambda)\|\left\|(T-\lambda)^{-1}\right\| = \|(T^*-\lambda)\|\left\|(T^*-\lambda)^{-1}\right\|$$

Apply the Theorem (4.7) to the operator T^* and the Banach space X^*. This completes the proof. □

Corollary 4.10. *Let X be a complex Banach space, $T \in B(X)$ and $0 < \epsilon < 1$. If either X (or) X^* is complex uniformly convex then $L_\epsilon(T)$ has empty interior in the resolvent set of T.*

Proof. Immediate consequence of Theorem 4.7 and corollary 4.9. □

Corollary 4.11. *Let $0 < \epsilon < 1$ and $\mathcal{A}$ be a unital C^* algebra. If $a \in \mathcal{A}^e$ then interior of $L_\epsilon(a)$ is empty.*

Proof. We know that there exists a C^* isomorphism ψ form $\mathcal{A}$ to C^* subalgebra of $B(H)$ for some Hilbert space H. For any $a \in \mathcal{A}$, we have $\sigma(a) = \sigma(\psi(a))$ and $\|a\| = \|\psi(a)\|$. These implies us

$$L_\epsilon(a) = L_\epsilon(\psi(a)).$$

Since, the Hilbert space H is complex uniformly convex and by the Theorem 4.7, we get interior of $L_\epsilon(\psi(a))$ is empty. Hence interior of $L_\epsilon(a)$ is empty. □

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