Matrix representation – Hajos stable graphs

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Matrix representation – Hajos stable graphs

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Abstract. Binary operations are in general more difficult to implement when compared with unary operations. Normally any operation on a graph involves vertices or edges or both together. In general two graphs are involved in binary operation. Hajos construction is one such operation which involves vertices and edges with three operations. So two graphs with three operations on vertices and edges increases the difficulty in implanting it. Computation time increases when this operation is to be verified for all possible edge pairs between the two graphs. A matrix representation for this purpose would enable easy computation. In this paper we have provided a matrix method of verifying if two graphs are Hajos stable.

1. Introduction

The eigenvalues of L (G) are called the Laplacian eigen values and denoted by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$. In particular λ_1 is called Laplacian spectral radius of G. In 2006, Lihua Feng et al. have studied the Laplacian spectral radius of trees on n vertices with the domination number γ, where $n = k\gamma$, $k \geq 2$ is an integer. Also they determined the extremal trees that attained the minimal Laplacian spectral radius when $\gamma = 2, 3, 4$ [1].

In [2], Dragan Stevanovic et al have characterized the graphs which achieved the maximum value of the spectral radius of the adjacency matrix in the sets of all graphs with a given domination number and graphs with no isolated vertices and a given domination number. Using the properties of graph theory and matrix, VenkataRao has provided a technique for material selection for given any engineering component [3]. Also, in [4] Venkata Rao et al have approached a new method using graph theory and matrix representation for selection of a rapid prototyping process.In [5], Kamal Kumar has found few bounds which are interrelated to the energy,domination number and rank of the incident matrix of G. In [6], Yang et al have provided a new method for recognizing the isomorphism of topological graph using incident matrices.

In 2014, Kamal Kumar has defined set energy and discussed its properties. Also he studied some special class of graphs with its dominating set [7]. In [8], Xianya Geng et al. determined the first, second, third smallest Laplacian permanents of trees in the collection of all trees with n – vertices and with the domination number. Also they characterized the corresponding extremal graphs. The energy, E (G) of a simple graph G is defined to be the sum of the absolute values of the eigen value of G.

In 2013, Rajesh Kanna et al have introduced minimum dominating energy of a graph. They computed minimum dominating energies of some class of graphs and provided the upper and lower bound for minimum dominating energies [9]. Using adjacency matrix, a new technique for generating a minimum weighted spanning tree was provided in [10]. Classification of domination dot stable graphs, γ - stable graphs and graph domination graphs using matrices were discussed in [11].

In graph theory, a branch of mathematics, the Hajos construction is an operation on graphs named after GyorgyHajos in 1961. That may be used to construct any critical graph or any graph whose chromatic number is at least some given threshold [12].

In 1979, Catlin has provided a conjecture for Hajos graph coloring. Also he provided a counterexample to Hajos conjecture [13]. In [14], Brown et al. have proved that the Hajos construction of two amenable k – critical graphs need be amenable for any k ≥5. In 2001, an analogue of Hajos theorem for the circular chromatic number was proved by Xuding Zhu [16]. In this paper, we obtain a method for finding G_1 and G_2 are Hajos stable graph.

2. Materials and methods

We consider only simple connected undirected graphs G = (V, E). The open neighborhood of vertex v ∈V (G) is denoted by N (v) = { u ∈ V (G) / (u v) ∈ E (G) } while its closed neighborhood is the set N [v] = N (v) ∪ { v }. We indicate that u is adjacent to v by writing u ⊥ v.

A set of vertices D, in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set. The cardinality of any minimum dominating set for G is called the domination number of G and it is denoted by γ (G). γ - set denotes a dominating set for G with minimum cardinality.

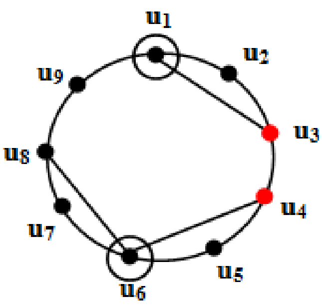
A vertex v is said to be good if there is a γ - set of G containing v. If there is no γ - set of G containing v, then v is said to be bad vertex. A vertex v is said to be a, up vertex if γ (G – v) >γ (G), down vertex if γ (G – v) <γ (G), level vertex if γ (G – v) = γ (G). A vertex v is said to be selfish in the γ - set D, if v is needed only to dominate itself. A vertex v ∈V – D is said to be p – dominated, if it is dominated by atleast p – vertices in D. A vertex v is said to be the private neighborhood of uin D which is defined as pn [u, D] = N (u) – N (D − { u }). For details of on domination we refer to [16].

Hajos construction

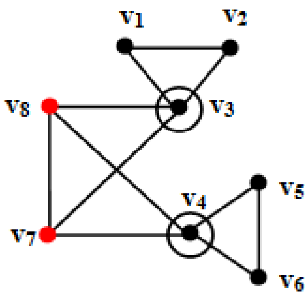
Let G_1 and G_2 be two graphs, (u_1 v_1) be an edge of G_1, and (u_2 v_2) be an edge of G_2. Then the Hajos construction forms a new graph H that combines the two graphs by merging vertices u_1 and u_2 into a single vertex u_{12}, removing the two edges (u_1 v_1) and (u_2 v_2), and adding a new edge (v_1 v_2) [12].

Hajos stable graphs

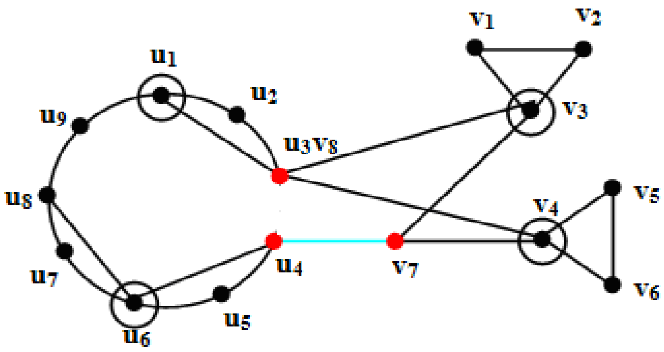
Let G_1 and G_2 be any two graphs. Let E (G_1) = { e_{11}, e_{12}, …, e_{1p} } and E (G_2) = { e_{21}, e_{22}, …, e_{2q} }. Let M = E (G_1) × E (G_2) = { (e_{1i} e_{2j}) / e_{1i}∈ E (G_1), e_{2j}∈ E (G_2) }, that is M is the cartesian product between sets E (G_1) and E (G_2). Let | M | = k. Let H_1, H_2, …, H_{4k} be the Hajos graphs generated by applying Hajos construction 4k times. If γ (H_i) = γ (G_1) + γ (G_2), for all i = 1, 2, …, 4k, then G_1 and G_2 are said to be Hajos stable graphs [17].



G_1



G_2



H

Figure 1.

In Figure 1, H is the Hajos graph obtained from G_1 and G_2 using the edge pairs (u_3 u_4) and (v_7 v_8), γ (H) = γ (G_1) + γ (G_2) = 4. Similarly γ (H_i) = γ (G_1) + γ (G_2), for all i = 1, 2, …, 4k, implies G_1 and G_2 are Hajos stable graphs.

The results R1 and R2 were proved in [17].

R1.

Let G_1 and G_2 be any two graphs. Let D_1 and D_2 be γ - sets for G_1 and G_2 respectively. Let H be the Hajos graph. Then γ (H) <γ (G_1) + γ (G_2) if and only if either

1. there is some (u_i v_i) $\in$ D_i such that $u_i \perp v_i$, i = 1, 2, or
2. there is a selfish vertex in G_i, i = 1, 2, or
3. both G_1 and G_2 have 2 – dominated vertices simultaneously together, or
4. if pn [u_i, D_i] = v_i in G_i, then G_j has 2 – dominated vertices, where i, j = 1, 2, i $\neq$ j.

R2

Let G_1 and G_2 be any two graphs. Let D_1 and D_2 be γ - sets for G_1 and G_2 respectively. Let H be the Hajos graph. Then γ (H) >γ (G_1) + γ (G_2) if and only if either

1. if u_i is an up vertex, u_j, v_i, v_j are bad vertices, then
   a. v_i is not a 2 – dominated vertex with respect to every D_i in G_i and
   b. v_j is not a good vertex in C_j – N [u_j] for all γ - sets D_3 for C_j – N [u_j] such that | D_3 | = | D_j |,

   where i, j = 1, 2, i $\neq$ j, or
2. if u_i are bad vertices, v_i are up vertices, then $u_i \in$pn [v_i, D_i] for all possible γ - sets in G_i, i = 1, 2.

R3.

If R1 and R2 are not satisfied, then G_1 and G_2 are said to be Hajos stable graphs.

R4.

If v is level vertex such that v is contained in every possible γ – sets for G, then no γ – set of G – v contains any $v_i \in$ N (v), i = 1, 2, …, k.

Proof

Let v $\in$V (G) be a level vertex in G and D be a γ – set for G. Assume that v belongs to all possible γ – sets in G. Since v is a level vertex, we know that γ (G) = γ (G – v). Let D' be a γ – set for G – v. Suppose that, there is some u $\in$ D', where u $\in$ N (v), implies D' is γ – set for both G – v and G, (since u dominates v in G and v $\notin$ D), implies as v belongs to all possible γ – sets.

3. Results and Discussions

An adjacency matrix of a graph G with n vertices that are assumed to be ordered from v_1 to v_n is defined by,

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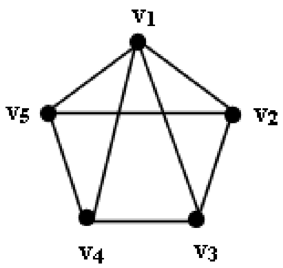
$$A = [a_{ij}]_{n\times n} = \begin{cases} 1, \text{ if there exist an edge between } v_i \text{ and } v_j \\ 0, \text{ otherwise.} \end{cases}$$

Let G be a graph with | V (G) | = n. Let N denote a n ×n matrix [12], where

$$N = [n_{ij}]_{n\times n} = \begin{cases} 1 & \text{if } i = j \\ a_{ij} & \text{the } (i, j)^{th} \text{ entry of the adjacency matrix.} \end{cases}$$

Let $x = \langle x(v_1), x(v_2), \ldots, x(v_n) \rangle^T$ be a { 0, 1 } vector. If x is any dominating set, then Nxis greater than or equal to one [12].

Example



G

Figure 2.

N **x** **Nx**

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

that is, { v_1, v_2 }is a dominating set for G [18].

X and N [v_i] denotes a dominating set and the number of non zero entries in any row of matrix N respectively.Nx is a column matrix. Every entry in Nx represents, the number of vertices dominating any vertex v_i. A vertex $v_i$in V – D is said to be a private neighborhood with respect to D, if row v_i entry in Nx is one. Similarly a vertex $v_i$in V – D is said to k – dominated with respect to x, if row v_i entry in Nxis greater than or equal to two.

We use the following notation for further discussion.

Notation

1. Consider a graph G with n vertices $v_1, v_2, \ldots, v_n$. Let γ (G) = k. Consider all possible subsets with k vertices. Label them as $S_1, S_2, \ldots, S_p$, where p = nC_k. Let X = { $x_1, x_2, \ldots, x_p$ } be a set of { 0, 1 } vectors defined by $x_i = \langle x_i(v_1), x_i(v_2), \ldots, x_i(v_n) \rangle^T$,where
$$x_i(v_i) = \begin{cases} 1 & \text{if } v_i \in S_i \\ 0 & \text{otherwise.} \end{cases}$$
2. Nx_i is a column matrix. Let us denote this as vector, $nx_i = \langle nx_i(v_1), nx_i(v_2), \ldots, nx_i(v_n) \rangle^T$.
3. Let V = [v_{ij}] = [$x_1, x_2, \ldots, x_p$] be a vector of a matrix. Each x_i, i = 1, 2, …, p in V, denotes a vector defined in Notation – 1. Find NV matrix. The matrix NV is a n × p matrix, every column in NV denotes vector nx_i, ie., the columns represents vector $nx_1, nx_2, \ldots, nx_p$.
4. Let S be the set of all vectors ∍Nx_i≥ 1, where S ⊆ X , ie., NS ≥ 1. Let q denotes the number of elements in S, p≥ q.

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A. Determination of 2 – dominated vertices

Consider matrices S and NS. A row in these two matrices represents vertices. A zero entry in vector x_i represents a vertex in $V_1 - D_1$. If any vertex is 2 – dominated, then we know that atleast two vertices in D_1 are adjacent to it. So, in the product matrix Nx_i, the corresponding entry is atleast two. With this observation we compare a zero entry in $x_i = \langle x_i(v_1), x_i(v_2), \ldots, x_i(v_n) \rangle^T$ and its corresponding entry in $nx_i = \langle nx_i(v_1), nx_i(v_2), \ldots, nx_i(v_n) \rangle^T$. If for any $x_i(v_i) = 0$, $nx_i(v_i) \geq 2$, then v_i is 2 – dominated. Let $X_1 = \{ x_{11}, x_{12}, \ldots, x_{1k_1} \} \subseteq V(G_1)$ be the set of 2 – dominated vertices for G_1, $1 \leq k_1 \leq n$.

A similar discussion is true for graph G_2 also. Let $Y_1 = \{ y_{11}, y_{12}, \ldots, y_{1k_2} \} \subseteq V(G_2)$ be the set of 2 – dominated vertices for G_2, $1 \leq k_2 \leq n$.

B. Determination of up vertices

We know that an up vertex is contained in every possible γ – sets for G [16]. This is not true only for up vertices, a vertex included in every γ – set may be a level vertex also.

Example

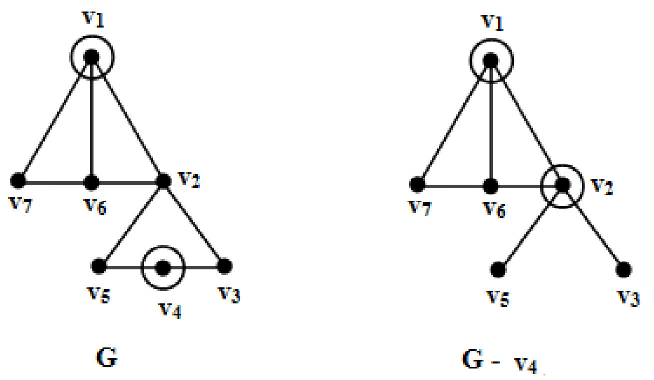

Figure 3.

In Figure 3, $\{ v_1, v_4 \}$, $\{ v_6, v_4 \}$ are the only possible γ – sets for G. $\gamma(G) = \gamma(G - v_4) = 2$, implies v_4 is a level vertex and it is included in every γ – set. Consider matrix S. If there is a row in S, with all non zero entries, then the corresponding vertex v_i may or may not be an up vertex. So we need to verify if vertex v_i is a level vertex.

If v is a level vertex, we know that $\gamma(G_1) = \gamma(G_1 - v)$. Also we need to note that, this specific level vertex is contained in every γ – set. By R4, we know that no γ – set of $G_1 - v$ contains any vertex from N (v). With this note, we consider matrix NV. Consider row v_i of matrix NV. If there is atleast one column such that row v_i entry is zero and the remaining entries are non – zero, then this means that the corresponding column vector in matrix V dominates $V - \{ v_i \}$. Hence vertex v_i is a level vertex. Else v_i is an up vertex with respect to G_i. Let $X_2 = \{ x_{21}, x_{22}, \ldots, x_{2k_3} \} \subseteq V(G_1)$ be the set of up vertices for G_1.

A similar discussion is true for graph G_2 also. Let $Y_2 = \{ y_{21}, x_{22}, \ldots, x_{2k_4} \} \subseteq V(G_2)$ be the set of up vertices for G_2.

C. Determination of bad edges

Consider matrix S. If a row of S contains all zero entries, means that the corresponding vertex v_i is not contained in any γ – set, implies v_i is a bad vertex. Let $X_3 = \{ x_{31}, x_{32}, \ldots, x_{3k_5} \} \subseteq V(G_1)$ be the set of bad vertices for G_1.

Consider the sub matrix $Y = [\, y_{ij} \,]_{k_5 \times k_5}$, where y_{ij} entry denotes the corresponding a_{ij} entry in matrix A, that is Y is a sub matrix of A with rows and columns representing bad vertices. If atleast one entry in matrix Y is non – zero, then there are two bad vertices adjacent to each other. Let

$X_4 = \{ e_{41,} e_{42}, ..., e_{4k_6} \} \subseteq E(G_1)$ be the set of edges such that end vertices are in X_3. Let $e_{4i} = (x_{3a_i}\ x_{3b_i})$.

A similar discussion is true for graph G_2 also. Let $Y_3 = \{ x_{31,} x_{32}, ..., x_{3k_7} \} \subseteq V(G_2)$ be the set of bad vertices for G_2.

$Y_4 = \{ e_{41,} e_{42}, ..., e_{4k_8} \} \subseteq E(G_2)$ be the set of edges such that end vertices are in Y_3. Let $e_{4i} = (y_{3a_i}\ y_{3b_i})$.

D. Determination of u, v ∈ D

Consider matrices S and NS. A row in these two matrices represents vertices. A nonzero entry in vector x_i represent a vertex in D_1. If any vertex in D_1 is adjacent to atleast another vertex in D_1, then the corresponding entry is atleast two. So, in the product matrix Nx_i, the corresponding entry is atleast two. With this observation we compare a nonzero entry in $x_i = \langle x_i(v_1), x_i(v_2), \ldots, x_i(v_n) \rangle^T$ and its corresponding entry in $nx_i = \langle nx_i(v_1), nx_i(v_2), \ldots, nx_i(v_n) \rangle^T$. If for any $x_i(v_i) = 1$, $nx_i(v_i) \geq 2$, then there is some x, y ∈ D_1 such that $x \perp y$. Let $X_5 = \{ x_{51,} x_{52}, ..., x_{5k_9} \} \subseteq V(G_1)$ be the set of vertices in D_1 such that every vertex in X_5 is adjacent to atleast one more vertex in D_1.

A similar discussion is true for graph G_2 also. Let $Y_5 = \{ y_{51,} y_{52}, ..., y_{5k_{10}} \} \subseteq V(G_2)$ be the set of vertices in D_2 such that every vertex in Y_5 is adjacent to atleast one more vertex in D_2.

E. Determination of single private neighbors of D

Let G_1 be a graph such that every vertex in $V_1 - D_1$ are private neighbors. We now redefine vector x_i as follows.

Let $y_i = \langle y_i(v_1), y_i(v_2), \ldots, y_i(v_n) \rangle$ be { 0, 1 } vector such that

$$y_i(v_i) = \begin{cases} 0 & \text{if } v_i \in S_i \\ 1 & \text{otherwise.} \end{cases}$$

With this new definition of y_i, we compare zero entry in y_i and its corresponding entry in ny_i. If for any $y_i(v_i) = 0$, $ny_i(v_i) \geq 2$, then there is atleast two vertices in $V_1 - D_1$ adjacent to v_i.

Create a new matrix S_1, where

$$S_1 = [s_{1ij}]_{n \times q} = \begin{cases} 1 & \text{if } s_{ij}^{th} \text{ entry of matrix S is equal to zero} \\ 0 & \text{if } s_{ij}^{th} \text{ entry of matrix S is equal to one.} \end{cases}$$

In product NS_1, for any zero entry in column y_i, the corresponding entry $ny_i(v_i) = 1$, then the vertex v_i in G_1 dominates only one vertex. Since in G_1 every vertex in $V_1 - D_1$ are private neighbors, implies G_1 has single private neighbor. Let $X_6 = \{ x_{61,} x_{62}, ..., x_{6k_{11}} \} \subseteq V(G_1)$ be the set of single private neighbors for G_1.

A similar discussion is true for graph G_2 also. Let $Y_6 = \{ y_{61,} y_{62}, ..., y_{6k_{12}} \} \subseteq V(G_2)$ be the set of single private neighbors for G_2.

Step 1

If $X_5 \neq \phi$, then every vertex in X_5 is adjacent to one more vertex in D_1 or
If $Y_5 \neq \phi$, then every vertex in Y_5 is adjacent to one more vertex in D_2.
This implies, G_1 and G_2 are not Hajos stable graphs. Else continue to step 3.

Step 2

If u is a selfish vertex, then we know that, there is one γ – set such that u, x ∈ D_1, $u \perp x$ or v, y ∈ D_2, $v \perp y$. So when condition 1 is verified, condition 2 of R1 is also verified. This implies, G_1 and G_2 are not Hajos stable graphs. Else continue to step 3.

Step 3

If X_1, $Y_1 \neq \phi$, then both G_1 and G_2 have 2 – dominated vertices simultaneously together. This implies, G_1 and G_2 are not Hajos stable graphs. Else continue to step 4.

Step 4

In this step we use vector Notation in E. From Step 3, we know that both G_1 and G_2 do not have 2 – dominated vertices simultaneously, that is either $X_1 = \phi$ or $Y_1 = \phi$. Assume that $X_1 = \phi$. If $X_6 \neq \phi$, $Y_1 \neq \phi$, then G_1 has a single private neighbor and G_2 has 2 – dominated vertices simultaneously together, implies G_1 and G_2 are not Hajos stable graphs. Similarly if $X_1 \neq \phi$, $Y_6 \neq \phi$, then G_1 and G_2 are not Hajos stable graphs. Else if

i. $X_2 = Y_2 = \phi$, G_1 and G_2 are Hajos stable graphs.
ii. if either $X_2 \neq \phi$ or $Y_2 \neq \phi$ continue to step 5.

Step 5

If $X_4 = \phi$ or $Y_4 = \phi$, then continue to step 6. Else proceed as follows.

Let $W_{2i} = N(x_{2i}) = \{w_1, w_2, \ldots, w_{ji}\}$, $i = 1, 2, \ldots, k_3$ be the open neighbors of the up vertices in set X_2. Let $R_{2i} = W_{2i} \cap X_1$. If

$$R_{2i} = \begin{cases} W_{2i} & \text{then continue to step 6} \\ \text{else} & \text{continue further.} \end{cases}$$

or

Let $Z_{2i} = N(y_{2i}) = \{z_1, z_2, \ldots, z_{ji}\}$, $i = 1, 2, \ldots, k_4$ be the open neighbors of the up vertices in set Y_2. Let $F_{2i} = Z_{2i} \cap Y_1$. If

$$F_{2i} = \begin{cases} Z_{2i} & \text{then continue to step 6} \\ \text{else} & \text{continue further.} \end{cases}$$

If $R_{2i} \neq W_{2i}$ and $F_{2i} \neq Z_{2i}$, we proceed as follows.

Let $X_7 \subseteq X_4$, be the set of all bad vertices of X_4 such that x_{3a_i} is a good vertex in $A_2 - N[x_{3b_i}]$ and vice – versa. If $|X_4| = |X_7|$, then continue to step 6, else G_1 and G_2 are not Hajos stable graphs.

A similar discussion is true for G_2 also. Let $Y_7 \subseteq Y_4$, be the set of all bad vertices of Y_4 such that y_{3a_i} is a good vertex in $A_1 - N[y_{3b_i}]$ and vice – versa. If $|Y_4| = |Y_7|$, then continue to step 6, else G_1 and G_2 are not Hajos stable graphs.

Step 6

If either $|R_{2i}| = |W_{2i}|$ or $|F_{2i}| = |Z_{2i}|$, then G_1 and G_2 are Hajos stable graphs. Else G_1 and G_2 are not Hajos stable graphs.

Example

Consider the graph G_1 in Fig. 1,

$$A = \begin{bmatrix} 0&1&1&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0&0 \\ 0&0&1&0&1&1&0&0&0 \\ 0&0&0&1&0&1&0&0&0 \\ 0&0&0&1&1&0&1&1&0 \\ 0&0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&1&1&0&1 \\ 1&0&0&0&0&0&0&1&0 \end{bmatrix}; \qquad N = \begin{bmatrix} 1&1&1&0&0&0&0&0&1 \\ 1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0 \\ 0&0&1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1&1&0 \\ 0&0&0&0&0&1&1&1&0 \\ 0&0&0&0&0&1&1&1&1 \\ 1&0&0&0&0&0&0&1&1 \end{bmatrix}$$

$\gamma(G_1) = 2$. consider $\{S_1, S_2, \ldots, S_{36}\} = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_1, v_7\}, \{v_1, v_8\}, \{v_1, v_9\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_6\}, \{v_2, v_7\}, \{v_2, v_8\}, \{v_2, v_9\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_3, v_7\}, \{v_3, v_8\}, \{v_3, v_9\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_4, v_7\}, \{v_4, v_8\}, \{v_4, v_9\}, \{v_5, v_6\}, \{v_5, v_7\}, \{v_5, v_8\}, \{v_5, v_9\}, \{v_6, v_7\}, \{v_6, v_8\}, \{v_6, v_9\}, \{v_7, v_8\}, \{v_7, v_9\}, \{v_8, v_9\}\}$.

$$NV = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&1 \end{bmatrix}$$

$$NV = \begin{bmatrix} 2&2&2&1&1&1&1&1&2&2&1&1&1&1&1&2&1&1&1&1&2&0&0&0&0&1&0&0&0&1&1&0&0&0&1&1 \\ 2&2&2&1&1&1&1&1&1&2&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&2&2&1&1&1&1&1&2&2&1&1&1&1&1&2&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&0&0&0&0&1&1&1&1&0&0&0&2&2&2&1&1&2&2&1&1&1&2&1&1&1&1&1&1&0&0&0 \\ 0&0&0&1&1&0&0&0&0&0&1&1&1&0&0&0&1&1&1&0&0&2&2&1&1&1&2&1&1&1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1&0&0&0&1&1&1&1&1&0&1&1&1&1&0&2&2&2&2&1&2&2&2&1&2&2&1&2&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&0&0&0&1&1&0&0&1&1&1&0&1&1&1&0&2&2&1&2&1&1 \\ 1&0&0&0&1&1&1&1&1&0&0&0&1&1&1&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&2&2&2&2&2&2 \\ 1&1&1&1&1&1&1&2&2&0&0&0&0&0&1&1&0&0&0&1&1&0&0&0&1&1&0&0&1&1&0&1&1&1&1&2 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \; NS = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Consider the graph G_2 in Fig. 1,

$$A = \begin{bmatrix} 0&1&1&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 1&1&0&0&0&0&1&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&1&0&0&0 \\ 0&0&1&1&0&0&0&1 \\ 0&0&1&1&0&0&1&0 \end{bmatrix}; \quad N = \begin{bmatrix} 1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&1&1 \\ 0&0&0&1&1&1&1&1 \\ 0&0&0&1&1&1&0&0 \\ 0&0&0&1&1&1&0&0 \\ 0&0&1&1&0&0&1&1 \\ 0&0&1&1&0&0&1&1 \end{bmatrix}$$

$\gamma (G_2) = 2$. Consider $\{ S_1, S_2, \ldots, S_{28} \} = \{ \{ v_1, v_2 \}, \{ v_1, v_3 \}, \{ v_1, v_4 \}, \{ v_1, v_5 \}, \{ v_1, v_6 \}, \{ v_1, v_7 \}, \{ v_1, v_8 \}, \{ v_2, v_3 \}, \{ v_2, v_4 \}, \{ v_2, v_5 \}, \{ v_2, v_6 \}, \{ v_2, v_7 \}, \{ v_2, v_8 \}, \{ v_3, v_4 \}, \{ v_3, v_5 \}, \{ v_3, v_6 \}, \{ v_3, v_7 \}, \{ v_3, v_8 \}, \{ v_4, v_5 \}, \{ v_4, v_6 \}, \{ v_4, v_7 \}, \{ v_4, v_8 \}, \{ v_5, v_6 \}, \{ v_5, v_7 \}, \{ v_5, v_8 \}, \{ v_6, v_7 \}, \{ v_6, v_8 \}, \{ v_7, v_8 \} \}$.

$$NV = \begin{bmatrix} 1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&1&1 \\ 0&0&0&1&1&1&1&1 \\ 0&0&0&1&1&1&0&0 \\ 0&0&0&1&1&1&0&0 \\ 0&0&1&1&0&0&1&1 \\ 0&0&1&1&0&0&1&1 \end{bmatrix} \begin{bmatrix} 1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0 \\ 0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1 \\ 0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&1 \end{bmatrix}$$

$$NV = \begin{bmatrix} 2&2&0&1&1&1&1&1&1&2&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \\ 2&2&0&0&1&1&1&1&1&2&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \\ 2&2&0&0&1&1&1&2&2&2&1&1&1&2&2&1&1&1&2&2&0&1&1&0&1&1&1&2 \\ 0&0&0&2&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1&1&2&2&2&2&2&2&2&2 \\ 0&0&0&1&1&1&1&0&0&0&1&1&1&0&0&1&1&1&0&0&2&1&1&2&1&1&2&0 \\ 0&0&1&0&1&1&1&0&1&0&1&1&1&0&0&1&1&1&0&0&2&1&1&2&1&1&2&0 \\ 0&1&1&0&1&0&0&1&1&1&1&0&0&1&1&2&1&1&2&2&1&2&2&0&1&1&2&2 \\ 0&1&0&1&1&0&0&0&1&1&1&0&0&1&1&2&1&1&2&2&1&2&2&0&1&1&2&2 \end{bmatrix}$$

$$S = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&1&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}; \quad NS = \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&2&1&1 \\ 1&1&2&1&1 \end{bmatrix}$$

A. Determination of 2 – dominated vertices

For graph G_1

$$NS = \begin{bmatrix} 1&1&1&0&0&0&0&0&1 \\ 1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0 \\ 0&0&1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1&1&0 \\ 0&0&0&0&0&1&1&1&0 \\ 0&0&0&0&0&1&1&1&1 \\ 1&0&0&0&0&0&0&1&1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Consider NS, for all 0 entry in x_5, the entries in the corresponding position of $Nx_5 \leq 1$, implies $X_1 = \phi$.

For graph G_2

$$NS = \begin{bmatrix} 1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&1&1 \\ 0&0&0&1&1&1&1&1 \\ 0&0&0&1&1&1&0&0 \\ 0&0&0&1&1&1&0&0 \\ 0&0&1&1&0&0&1&1 \\ 0&0&1&1&0&0&1&1 \end{bmatrix} \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&1&0 \\ 1&1&1&0&0 \\ 0&0&1&1&1 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} = \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&2&1&1 \\ 1&1&2&1&1 \end{bmatrix}$$

In NS, for all zero entries in x_{14}, the entries in the corresponding position of $Nx_{14} \geq 2$, implies $Y_1 = \{ 7, 8 \}$.

B. Determination of up vertices

For graph G_1, $\gamma (G_1) = \{ 1, 6 \}$. We consider matrix NV. Consider row v_i, i = 1, 6 of matrix NV. There exist no column in NV such that row v_i, i = 1, 6 entry is zero and the remaining entries are non – zero. Hence vertex v_i is an up vertex, i = 1, 6, implies $X_2 = \{ 1, 6 \}$.

Similar discussion for G_2, implies $Y_2 = \phi$.

C. Determination of bad edges

Consider the graph G_1.

$$S = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad Y = \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \\ 7 \\ 8 \\ 9 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

From S and Y, $X_3 = \{ 2, 3, 4, 5, 7, 8, 9 \}$ and $X_4 = \{ (2\ 3), (3\ 4), (4\ 5), (7\ 8), (8\ 9) \}$.

Consider the graph G_2.

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad Y = \begin{matrix} 7 \\ 8 \end{matrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

From S and Y, $Y_3 = \{ 7, 8 \}$ and $Y_4 = \{ (7\ 8) \}$.

D. Determination of u, v ∈ D

Consider the graph G_1.

$$NS = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Consider NS, for all non zero entry in x_5, the entry in the corresponding position of $Nx_5 \leq 1$, implies $X_5 = \phi$.

Consider the graph G_2.

$$NS = \begin{bmatrix} 1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&1&1 \\ 0&0&0&1&1&1&1&1 \\ 0&0&0&1&1&1&0&0 \\ 0&0&0&1&1&1&0&0 \\ 0&0&1&1&0&0&1&1 \\ 0&0&1&1&0&0&1&1 \end{bmatrix} \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&1&0 \\ 1&1&1&0&0 \\ 0&0&1&1&1 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} = \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&2&1&1 \\ 1&1&2&1&1 \end{bmatrix}$$

Consider NS, for all non zero entry in x_i, the entry in the corresponding position of $Nx_i \leq 1$, where i = 5, 11, 16, 17, implies $Y_5 = \phi$.

E. Determination of single private neighbors of D

Consider the graph G_1.

$$NS_1 = \begin{bmatrix} 1&1&1&0&0&0&0&0&1 \\ 1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0 \\ 0&0&1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1&1&0 \\ 0&0&0&0&0&1&1&1&0 \\ 0&0&0&0&0&1&1&1&1 \\ 1&0&0&0&0&0&0&1&1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 2 \\ 1 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

In NS_1, for every zero entry in x_5, the entry in the corresponding position of $Nx_5 \neq 1$, implies $X_6 = \phi$.

Consider the graph G_2.

$$NS = \begin{bmatrix} 1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&1&1 \\ 0&0&0&1&1&1&1&1 \\ 0&0&0&1&1&1&0&0 \\ 0&0&0&1&1&1&0&0 \\ 0&0&1&1&0&0&1&1 \\ 0&0&1&1&0&0&1&1 \end{bmatrix} \begin{bmatrix} 1&1&1&1&0 \\ 1&1&1&0&1 \\ 0&0&0&1&1 \\ 1&1&0&0&0 \\ 1&0&1&1&1 \\ 0&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix} = \begin{bmatrix} 2&2&2&1&2 \\ 2&2&2&2&2 \\ 4&4&4&4&4 \\ 4&4&4&4&4 \\ 2&2&2&2&2 \\ 2&2&2&2&2 \\ 3&3&2&3&3 \\ 3&3&2&3&3 \end{bmatrix}$$

Consider NS, for all non zero entry in x_i, the entry in the corresponding position of $Nx_i \neq 1$, where i = 5, 11, 16, 17, implies $Y_6 = \phi$.

Step 1

$X_5 = \phi$ and $Y_5 = \phi$, implies there is no $(u_i, v_i) \in D_i$ such that $u_i \perp v_i$, i = 1, 2.

Step 2

There is no selfish vertex in G_i, i = 1, 2.

Step 3

$X_1 = \phi$, $Y_1 \neq \phi$, implies both G_1 and G_2 do not have 2 – dominated vertices simultaneously together.

Step 4

$Y_1 \neq \phi$ and $X_6 = \phi$. G_2 has no 2 – dominated vertices and pn $[u_1, D_1] = \{ v_1 \}$ in G_1.

Step 5

$Y_7 = (7 \;\; 8)$, $| Y_4 | = | Y_7 |$. u_1 is an up vertex in G_1, v_j is a bad vertex in G_2 and v_j is a good vertex in $A – N [v_j]$, j = 7, 8 in G_2.

Step 6

$X_2 \neq \phi$, $Y_2 = \phi$. Both G_1 and G_2 do not have up vertices simultaneously together.

Hence G_1 and G_2 are Hajos stable graphs.

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