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MAJORIZATION OF STARLIKE AND CONVEX FUNCTIONS OF COMPLEX ORDER INVOLVING LINEAR OPERATORS

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Dedicated to the Mathematician Srinivasan Ramanujan on his 125^{th} Birth Anniversary

(*Communicated by Stanisława Kanas*)

ABSTRACT. The main object of this present paper is to investigate the problem of majorization of certain class of analytic functions of complex order defined by the Dziok-Raina linear operator. Moreover we point out some new or known consequences of our main result.

1. Introduction

Let $\mathcal{A}$ be the class of functions which are analytic in the open unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$$

of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

For two analytic functions $f, g \in \mathcal{A}$ we say that f is subordinate to g denoted by $f \prec g$ (see [12]), if there exists a Schwarz function ω which is analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathcal{U}$, such that

$$f(z) = g(\omega(z)), \qquad (z \in \mathcal{U}).$$

We denote this subordination by $f \prec g$. Furthermore, if the function g is univalent in $\mathcal{U}$, we have

$$f \prec g \Longleftrightarrow f(0) = g(0) \qquad \text{and} \qquad f(\mathcal{U}) \subset g(\mathcal{U}).$$

If f and g are analytic functions in $\mathcal{U}$, following T. H. MacGregor [11], (also see [14]) we say that f is majorized by g in $\mathcal{U}$ and we write

$$f(z) \ll g(z), \qquad (z \in \mathcal{U}) \tag{1.2}$$

if there exists a function ϕ, analytic in $\mathcal{U}$, such that

$$|\phi(z)| < 1 \qquad \text{and} \qquad f(z) = \phi(z)g(z), \qquad (z \in \mathcal{U}). \tag{1.3}$$

It may be noted here that (1.2) is closely related to the concept of quasi-subordination between analytic functions.

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The study of operators plays an important role in the geometric function theory and its related fields.Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better.

The convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \tag{1.4}$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

For complex parameters $\alpha_1, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m$, ($\beta_j \neq 0, -1, \dots;\ j = 1, 2, \dots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{gathered} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1; \quad l, m \in N_0 = N \cup \{0\}; \quad z \in \Delta) \end{gathered} \tag{1.5}$$

where N denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0, \\ a(a+1)(a+2)\dots(a+n-1), & n \in N. \end{cases} \tag{1.6}$$

For positive real values of $\alpha_1, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m$ ($\beta_j \neq 0, -1, \dots;\ j = 1, 2, \dots, m$), let

$$\mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m) \colon \mathcal{A} \to \mathcal{A}$$

be a linear operator defined by

$$\begin{gathered} \left[(\mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m))(f)\right](z) = z\ {}_lF_m(\alpha_1, \alpha_2, \dots \alpha_l; \beta_1, \beta_2 \dots, \beta_m; z) * f(z) \\ \mathcal{H}_m^l[\alpha_1] f(z) = z + \sum_{n=2}^{\infty} \Gamma_n\ a_n\ z^n \end{gathered} \tag{1.7}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \frac{1}{(n-1)!} \tag{1.8}$$

$\alpha_i > 0$, $(i = 1, 2, \dots, l)$, $\beta_j > 0$, $(j = 1, 2, \dots, m)$, $l \leq m+1$; $l, m \in N_0 = N \cup \{0\}$.

For notational simplicity, we use a shorter notation

$$\mathcal{H}_m^l[\alpha_1] \quad \text{for} \quad \mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m)$$

in the sequel. It follows from (1.7) that

$$\mathcal{H}_1^2[1] f(z) = f(z), \qquad \mathcal{H}_1^2[2] f(z) = z f'(z).$$

Further, it is easy to verify from (1.7) that

$$z(\mathcal{H}_m^l[\alpha_1] f(z))' = \alpha_1 \mathcal{H}_m^l[\alpha_1 + 1] f(z) - (\alpha_1 - 1)\mathcal{H}_m^l[\alpha_1] f(z). \tag{1.9}$$

The linear operator $\mathcal{H}_m^l[\alpha_1]$ is called Dziok-Srivastava operator (see [4, 13]).

Following J. Dziok and H. M. Srivastava [4], using Wright's generalized hypergeometric function [20], J. Dziok and R. K. Raina [5] defined another linear operator for positive real parameters $\alpha_1, A_1 \dots, \alpha_l, A_l$, $\beta_1, B_1 \dots, \beta_m, B_m$ with

$$1 + \sum_{n=1}^{m} B_n - \sum_{n=1}^{l} A_n \geq 0, \qquad (l \leq m+1; \quad l, m \in N_0 = N \cup \{0\})$$

and for suitably bounded values of $|z|$, given by

$$\mathcal{W}[\alpha_1]f(z) = z + \sum_{n=2}^{\infty} \sigma_n \; a_n z^n, \qquad (z \in \mathcal{U}) \tag{1.10}$$

where

$$\sigma_n = \frac{\Theta \, \Gamma(\alpha_1 + A_1(n-1)) \ldots \Gamma(\alpha_l + A_l(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1)) \ldots \Gamma(\beta_m + B_m(n-1))} \tag{1.11}$$

and Θ is given by $\Theta = \prod\limits_{t=0}^{m} \Gamma(\beta_t) \Big(\prod\limits_{t=0}^{l} \Gamma(\alpha_t) \Big)^{-1}$.

The operator $\mathcal{W}[\alpha_1]f(z)$ is called Dziok-Raina operator [5]. It is easy to verify from (1.10) that

$$zA_1(\mathcal{W}[\alpha_1]f(z))' = \alpha_1 \mathcal{W}[\alpha_1 + 1]f(z) - (\alpha_1 - A_1)\mathcal{W}[\alpha_1]f(z). \tag{1.12}$$

Remark 1. For $A_i = B_j = 1$, $(i = 1, 2, \ldots, l;\ j = 1, 2, 3, \ldots, m)$ the Dziok-Raina operator $\mathcal{W}[\alpha_1]f(z)$ yields the Dziok-Srivastava operator $\mathcal{H}_m^l[\alpha_1]$ [4] as given in (1.7).

Note that if $l = 2$ and $m = 1$ with $\alpha_1 = 1$; $\alpha_2 = 1$; $\beta_1 = 1$ then $\mathcal{W}[\alpha_1]f(z) = f(z)$. By using the Gaussian hypergeometric function given by (1.7), Y. E. Hohlov [7] introduced a generalized convolution operator $H_{a,b,c}$ as

$$H_{a,b,c}f(z) = z_2F_1(a, b, c; z) * f(z),$$

contains as special cases most of the known linear integral or differential operators. Further, the suitable choices of l, m the operator $\mathcal{H}_m^l[\alpha_1]$ in turn it includes various operators as remarked below:

Remark 2. For $f \in \mathcal{A}$,

$$\mathcal{H}_1^2(a, 1; c)f(z) = \mathcal{L}(a, c)f(z) = \left(z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \right) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n$$

was considered by B. C. Carlson and D. B. Shaffer [3].

Remark 3. For $f \in \mathcal{A}$,

$$\mathcal{H}_1^2(\delta + 1, 1; 1)f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = \mathcal{D}^{\delta} f(z), \qquad (\delta > -1)$$

and was given by $\mathcal{D}^{\delta} f(z) = z + \sum\limits_{n=2}^{\infty} \binom{\delta+n-1}{n-1} a_n z^n$, called Ruscheweyh derivative operator [16].

Remark 4. For $f \in \mathcal{A}$, and

$$\mathcal{H}_1^2(c + 1, 1; c + 2)f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) \mathrm{d}t = \mathcal{J}_c f(z)$$

where $c > -1$. The operator $\mathcal{J}_c$ was introduced by S. D. Bernardi [2]. In particular, the operator $\mathcal{J}_1$ was studied earlier by R. J. Libera [8] and A. E. Livingston [10].

Remark 5. For

$$f \in \mathcal{A}, \mathcal{H}_1^2(2, 1; 2 - \lambda)f(z) = \Gamma(2 - \lambda) z^{\lambda} \mathcal{D}_z^{\lambda} f(z) = \Omega^{\lambda} f(z), \qquad \lambda \neq 2, 3, 4, \ldots$$

called Owa-Srivastava operator [18] and Ω^{λ} is also called Srivastava-Owa fractional derivative operator, where $\mathcal{D}_z^{\lambda} f(z)$ denotes the fractional derivative of $f(z)$ of order λ, studied by S. Owa [15].

Denote by $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ the class of starlike and convex functions of complex order γ ($\gamma \in \mathbb{C} \setminus \{0\}$, satisfying the following conditions

$$\frac{f(z)}{z} \neq 0 \qquad \text{and} \qquad \Re\Big(1 + \frac{1}{\gamma}\Big[\frac{zf'(z))}{f(z)} - 1\Big]\Big) > 0$$

and

$$f'(z) \neq 0 \qquad \text{and} \qquad \Re\Big(1 + \frac{1}{\gamma}\Big[\frac{zf''(z))}{f'(z)}\Big]\Big) > 0, \qquad (z \in \mathcal{U}),$$

respectively. Further,

$$\mathcal{S}^*((1-\alpha)\cos\lambda\ \mathrm{e}^{-\mathrm{i}\lambda}) = S^*(\alpha, \lambda) \qquad |\lambda| < \frac{\pi}{2}; \ \ 0 \leq \alpha \leq 1$$

and

$$\mathcal{S}^*(\cos\lambda\ \mathrm{e}^{-\mathrm{i}\lambda}) = S^*(\lambda), \qquad |\lambda| < \frac{\pi}{2}; \ \ 0 \leq \alpha \leq 1$$

where denotes $S^*(\alpha, \lambda)$ the class of λ-Spiral-like function of order α investigated by R. J. Libera [9] and $S^*(\lambda)$ Spiral-like functions introduced by L. Spacek [17] (see [19]). Finally,due to O. Alitintas et al [1], we let $\mathcal{R}(\mu, \gamma)$ denote the class of functions $h(z)$ of the form

$$h(z) = 1 - \sum_{n=1}^{\infty} c_n z^n, \qquad (c_n \geq 0; \quad z \in \mathcal{U}) \tag{1.13}$$

which are analytic in $\mathcal{U}$ and satisfy the inequality

$$|h(z) + \mu z h^{'}(z) - 1| < |\gamma|, \qquad (\gamma \in \mathbb{C} \setminus \{0\}; \ \ \Re(\mu) \geq 0).$$

A mojorization problem for the class S^* have been investigated by T. H. MacGregor [11], and O. Altintas et al. [1]. Recently, S. P. Goyal and P. Goswami [6] extended these results for the fractional derivative operator and a multiplier transformation, respectively. In the present paper we investigate a majorization problem for the class $S^*(\gamma)$ associated with the generalized linear operator called Dziok-Raina operator $\mathcal{W}[\alpha_1]f(z)$.

Definition 1.1. A function $f(z) \in A$ is said to in the class $S^l_m(\gamma)$ of univalent function of complex order $\gamma(\gamma \in \mathbb{C} \setminus \{0\})$ in $\mathcal{U}$ if and only if

$$\frac{\mathcal{W}[\alpha_1]f(z)}{z} \neq 0 \qquad \text{and} \qquad \Re\Big(1 + \frac{1}{\gamma}\Big[\frac{z(\mathcal{W}[\alpha_1]f(z))'}{\mathcal{W}[\alpha_1]f(z)} - 1\Big]\Big) > 0 \tag{1.14}$$

where $\mathcal{W}[\alpha_1]f(z)$ is given by (1.10) and $z \in \mathcal{U}$.

Remark 6. Putting $\gamma = (1-\alpha)\cos\lambda\ \mathrm{e}^{-\mathrm{i}\lambda}$, $|\lambda| < \frac{\pi}{2}$; $0 \leq \alpha < 1$ the class

$$\mathcal{S}^l_m(\gamma) = \mathcal{S}^l_m((1-\alpha)\cos\lambda\ \mathrm{e}^{-\mathrm{i}\lambda}) \equiv \mathcal{S}^l_m(\alpha, \lambda)$$

called the generalized class of λ-spiral-like functions of order α $(0 \leq \alpha < 1)$ and $z \in \mathcal{U}$.

2. A majorization problem for the classes $S^l_m(\gamma)$

Theorem 2.1. *Let the function $f(z) \in \mathcal{A}$ and $g(z) \in S^l_m(\gamma)$ if $\mathcal{W}[\alpha_1]f(z)$ is majorized by $\mathcal{W}[\alpha_1]g(z)$ in $\mathcal{U}$ then*

$$|(\mathcal{W}[\alpha_1]f(z))'| \leq |(\mathcal{W}[\alpha_1]g(z))'|, \qquad |z| \leq r_1, \tag{2.1}$$

where r_1 is given by

$$r_1(\alpha_1,\gamma)=\frac{L-\sqrt{L^2-4\alpha_1|2\gamma A_1-\alpha_1|}}{2|2\gamma A_1-\alpha_1|} \tag{2.2}$$

and

$$L=2A_1+\alpha_1+|2\gamma A_1-\alpha_1|$$

is the smallest root of the equation

$$\rho|2\gamma A_1-\alpha_1|r^3+(A_1-A_1\rho^2-\rho\alpha_1)r^2+[A_1-A_1\rho^2-|2\gamma A_1-\alpha_1|\rho]r+\rho\alpha_1=0 \tag{2.3}$$

and $|z|=r$, $0\le\rho\le 1$.

Proof. Since $\mathcal{W}[\alpha_1]g(z)\in S_m^l(\gamma)$, we readily obtain from (1.14)that, if

$$h(z)=1+\frac{1}{\gamma}\Big(\frac{z(\mathcal{W}[\alpha_1]g(z))'}{\mathcal{W}[\alpha_1]g(z)}-1\Big) \tag{2.4}$$

then,

$$\Re(h(z))>0,\qquad (z\in\mathcal{U}) \tag{2.5}$$

and

$$h(z)=\frac{1+w(z)}{1-w(z)}, \tag{2.6}$$

where w denotes the well known class of bounded analytic functions in $\mathcal{U}$ and

$$w(0)=0\qquad\text{and}\qquad |w(z)|\le|z|\quad (z\in\mathcal{U}). \tag{2.7}$$

From (2.4), (2.6) and making use of (2.7), we get

$$\frac{z(\mathcal{W}[\alpha_1]g(z))'}{\mathcal{W}[\alpha_1]g(z)}\le\frac{1+(2\gamma-1)|z|}{1-|z|}. \tag{2.8}$$

Hence

$$|\mathcal{W}[\alpha_1]g(z)|\le\frac{\alpha_1(1+|z|)|z|}{\alpha_1-|(2\gamma A_1-\alpha_1)|\,|z|}|(\mathcal{W}[(\alpha_1+1)]g(z))'|. \tag{2.9}$$

Since $\mathcal{W}[\alpha_1]f(z)$ is majorized by $\mathcal{W}[\alpha_1]g(z)$ in $\mathcal{U}$ from (1.3), we have

$$z((\mathcal{W}[\alpha_1]f(z))'=z\phi'(z)(\mathcal{W}[\alpha_1]g(z))+z\phi(z)(\mathcal{W}[\alpha_1]g'(z)). \tag{2.10}$$

Noting that the Schwarz function $\phi(z)$ satisfies

$$|\phi^{'}(z)|\le\frac{1-|\phi(z)|^2}{1-|z|^2} \tag{2.11}$$

and using (1.12), (2.9) and (2.11) in (2.10) we have

$$|(\mathcal{W}[\alpha_1]f(z))'|\le\left(|\phi(z)|+\frac{(1-|\phi(z)|^2)}{(1-|z|)}\cdot\frac{A_1|z|}{\alpha_1-|(2\gamma A_1-\alpha_1)|\,|z|}\right)|(\mathcal{W}[\alpha_1]g(z))'|$$

which upon setting

$$|z|=r\qquad\text{and}\qquad|\phi(z)|=\rho\quad(0\le\rho\le1)$$

leads us to the inequality

$$|(\mathcal{W}[\alpha_1]f(z))'|\le\frac{\psi(\rho)}{(1-r)(\alpha_1-|(2\gamma A_1-\alpha_1)|r)}|(\mathcal{W}[\alpha_1]g(z))'|, \tag{2.12}$$

where

$$\psi(\rho)=-A_1r\rho^2+(1-r)(\alpha_1-|(2\gamma A_1-\alpha_1|r)\rho+A_1r$$

takes its maximum value at $\rho=1$. Furthermore, if $0\le\sigma\le r_1$, the function $\varphi(\rho)$ defined by

$$\varphi(\rho)=-A_1\sigma\rho^2+(1-\sigma)(\alpha_1-|(2\gamma A_1-\alpha_1|\sigma)\rho+A_1\sigma$$

is an increasing function on $(0 \leq \rho \leq 1)$ so that

$$\varphi(\rho) \leq \varphi(1) = (1-\sigma)(\alpha_1 - |2\gamma A_1|\sigma), \qquad 0 \leq \rho \leq 1, \;\; 0 \leq \sigma \leq r_1. \tag{2.13}$$

Hence, by setting $\rho = 1$ in we conclude that Theorem 2.1 holds true for for $|z| \leq r_1(\alpha_1, \gamma)$ is given by (2.2). This completes the proof of Theorem 2.1. □

By taking $A_1 = 1$ and $\alpha_1 = 1$ and $\gamma = (1-\alpha)\cos\lambda\ \mathrm{e}^{-\mathrm{i}\lambda}$, $|\lambda| < \frac{\pi}{2}$; $0 \leq \alpha < 1$ in Theorem 2.1 we state the following corollary without proof.

Corollary 2.1.1. *Let the function $f \in \mathcal{A}$ and $g \in S_m^l(\alpha, \lambda)$ $(|\lambda| < \frac{\pi}{2},\ 0 \leq \alpha < 1)$, if $\mathcal{W}[\alpha_1]f(z)$ is majorized by $\mathcal{W}[\alpha_1]g(z)$ in $\mathcal{U}$, then*

$$|(\mathcal{W}[\alpha_1]f(z))'| \leq |(\mathcal{W}[\alpha_1]g(z))'|, \qquad |z| \leq r_2, \tag{2.14}$$

where $r_2 = r_2(\delta, \lambda)$ is given by

$$r_2 = \frac{\delta - \sqrt{\delta^2 - 4|2(1-\alpha)\cos\lambda\ \mathrm{e}^{-\mathrm{i}\lambda} - 1|}}{2|2(1-\alpha)\cos\lambda\ \mathrm{e}^{-\mathrm{i}\lambda} - 1|} \tag{2.15}$$

and

$$\delta = |2(1-\alpha)\cos\lambda \mathrm{e}^{-\mathrm{i}\lambda} - 1| + 3.$$

The proof of our next result is essentially based upon the following lemma.

Lemma 2.1. ([1]) *If $f \in C(\gamma)$, then $f \in S(\frac{1}{2}\gamma)$, that is $C(\gamma) \subset S(\frac{1}{2}\gamma)$ $(\gamma \in C \setminus \{0\})$.*

Theorem 2.2. *Let the function $f \in \mathcal{A}$ and $g \in S_m^l(\gamma)$ if $\mathcal{W}[\alpha_1]f(z)$ is majorized by $\mathcal{W}[\alpha_1]g(z)$ in $\mathcal{U}$ then*

$$|(\mathcal{W}[\alpha_1 + 1]f(z))| \leq |(\mathcal{W}[\alpha_1 + 1]g(z))|, \qquad |z| \leq r_3, \tag{2.16}$$

where r_3 is given by

$$r_3 = \frac{T - \sqrt{T^2 - 4\alpha_1|4\gamma A_1 - \alpha_1|}}{2|4\gamma A_1 - \alpha_1|}$$

and

$$T = 2A_1 + \alpha_1 + |4\gamma A_1 - \alpha_1|$$

is the smallest root of the equation

$$\rho|4\gamma A_1 - \alpha_1|r^3 + (A_1 - A_1\rho^2 - \rho\alpha_1)r^2 + [A_1 - A_1\rho^2 - |4\gamma A_1 - \alpha_1|\rho]r + \rho\alpha_1 = 0 \tag{2.17}$$

$|z| = r$, $0 \leq \rho \leq 1$.

3. A majorization problem for the class $\mathcal{R}_m^l(\mu, \gamma)$

We recall the following lemmas, which will be required in our investigation of the majorization problem for the class $\mathcal{R}_m^l(\mu, \gamma)$.

Lemma 3.1. ([1]) *If the function $h(z)$ defined by (1.13) is in the class $\mathcal{R}(\mu, \gamma)$ then*

$$\sum_{k=1}^{\infty} a_n \leq \frac{|\gamma|}{1 + \Re(\mu)}. \tag{3.1}$$

Lemma 3.2. ([1]) *If the function $h(z)$ defined by is in the class $\mathcal{R}(\mu, \gamma)$ then*

$$1 - \frac{|\gamma|}{1 + \Re(\mu)}|z| \leq |h(z)| \leq 1 + \frac{|\gamma|}{1 + \Re(\mu)}|z|, \qquad z \in \mathcal{U}. \tag{3.2}$$

Theorem 3.1. *Let the function f and g be analytic in $\mathcal{U}$ and suppose that the function g is so normalized that it also satisfies the following inclusion property:*

$$\frac{z(\mathcal{W}[\alpha_1]g(z))'}{\mathcal{W}[\alpha_1]g(z)} \in \mathcal{R}_m^l(\mu,\gamma).$$

If $\mathcal{W}[\alpha_1]f(z)$ is majorized by $\mathcal{W}[\alpha_1]g(z)$ in $\mathcal{U}$ then

$$|(\mathcal{W}[\alpha_1+1]f(z))| \le |(\mathcal{W}[\alpha_1+1]g(z))|, \qquad |z| \le r_4, \tag{3.3}$$

where r_4 is given by $r_4 = r_4(\mu,\gamma)$ is the root of the cubic equation

$$|\gamma|r^3 - (1+\Re(\mu)r^2) - [2+|\gamma|+2\Re(\mu)]r + 1 + \Re(\mu) = 0 \tag{3.4}$$

which lies in the open interval $(0,1)$.

Proof. For an appropriately normalized analytic function $g(z)$ satisfying the inclusion property we find from the assertion of Lemma 3.2 that

$$\left|\frac{z(\mathcal{W}[\alpha_1]g(z))'}{\mathcal{W}[\alpha_1]g(z)}\right| \ge 1 - \frac{|\gamma|}{1+\Re(\mu)}r \qquad (|z|=r;\ \ 0<r<1) \tag{3.5}$$

or, equivalently, that

$$|\mathcal{W}[\alpha_1]g(z)| \le \frac{(1+\Re(\mu))r}{1+\Re(\mu)-|\gamma|r}|(\mathcal{W}[\alpha_1]g(z))'| \qquad (|z|=r;\ \ 0<r<1). \tag{3.6}$$

Since

$$\mathcal{W}[\alpha_1]f(z) \ll \mathcal{W}[\alpha_1]g(z)$$

in $\mathcal{U}$, there exists an analytic function w such that

$$\mathcal{W}[\alpha_1]f(z) = w(z)\mathcal{W}[\alpha_1]g(z) \qquad \text{and} \qquad |w(z)|<1.$$

Thus in view of (3.6) and just as in the proof of Theorem 2.1, we have

$$|w(z)| \le \frac{1-|w(z)|^2}{1-|z|^2}, \qquad (z\in\mathcal{U})$$

and

$$\begin{aligned}|\mathcal{W}[\alpha_1+1]f(z)| &\le \Big(|w(z)| + \frac{1-|w(z)|^2}{1-r^2}\cdot\frac{\{1+\Re(\mu)\}r}{1+\Re(\mu)-|\gamma|r}\Big)|\mathcal{W}[\alpha_1+1]g(z)|\\ &= \Big(\frac{\theta(\rho)}{(1-r^2)\{1+\Re(\mu)-|\gamma|r\}}\Big)|\mathcal{W}[\alpha_1+1]g(z)|,\end{aligned} \tag{3.7}$$

where $z\in\mathcal{U}$; $0<r<1$, we have set $|w(z)|=\rho$ and the function $\theta(\rho)$ defined by

$$\theta(\rho) = \{1+\Re(\mu)\} + (1-r^2)\{1+\Re(\mu)-|\gamma|r\}\rho - \{1+\Re(\mu)\}\rho^2, \qquad (0\le\rho\le1)$$

takes on its maximum value at $\rho=1$ with $r=r_4(\mu,\gamma)$ given by (3.4). Moreover, if $0\le\eta\le r_4(\mu,\gamma)$ and $r_4(\mu,\gamma)$ is the root of the cubic equation (3.4) such that $0<r_4(\mu,\gamma)<1$ then the function $\vartheta(\rho)$ defined by

$$\vartheta(\rho) = \{1+\Re(\mu)\}\eta + (1-\eta^2)\{1+\Re(\mu)-|\gamma|\eta\}\rho - \{1+\Re(\mu)\}\eta\rho^2, \qquad (0\le\rho\le1)$$

is seen to be increasing function on the interval $0\le\rho\le1$, so that

$$\vartheta(\rho) = \vartheta(1) = (1-\eta^2)\{1+\Re(\mu)-|\gamma|\eta\}, \qquad (0\le\eta\le r_4(\mu,\gamma)).$$

Consequently, upon setting $\rho=1$ in (3.7), we complete the proof of Theorem 3.1. □

Concluding remarks. Further specializing the parameters l, m one can define the various other interesting subclasses of $S^l_m(\gamma)$ and $\mathcal{R}^l_m(\mu,\gamma)$involving the differential operators as stated in Remarks 1 to 5, and results (as in Theorems 2.1 and 2.2) and the corresponding corollaries as mentioned above can be derived easily. The details involved may be left as an exercise for the interested reader.

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