Research Article

Meromorphic Parabolic Starlike Functions Associated with q-Hypergeometric Series

G. Murugusundaramoorthy and T. Janani

School of Advanced Sciences, VIT University, Vellore 632014, India

Correspondence should be addressed to G. Murugusundaramoorthy; gmsmoorthy@yahoo.com

We introduce a new class of meromorphic parabolic starlike functions with a fixed point defined in the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ involving the q-hypergeometric functions. We obtained coefficient inequalities, growth and distortion inequalities, and closure results for functions $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$. We further established some results concerning convolution and the partial sums.

1. Introduction

Let ξ be a fixed point in the unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{H}(\Delta)$ the class of functions which are regular and

$$\mathcal{A}(\xi) = \left\{ f \in H(\Delta) : f(\xi) = f'(\xi) - 1 = 0 \right\}. \quad (1)$$

Also denote by $\mathcal{S}_\xi = \{f \in \mathcal{A}(\xi) : f \text{ is univalent in } \Delta\}$, the subclass of $\mathcal{A}(\xi)$ consisting of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n (z - \xi)^n \quad (2)$$

which are analytic in Δ. Note that $\mathcal{S}_0 = \mathcal{S}$ is subclasses of $\mathcal{A}$ consisting of univalent functions in Δ. By $\mathcal{S}_w^*(\beta)$ and $\mathcal{K}_w(\beta)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions $\Re\{(z - \xi) f'(z)/f(z)\} > \beta$, and $\Re\{1 + ((z - \xi) f''(z)/f'(z))\} > \beta$, $(z - w) \in \Delta$ for $0 \leq \beta < 1$ introduced and studied by Kanas and Ronning [1]. The class $\mathcal{S}_\xi^*(0)$ is defined by geometric property that the image of any circular arc centered at ξ is starlike with respect to $f(\xi)$ and the corresponding class $\mathcal{K}_\xi^*(0)$ is defined by the property that the image of any circular arc centered at ξ is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [2, 3] for uniformly starlike and convex functions, except that in this case the point ξ is fixed.

In particular, $\mathcal{K} = \mathcal{K}_0(0)$ and $\mathcal{S}_0^* = \mathcal{S}^*(0)$, respectively, are the well-known standard classes of convex and starlike functions.

Let Σ denote the class of meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (3)$$

defined on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Denote by Σ_ξ the subclass of Σ consisting of the functions of the form

$$f(z) = \frac{1}{z - \xi} + \sum_{n=1}^{\infty} a_n (z - \xi)^n, \quad a_n \geq 0; \ z \neq \xi. \quad (4)$$

A function f of the form (4) is in the class of *meromorphic starlike of order* γ $(0 \leq \gamma < 1)$ denoted by $\Sigma_\xi^*(\gamma)$, if

$$-\Re\left(\frac{(z - \xi) f'(z)}{f(z)} \right) > \gamma, \qquad z - \xi \in \Delta := \Delta^* \cup \{0\}, \quad (5)$$

and is in the class of *meromorphic convex of order γ* ($0 \leq \gamma < 1$) denoted by $\Sigma_\xi^K(\gamma)$, if

$$-\Re\left(1 + \frac{(z-\xi)f''(z)}{f'(z)}\right) > \gamma, \quad z - \xi \in \Delta := \Delta^* \cup \{0\}. \tag{6}$$

For functions $f(z)$ given by (4) and $g(z) = (1/(z-\xi)) + \sum_{n=1}^\infty b_n(z-\xi)^n$, ($b_n \geq 0$) we define the Hadamard product or convolution of f and g by

$$(f * g)(z) := \frac{1}{z-\xi} + \sum_{n=1}^\infty a_n b_n(z-\xi)^n. \tag{7}$$

More recently, Purohit and Raina [4] introduced a generalized q-Taylor's formula in fractional q-calculus and derived certain q-generating functions for q-hypergeometric functions. In this work we proceed to derive a generalized differential operator on meromorphic functions in $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ involving these functions and discuss some of their properties.

For complex parameters $a_1, \ldots, a_l$ and $b_1, \ldots, b_m$ ($b_j \neq 0$, $-1, \ldots$; $j = 1, 2, \ldots, m$) the q-hypergeometric function $_l\Psi_m(z)$ is defined by

$$_l\Psi_m(a_1, \ldots a_l; b_1, \ldots, b_m; q, z)$$
$$:= \sum_{n=0}^\infty \frac{(a_1, q)_n \cdots (a_l, q)_n}{(q, q)_n (b_1, q)_n \cdots (b_m, q)_n} \tag{8}$$
$$\times \left[(-1)^n q^{\binom{n}{2}}\right]^{1+m-l} z^n,$$

with $\binom{n}{2} = n(n-1)/2$ where $q \neq 0$ when $l > m + 1$ ($l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $z \in \mathbb{U}$).

The q-shifted factorial is defined for $a, q \in \mathbb{C}$ as a product of n factors by

$$(a; q)_n = \begin{cases} 1 & n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & n \in \mathbb{N}, \end{cases} \tag{9}$$

and in terms of basic analogue of the gamma function

$$(q^a; q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0. \tag{10}$$

It is of interest to note that $\lim_{q \to 1^-}((q^a; q)_n/(1-q)^n) = (a)_n = a(a+1)\cdots(a+n-1)$ is the familiar Pochhammer symbol and

$$_l\Psi_m(a_1, \ldots, a_l; b_1, \ldots, b_m; z) = \sum_{n=0}^\infty \frac{(a_1)_n \cdots (a_l)_n}{(b_1)_n \cdots (b_m)_n} \frac{z^n}{n!}. \tag{11}$$

Now for $z \in \mathbb{U}$, $0 < |q| < 1$, and $l = m + 1$, the basic hypergeometric function defined in (8) takes the form

$$_l\Psi_m(a_1; \ldots a_l; b_1, \ldots, b_m; q, z)$$
$$= \sum_{n=0}^\infty \frac{(a_1, q)_n \cdots (a_l, q)_n}{(q, q)_n (b_1, q)_n \cdots (b_m, q)_n} z^n, \tag{12}$$

which converges absolutely in the open unitdisk $\mathbb{U}$.

Corresponding to the function $_l\Psi_m(a_1; \ldots a_l; b_1, \ldots, b_m; q, z)$ recently for meromorphic functions $f \in \Sigma_0$ consisting functions of the form (3), Huda and Darus [5] introduce q-analogue of Liu-Srivastava operator as below:

$$_l\Psi_m(a_1; \ldots a_l; b_1, \ldots, b_m; q, z) * f(z)$$
$$= \frac{1}{z}\,_l\Psi_m(a_1; \ldots a_l; b_1, \ldots, b_m; q, z) * f(z) \tag{13}$$
$$= \frac{1}{z} + \sum_{n=1}^\infty \frac{(a_1; q)_{n+1} \cdots (a_l; q)_{n+1}}{(q; q)_{n+1}(b_1; q)_{n+1} \cdots (b_m; q)_{n+1}} a_n z^n,$$

where $z \in \Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

In this paper for functions $f \in \Sigma_\xi$ and for real parameters $a_1, \ldots, a_l$ and $b_1, \ldots, b_m$ ($b_j \neq 0$, $-1, \ldots$; $j = 1, 2, \ldots, m$) we define the following new linear operator:

$$\mathscr{I}_m^l(a_1; \ldots a_l; b_1, \ldots, b_m; q, z - \xi) : \Sigma_\xi \longrightarrow \Sigma_\xi, \tag{14}$$

as

$$\mathscr{I}_m^l(a_1; \ldots a_l; b_1, \ldots, b_m; q, z - \xi)$$
$$= \frac{1}{z-\xi}\,_l\Psi_m(a_1; \ldots a_l; b_1, \ldots, b_m; q, z - \xi) \tag{15}$$
$$\mathscr{I}_m^l[a_l, q] = \frac{1}{z-\xi} + \sum_{n=1}^\infty \Upsilon_n^{l,m}[a_1, q](z-\xi)^n,$$

where

$$\Upsilon_n^{l,m}[a_1, q]$$
$$= \frac{(a_1; q)_{n+1} \cdots (a_l; q)_{n+1}}{(q; q)_{n+1}(b_1; q)_{n+1} \cdots (b_m; q)_{n+1}}. \tag{16}$$

Throughout our study for $f \in \Sigma_\xi$, we let

$$\mathscr{I}_m^l f(z) = \mathscr{I}_m^l[a_l, q] * f(z)$$
$$= \frac{1}{z-\xi} + \sum_{n=1}^\infty \Upsilon_m^l(n) a_n(z-\xi)^n, \tag{17}$$
$$\Upsilon_m^l(n) = \Upsilon_n^{l,m}[a_1, q]$$
$$= \frac{(a_1; q)_{n+1} \cdots (a_l; q)_{n+1}}{(q; q)_{n+1}(b_1; q)_{n+1} \cdots (b_m; q)_{n+1}}, \tag{18}$$

unless otherwise stated.

Motivated by earlier works on meromorphic functions by function theorists (see [6–14]), we define the following new subclass of functions in Σ_ξ by making use of the generalized operator $\mathscr{I}_m^l$.

For $0 \leq \gamma < 1$ and $0 \leq \lambda < 1/2$, we let $\mathscr{M}_m^l(\lambda, \beta, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (4) satisfying the condition that

$$
\begin{aligned}
&-\Re\left(\frac{(z-\xi)\left(\mathscr{I}_m^l f(z)\right)' + \lambda(z-\xi)^2\left(\mathscr{I}_m^l f(z)\right)''}{(1-\lambda)\,\mathscr{I}_m^l f(z) + \lambda(z-\xi)\left(\mathscr{I}_m^l f(z)\right)'}\right) \\
&> \beta\left|\frac{(z-\xi)\left(\mathscr{I}_m^l f(z)\right)' + \lambda(z-\xi)^2\left(\mathscr{I}_m^l f(z)\right)''}{(1-\lambda)\,\mathscr{I}_m^l f(z) + \lambda(z-\xi)\left(\mathscr{I}_m^l f(z)\right)'} + 1\right| \\
&+ \gamma,
\end{aligned}
\tag{19}
$$

where $\mathscr{I}_m^l f$ is given by (17).

Further, shortly we can state this condition by

$$
-\Re\left(\frac{(z-\xi)\,G'(z)}{G(z)}\right) > \beta\left|\frac{(z-\xi)\,G'(z)}{G(z)} + 1\right| + \gamma, \tag{20}
$$

where

$$
\begin{aligned}
G(z) &= (1-\lambda)\,\mathscr{I}_m^l f(z) + \lambda(z-\xi)\left(\mathscr{I}_m^l f(z)\right)' \\
&= \frac{1-2\lambda}{z-\xi} + \sum_{n=1}^{\infty} (n\lambda - \lambda + 1)\,\Upsilon_m^l(n)\,a_n(z-\xi)^n, \tag{21}
\end{aligned}
$$

$$
a_n \geq 0.
$$

It is of interest to note that, on specializing the parameters λ, β and l, m, we can define various new subclasses of Σ_ξ. We illustrate two important subclasses in the following examples.

Example 1. For $\lambda = 0$, we let $\mathscr{M}_m^l(0, \beta, \gamma) = \mathscr{M}_m^l(\beta, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (4) satisfying the condition that

$$
\begin{aligned}
&-\Re\left(\frac{(z-\xi)\left(\mathscr{I}_m^l f(z)\right)'}{\mathscr{I}_m^l f(z)}\right) \\
&> \beta\left|\frac{(z-\xi)\left(\mathscr{I}_m^l f(z)\right)'}{\mathscr{I}_m^l f(z)} + 1\right| + \gamma,
\end{aligned}
\tag{22}
$$

where $\mathscr{I}_m^l f(z)$ is given by (17).

Example 2. For $\lambda = 0$, $\beta = 0$ we let $\mathscr{M}_m^l(0,0,\gamma) = \mathscr{M}_m^l(\gamma)$ denote a subclass of Σ_ξ consisting functions of the form (4) satisfying the condition that

$$
-\Re\left(\frac{(z-\xi)\left(\mathscr{I}_m^l f(z)\right)'}{\mathscr{I}_m^l f(z)}\right) > \gamma, \tag{23}
$$

where $\mathscr{I}_m^l f(z)$ is given by (17).

In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, and closure results for the function class $\mathscr{M}_m^l(\lambda, \beta, \gamma)$. Properties of certain integral operator and convolution properties of the new class $\mathscr{M}_m^l(\lambda, \beta, \gamma)$ are also discussed.

2. Coefficients Inequalities

In order to obtain the necessary and sufficient condition for a function, $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$, we recall the following lemmas.

Lemma 3. *If γ is a real number and w is a complex number, then $\Re(w) \geq \gamma \Leftrightarrow |w + (1-\gamma)| - |w - (1+\gamma)| \geq 0$.*

Lemma 4. *If w is a complex number and γ, k are real numbers, then*

$$
\Re(w) \geq k\,|w-1| + \gamma \iff \Re\left\{w\left(1 + ke^{i\theta}\right) - ke^{i\theta}\right\} \geq \gamma,
$$
$$
-\pi \leq \theta \leq \pi. \tag{24}
$$

Analogous to the lemma proved by Dziok et al. [8], we state the following lemma without proof.

Lemma 5. *Suppose that $\gamma \in [0,1)$, $r \in (0,1]$, and the function $H \in \Sigma_\xi(\gamma)$ is of the form $H(z) = (1/(z-\xi)) + \sum_{n=1}^{\infty} b_n(z-\xi)^n$, $0 < |z-\xi| < r < 1$, with $b_n \geq 0$, then*

$$
\sum_{n=1}^{\infty} (n+\gamma)\,b_n r^{n+1} \leq 1 - \gamma. \tag{25}
$$

Theorem 6. *Let $f \in \Sigma_\xi$ be given by (4). Then $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$ if and only if*

$$
\sum_{n=1}^{\infty} \left[n(1+\beta) + (\gamma+\beta)\right](1+n\lambda-\lambda)\,\Upsilon_m^l(n)\,a_n \tag{26}
$$
$$
\leq (1-2\lambda)(1-\gamma).
$$

Proof. If $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$, then by (20) we have

$$
-\Re\left(\frac{(z-\xi)\,G'(z)}{G(z)}\right) > \beta\left|\frac{(z-\xi)\,G'(z)}{G(z)} + 1\right| + \gamma. \tag{27}
$$

Making use of Lemma 4,

$$
-\Re\left(\frac{(z-\xi)\left(1 + \beta e^{i\theta}\right)G'(z) + \beta e^{i\theta}G(z)}{G(z)}\right) > \gamma, \tag{28}
$$

where $G(z)$ is given by (21). Substituting $G(z)$, $G'(z)$ and letting $|z-\xi| < r \to 1^-$, we have

$$
\begin{aligned}
&\left\{\left((1-2\lambda)(1-\gamma) - \sum_{n=1}^{\infty}\left[n(1+\beta) + (\gamma+\beta)\right]\right.\right. \\
&\left.\times (1+n\lambda-\lambda)\,\Upsilon_m^l(n)\,a_n\right) \tag{29} \\
&\left.\times\left((1-2\lambda) - \sum_{n=1}^{\infty}(1+n\lambda-\lambda)\,\Upsilon_m^l(n)\,a_n\right)^{-1}\right\} > 0.
\end{aligned}
$$

This shows that (26) holds.

Conversely, assume that (26) holds. Since $-\Re(w) > \gamma$, if and only if $|w + 1| < |w - (1 - 2\gamma)|$, it is sufficient to show that

$$\left| \frac{w+1}{w-(1-2\gamma)} \right| < 1, \qquad |w - (1-2\gamma)| \neq 0 \tag{30}$$

$$\text{for } |z - \xi| < r \leq 1, \ (z - \xi) \in \Delta.$$

Using (26) and taking $w(z) = ((z - \xi)(1 + \beta e^{i\theta})G'(z) + \beta e^{i\theta}G(z))/G(z)$, we get

$$\left| \frac{w+1}{w-(1-2\gamma)} \right|$$

$$\leq \left(\left(\sum_{n=1}^{\infty} (1 + n\lambda - \lambda) \left[(n+1)(1+\beta) \right] \Upsilon_m^l(n) a_n \right) \right.$$

$$\times \left(2(1-\gamma)(1-2\lambda) - \sum_{n=1}^{\infty} (1 + n\lambda - \lambda) \right.$$

$$\left. \times \left[n(1+\beta) + (\beta + 2\gamma - 1) \right] \Upsilon_m^l(n) a_n \right)^{-1} \right)$$

$$\leq 1. \tag{31}$$

Thus we have $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$.   □

For the sake of brevity throughout this paper we let

$$d_n(\lambda, \beta, \gamma) = \left[n(1+\beta) + (\gamma + \beta) \right](1 + n\lambda - \lambda),$$
$$d_1(\lambda, \beta, \gamma) = (1 + \gamma + 2\beta), \tag{32}$$

unless otherwise stated.

Our next result gives the coefficient estimates for functions in $\mathscr{M}_m^l(\lambda, \beta, \gamma)$.

Theorem 7. *If $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$, then*

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n)}, \qquad n = 1, 2, 3, \dots. \tag{33}$$

The result is sharp for the functions $f_n(z)$ given by

$$f_n(z) = \frac{1}{z-\xi} + \frac{1-\gamma}{d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n)}(z-\xi)^n, \tag{34}$$

$$n = 1, 2, 3, \dots.$$

Proof. If $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$, then we have, for each n,

$$d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n) a_n \leq \sum_{n=1}^{\infty} d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n) a_n \tag{35}$$

$$\leq (1-\gamma)(1-2\lambda).$$

Therefore we have

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n)}. \tag{36}$$

Since

$$f_n(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n)}(z-\xi)^n \tag{37}$$

satisfies the conditions of Theorem 6, $f_n(z) \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$ and the equality is attained for this function.   □

Theorem 8. *Suppose that there exists a positive number ν:*

$$\nu = \inf_{n \in \mathbb{N}} \left\{ d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n) \right\}. \tag{38}$$

If $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$, then

$$\left| \frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)}{\nu} r \right|$$

$$\leq |f(z)| \leq \frac{1}{r}$$

$$+ \frac{(1-\gamma)(1-2\lambda)}{\nu} r, \quad (|z - \xi| = r),$$

$$\left| \frac{1}{r^2} - \frac{(1-\gamma)(1-2\lambda)}{\nu} \right| \tag{39}$$

$$\leq |f'(z)| \leq \frac{1}{r^2}$$

$$+ \frac{(1-\gamma)(1-2\lambda)}{\nu} \quad (|z - \xi| = r).$$

If $\nu = d_1(\lambda, \beta, \gamma)\Upsilon_m^l(1) = (1 + \gamma + 2\beta)\Upsilon_m^l(1)$, then the result is sharp for

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{(1+\gamma+2\beta)r^2\Upsilon_m^l(1)}(z-\xi). \tag{40}$$

Proof. Let $f \in \sum_\xi$ and be given by (4)

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r \sum_{n=1}^{\infty} a_n. \tag{41}$$

Since $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$, and by Theorem 6,

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\gamma)(1-2\lambda)}{\nu}. \tag{42}$$

Using this, we have

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu} r. \tag{43}$$

Similarly

$$|f(z)| \geq \left| \frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)}{\nu} r \right|. \tag{44}$$

The result is sharp for function (40) with

$$\nu = d_1(\lambda, \beta, \gamma)\Upsilon_m^l(1) = (1 + \gamma + 2\beta)\Upsilon_m^l(1). \tag{45}$$

Similarly we can prove the other inequality $|f'(z)|$.   □

3. Order of Starlikeness

In the following theorem we obtain the order of starlikeness for the class $\mathcal{M}_m^l(\lambda, \beta, \gamma)$. We say that f given by (4) is meromorphically starlike of order ρ, $(0 \le \rho < 1)$, in $|z - \xi| < r$ when it satisfies condition (5) in $|z - \xi| < r$.

Theorem 9. *Let the function f given by (4) be in the class $\mathcal{M}_m^l(\lambda, \beta, \gamma)$. Then, if there exists*

$$r = r_1(\lambda, \gamma, \rho) = \inf_{n \ge 1} \left[\frac{(1 - \rho) d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n)}{(n + \rho)(1 - \gamma)(1 - 2\lambda)} \right]^{1/(n+1)} \quad (46)$$

and it is positive, then f is meromorphically starlike of order ρ in $|z - \xi| < r \le r_1(\lambda, \gamma, \rho)$.

Proof. Let the function $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$ be of the form (4). If $0 < r \le r_1(\lambda, \gamma, \rho)$, then by (46)

$$r^{n+1} \le \frac{(1 - \rho) d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n)}{(n + \rho)(1 - \gamma)(1 - 2\lambda)}, \quad (47)$$

for all $n \in \mathbb{N}$. From (47) we get

$$\frac{n + \rho}{1 - \rho} r^{n+1} \le \frac{d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n)}{(1 - \gamma)(1 - 2\lambda)}, \quad (48)$$

for all $n \in \mathbb{N}$, and thus

$$\sum_{n=1}^{\infty} \frac{n + \rho}{1 - \rho} a_n r^{n+1} \le \sum_{n=1}^{\infty} \frac{d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n)}{(1 - \gamma)(1 - 2\lambda)} a_n \le 1, \quad (49)$$

because of (26). Hence, from (49) and (25), f is meromorphically starlike of order ρ in $|z - \xi| < r \le r_1(\lambda, \gamma, \rho) = r$. $\square$

Suppose that there exists a number $\tilde{r}$, $\tilde{r} > r_1(\lambda, \gamma, \rho)$, such that each $f \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$ is meromorphically starlike of order ρ in $|z - \xi| < \tilde{r} \le 1$. The function

$$f(z) = \frac{1}{z - \xi} + \frac{(1 - \gamma)(1 - 2\lambda)}{d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n)} (z - \xi)^n \quad (50)$$

is in the class $\mathcal{M}_m^l(\lambda, \beta, \gamma)$; thus it should satisfy (25) with $\tilde{r}$:

$$\sum_{n=1}^{\infty} (n + \rho) a_n \tilde{r}^{n+1} \le 1 - \rho, \quad (51)$$

while the left–hand side of (51) becomes

$$(n + \rho) \frac{(1 - \gamma)(1 - 2\lambda)}{d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n)} \tilde{r}^{n+1}$$

$$> (n + \rho) \frac{(1 - \gamma)(1 - 2\lambda)}{d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n)} \frac{(1 - \rho) d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n)}{(n + \rho)(1 - \gamma)(1 - 2\lambda)}$$

$$= 1 - \rho,$$

(52)

which contradicts (51). Therefore the number $r_1(\lambda, \gamma, \rho)$ in Theorem 9 cannot be replaced with a greater number. This means that $r_1(\lambda, \gamma, \rho)$ is called radius of meromorphically starlikeness of order ρ for the class $\mathcal{M}_m^l(\lambda, \beta, \gamma)$.

4. Results Involving Modified Hadamard Products

For functions

$$f_j(z) = \frac{1}{z - \xi} + \sum_{n=1}^{\infty} a_{n,j}(z - \xi)^n, \quad a_{n,j} \ge 0, \quad (53)$$

we define the Hadamard product or convolution of f_1 and f_2 by

$$(f_1 * f_2)(z) := \frac{1}{z - \xi} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} (z - \xi)^n. \quad (54)$$

Let

$$\Psi(n, \lambda) = \frac{(n\lambda - \lambda + 1)}{(1 - 2\lambda)} \Upsilon_m^l(n). \quad (55)$$

Theorem 10. *For functions f_j $(j = 1, 2)$ defined by (53), let $f_1 \in \mathcal{M}_m^l(\lambda, \beta, \gamma)$ and $f_2 \in \mathcal{M}_m^l(\lambda, \beta, \delta)$. Then $f_1 * f_2 \in \mathcal{M}_m^l(\lambda, \beta, \eta)$ where*

$$\eta$$

$$= 1 - \frac{(1 - \gamma)(1 - \delta)(3 + \beta)}{(1 + \gamma + 2\beta)(1 + \delta + 2\beta) \Psi(1, \lambda) - 2(1 - \gamma)(1 - \delta)}, \quad (56)$$

and $\Psi(1, \lambda) = \Upsilon_m^l(1)/(1 - 2\lambda)$. The results are the best possible for

$$f_1(z) = \frac{1}{z - \xi} + \frac{1 - \gamma}{(1 + \gamma + 2\beta) \Psi(1, \lambda)} (z - \xi),$$

$$f_2(z) = \frac{1}{z - \xi} + \frac{1 - \delta}{(1 + \delta + 2\beta) \Psi(1, \lambda)} (z - \xi),$$

(57)

where $\Psi(1, \lambda) = \Upsilon_m^l(1)/(1 - 2\lambda)$.

Proof. In view of Theorem 6, it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{[n(1 + \beta) + (\eta + \beta)]}{(1 - \eta)} \Psi(n, \lambda) a_{n,1} a_{n,2} \le 1, \quad (58)$$

where η is defined by (56) under the hypothesis. It follows from (26) and the Cauchy-Schwarz inequality that

$$\sum_{n=1}^{\infty} \frac{[n(1 + \beta) + (\gamma + \beta)]^{1/2} [n(1 + \beta) + (\delta + \beta)]^{1/2}}{\sqrt{(1 - \gamma)(1 - \delta)}}$$

$$\times \Psi(n, \lambda) \sqrt{a_{n,1} a_{n,2}} \le 1. \quad (59)$$

Thus we need to find the largest η such that

$$
\sum_{n=1}^{\infty} \frac{[n(1+\beta)+(\eta+\beta)]}{(1-\eta)} \Psi(n,\lambda) a_{n,1} a_{n,2}
$$

$$
\leq \sum_{n=1}^{\infty} \frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}}
$$

$$
\times \Psi(n,\lambda) \sqrt{a_{n,1} a_{n,2}}
$$

$$
\leq 1. \tag{60}
$$

By virtue of (59) it is sufficient to find the largest η, such that

$$
\frac{\sqrt{(1-\gamma)(1-\delta)}}{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}\Psi(n,\lambda)}
$$

$$
\leq \frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}}
$$

$$
\times \frac{1-\eta}{[n(1+\beta)+(\eta+\beta)]}, \tag{61}
$$

which yields

$$
\eta \leq 1 - \Big(\big((1-\gamma)(1-\delta)(2n+1+\beta)\big)
$$

$$
\times \big([n(1+\beta)+(\gamma+\beta)][n(1+\beta)+(\delta+\beta)]
$$

$$
\times \Psi(n,\lambda) - (1-\gamma)(1-\delta)(n+1)\big)^{-1}\Big), \tag{62}
$$

for $n \geq 1$ where $\Psi(n,\lambda)$ is given by (55) and, since $\Psi(n,\lambda)$ is a decreasing function of n $(n \geq 1)$, we have

$$
\eta = 1
$$

$$
- \frac{(1-\gamma)(1-\delta)(3+\beta)}{(1+\gamma+2\beta)(1+\delta+2\beta)\Psi(1,\lambda) - 2(1-\gamma)(1-\delta)}, \tag{63}
$$

and $\Psi(1,\lambda) = \Upsilon_m^l(1)/(1-2\lambda)$, which completes the proof. □

Theorem 11. *Let the functions f_j, $(j = 1, 2)$, defined by (53) be in the class $\mathscr{M}_m^l(\lambda, \beta, \gamma)$. Then $(f_1 * f_2)(z) \in \mathscr{M}_m^l(\lambda, \beta, \eta)$ where*

$$
\eta = 1 - \frac{(1-\gamma)^2(3+\beta)}{(1+\gamma+2\beta)^2\Psi(1,\lambda) - 2(1-\gamma)^2} \tag{64}
$$

with $\Psi(1,\lambda) = \Upsilon_m^l(1)/(1-2\lambda)$.

Proof. By taking $\delta = \gamma$ in the above theorem, the results follow. □

For functions in the class $\mathscr{M}_m^l(\lambda, \beta, \gamma)$, we can prove the following inclusion property.

Theorem 12. *Let the functions f_j $(j = 1, 2)$ defined by (53) be in the class $\mathscr{M}_m^l(\lambda, \beta, \gamma)$. Then the function h, defined by*

$$
h(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} \big(a_{n,1}^2 + a_{n,2}^2\big)(z-\xi)^n, \tag{65}
$$

is in the class $\mathscr{M}_m^l(\lambda, \beta, \delta)$ where

$$
\delta \leq 1 - \frac{4(1-\gamma)^2(1+\beta)}{[1+\gamma+2\beta]^2\Psi(1,\lambda) + 2(1-\gamma)^2}, \tag{66}
$$

and $\Psi(1,\lambda) = \Upsilon_m^l(1)/(1-2\lambda)$.

Proof. In view of Theorem 6, it is sufficient to prove that

$$
\sum_{n=2}^{\infty} \Psi(n,\lambda) \frac{[n(1+\beta)+(\delta+\beta)]}{(1-\delta)}\big(a_{n,1}^2 + a_{n,2}^2\big) \leq 1, \tag{67}
$$

where $f_j \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$ $(j = 1, 2)$; we find from (53) and Theorem 6 that

$$
\sum_{n=1}^{\infty} \left[\Psi(n,\lambda) \frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2 a_{n,j}^2
$$

$$
\leq \sum_{n=1}^{\infty} \left[\Psi(n,\lambda) \frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}a_{n,j}\right]^2 \leq 1, \tag{68}
$$

which would yield

$$
\sum_{n=2}^{\infty} \frac{1}{2}\left[\Psi(n,\lambda) \frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2 \big(a_{n,1}^2 + a_{n,2}^2\big) \leq 1. \tag{69}
$$

On comparing (67) and (69) it can be seen that inequality (66) will be satisfied if

$$
\Psi(n,\lambda) \frac{[n(1+\beta)+(\delta+\beta)]}{1-\delta}\big(a_{n,1}^2 + a_{n,2}^2\big)
$$

$$
\leq \frac{1}{2}\left[\Psi(n,\lambda) \frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2 \tag{70}
$$

$$
\times \big(a_{n,1}^2 + a_{n,2}^2\big).
$$

That is, if

$$
\delta \leq 1 - \frac{2(1-\gamma)^2[(n+1)(1+\beta)]}{[n(1+\beta)+(\gamma+\beta)]^2\Psi(n,\lambda) + 2(1-\gamma)^2}, \tag{71}
$$

where $\Psi(n,\lambda)$ is given by (55) and $\Psi(n,\lambda)$ is a decreasing function of n $(n \geq 1)$, we get (66), which completes the proof. □

5. Closure Theorems

We state the following closure theorems for $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$ without proof (see [8–10]).

Theorem 13. *Let the function $f_k(z) = (1/(z - \xi)) + \sum_{n=1}^{\infty} a_{n,k}(z - \xi)^n$ be in the class $\mathscr{M}_m^l(\lambda, \beta, \gamma)$ for every $k = 1, 2, \ldots, m$. Then the function f defined by*

$$f(z) = \frac{1}{z - \xi} + \sum_{n=1}^{\infty} a_{n,k}(z - \xi)^n, \quad (a_{n,k} \geq 0) \quad (72)$$

belongs to the class $\mathscr{M}_m^l(\lambda, \beta, \gamma)$, where $a_{n,k} = (1/m) \sum_{k=1}^{m} a_{n,k}$, $(n = 1, 2, \ldots)$.

Theorem 14. *Let $f_0(z) = 1/(z - \xi)$ and $f_n(z) = (1/(z - \xi)) + ((1-\gamma)(1-2\lambda)/d_n(\lambda, \beta, \gamma)\Upsilon_m^l(n))(z-\xi)^n$ for $n = 1, 2, \ldots$. Then $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$ if and only if f can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \eta_n f_n(z)$ where $\eta_n \geq 0$ and $\sum_{n=0}^{\infty} \eta_n = 1$.*

Theorem 15. *The class $\mathscr{M}_m^l(\lambda, \beta, \gamma)$ is closed under convex linear combination.*

Now, we prove that the class is $\mathscr{M}_m^l(\lambda, \beta, \gamma)$ closed under integral transforms.

Theorem 16. *Let the function $f(z)$ given by (4) be in $\mathscr{M}_m^l(\lambda, \beta, \gamma)$. Then the integral operator*

$$F(z) = c \int_0^1 u^c f(uz) \, du \quad (0 < u \leq 1, 0 < c < \infty) \quad (73)$$

is in $\mathscr{M}_m^l(\lambda, \beta, \delta)$, where

$$\delta \leq \left(n^2 (1 + \beta) + n [(\gamma + \beta) + (1 + \beta)(1 + c\gamma)] \right.$$
$$+ (c + 1)(\gamma + \beta) + c\beta(1 - \gamma) \right)$$
$$\times \left(n^2 (1 + \beta) + n [(\gamma + \beta) + (1 + c)(1 + \beta)] \right. \quad (74)$$
$$+ (1 + c)(\gamma + \beta) + c(1 - \gamma) \left. \right)^{-1}.$$

The result is sharp for the function $f(z) = (1/(z - \xi)) + ((1 - \gamma)(1 - 2\lambda)/(1 + \gamma + 2\beta)\Upsilon_m^l(1))(z - \xi)$.

Proof. Let $f(z) \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$. Then

$$F(z) = c \int_0^1 u^c f(uz) \, du = \frac{1}{z - w} + \sum_{n=1}^{\infty} \frac{c}{c + n + 1} a_n(z - \xi)^n. \quad (75)$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c d_n(\lambda, \beta, \delta) \Upsilon_m^l(n)}{(c + n + 1)(1 - \delta)} a_n \leq 1. \quad (76)$$

Since $f \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$, we have

$$\sum_{n=1}^{\infty} \frac{d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n)}{(1 - \gamma)(1 - 2\lambda)} a_n \leq 1. \quad (77)$$

Note that (76) is satisfied if

$$\frac{c d_n(\lambda, \beta, \delta) \Upsilon_m^l(n)}{(c + n + 1)(1 - \delta)} \leq \frac{d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n)}{(1 - \gamma)(1 - 2\lambda)}. \quad (78)$$

From (78), we have

$$\delta \leq \left(\left(n^2 (1 + \beta) + n [(\gamma + \beta) + (1 + \beta)(1 + c\gamma)] \right. \right.$$
$$+ (c + 1)(\gamma + \beta) + c\beta(1 - \gamma) \left. \right)$$
$$\times \left(n^2 (1 + \beta) + n [(\gamma + \beta) + (1 + c)(1 + \beta)] \right. \quad (79)$$
$$+ (1 + c)(\gamma + \beta) + c(1 - \gamma) \left. \right)^{-1} \right) = \Phi(n).$$

A simple computation will show that $\Phi(n)$ is increasing and $\Phi(n) \geq \Phi(1)$. Using this, the results follow. $\square$

6. Partial Sums

Silverman [15] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested in searching results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman [15] and Cho and Owa [16], we will investigate the ratio of a function of the form (4) to its sequence of partial sums. Consider

$$f_k(z) = \frac{1}{z - \xi} + \sum_{n=1}^{k} a_n(z - \xi)^n, \quad (80)$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} d_n(\lambda, \beta, \gamma) \Upsilon_m^l(n) a_n \leq (1 - \gamma)(1 - 2\lambda). \quad (81)$$

More precisely we will determine sharp lower bounds for $\mathfrak{R}(f(z)/f_k(z))$ and $\mathfrak{R}(f_k(z)/f(z))$. In this connection we make use of the well-known results that $\mathfrak{R}((1 + w(z))/(1 - w(z))) > 0$, $(z - \xi \in \Delta)$, if and only if $w(z) = \sum_{n=1}^{\infty} c_n(z - \xi)^n$ satisfies the inequality $|w(z)| \leq |z - \xi|$.

Unless otherwise stated, we will assume that f is of the form (4) and its sequence of partial sums is denoted by (80).

Theorem 17. *Let $f(z) \in \mathscr{M}_m^l(\lambda, \beta, \gamma)$ be given by (4) which satisfies condition (26) and suppose that all of its partial sums (80) do not vanish in Δ. Moreover, suppose that*

$$2 - 2 \sum_{n=1}^{k} |a_n| - \frac{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k + 1)}{(1 - \gamma)(1 - 2\lambda)} \sum_{n=k+1}^{\infty} |a_n| > 0, \quad (82)$$
$$\forall k \in \mathbb{N}.$$

Then,

$$\mathfrak{R}\left(\frac{f(z)}{f_k(z)} \right) \geq 1 - \frac{(1 - \gamma)(1 - 2\lambda)}{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k + 1)} \quad (z - \xi \in \Delta), \quad (83)$$

where

$$d_n(\lambda, \beta, \gamma)$$

$$\geq \begin{cases} (1-\gamma)(1-2\lambda), & \text{if } n = 1, 2, 3, \dots, k \\ d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1), & \text{if } n = k+1, k+2, \dots. \end{cases}$$

$$(84)$$

The result (83) *is sharp with the function given by*

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)} (z-\xi)^{k+1}. \quad (85)$$

Proof. Define the function $w(z)$ by

$$w(z)$$

$$= \frac{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)}{(1-\gamma)(1-2\lambda)}$$

$$\times \left[\frac{f(z)}{f_k(z)} - \left(1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)} \right) \right]$$

$$= 1$$

$$+ \left(\left(\left(d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1) \right) \right. \right. \quad (86)$$

$$\times \left((1-\gamma)(1-2\lambda) \right)^{-1}$$

$$\times \sum_{n=k+1}^{\infty} a_n(z-\xi)^{n+1} \right)$$

$$\left. \times \left(1 + \sum_{n=1}^{k} a_n(z-\xi)^{n+1} \right)^{-1} \right).$$

It suffices to show that $\Re(w(z)) > 0$; hence we find that

$$\left| \frac{1+w(z)}{1-w(z)} \right|$$

$$\leq \left(\left(\left(d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1) \right) \times \left((1-\gamma)(1-2\lambda) \right)^{-1} \right. \right.$$

$$\times \sum_{n=k+1}^{\infty} |a_n| \right)$$

$$\times \left(2 - 2\sum_{n=1}^{k} |a_n| \right.$$

$$- \left(d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1) \right)$$

$$\times \left((1-\gamma)(1-2\lambda) \right)^{-1}$$

$$\left. \left. \times \sum_{n=k+1}^{\infty} |a_n| \right)^{-1} \right) \leq 1.$$

$$(87)$$

From condition (26), it readily yields the assertion (83) of Theorem 17.

To see that the function given by (85) gives the sharp result, we observe that for $z = re^{i\pi/(k+2)}$

$$\frac{f(z)}{f_k(z)} = 1 + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)} (z-\xi)^n$$

$$\longrightarrow 1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)},$$

$$(88)$$

when $r \to 1^-$ which shows that the bound (83) is the best possible for each $k \in \mathbb{N}$. □

We next determine bounds for $f_k(z)/f(z)$.

Theorem 18. *Under the assumptions of Theorem 17, we have*

$$\Re \left(\frac{f_k(z)}{f(z)} \right) \geq \frac{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)}{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1) + (1-\gamma)(1-2\lambda)}$$

$$(z - w \in \Delta),$$

$$(89)$$

The result (89) *is sharp with the function given by* (85).

Proof. By setting

$$w(z)$$

$$= \left(1 + \frac{d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1)}{(1-\gamma)(1-2\lambda)} \right)$$

$$\times \left[\frac{f_k(z)}{f(z)} \right.$$

$$\left. - \frac{\left(d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1) / (1-\gamma)(1-2\lambda) \right)}{1 + \left(d_{k+1}(\lambda, \beta, \gamma) \Upsilon_m^l(k+1) / (1-\gamma)(1-2\lambda) \right)} \right]$$

$$(90)$$

and proceeding as in Theorem 17, we get the desired result and so we omit the details. □

Concluding Remark. We observe that, if we specialize the parameters λ and β as mentioned in Examples 1 and 2, we obtain the analogous results for the classes $\mathcal{M}_m^l(\beta, \gamma)$ and $\mathcal{M}_m^l(\gamma)$. Further specializing the parameters l, m various other interesting results (as in Theorems 6–18) can be derived easily for the function class based on interesting differential operators as illustrated below.

(1) For $a_i = q^{a_i}$, $b_j = q^{b_j}$, $a_i > 0$, $b_j > 0$, $(i = 1, \dots, l;\ j = 1, \dots, m,\ l = m+1)$, $q \to 1$, the operator $\mathcal{I}_m^l f(z) = \mathcal{H}_m^l[a_1] f(z)$ defined by Liu and Srivastava [10].

(2) For $l = 2$, $m = 1$, $a_2 = q$, $q \to 1$, the operator $\mathcal{L}_1^2[a_1, q, b_1, q] f(z) = \mathcal{L}[a_1; b_1] f(z)$ was introduced and studied by Liu and Srivastava [9].

(3) For $l = 1$, $m = 0$, $a_1 = \delta + 1$, $q \to 1$, the operator $\mathcal{L}[a_1; b_1] f(z) = D^\delta f(z) = (1/z(1-z)^{\delta+1}) * f(z)$, $(\delta > -1)$

where D^δ is the differential operator which was introduced by Ganigi and Uralegaddi [17].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors thank the referee for their valuable suggestions.

References

[1] S. Kanas and F. Ronning, "Uniformaly starlike and convex function and other related classes of univalent functions," *Annales Universitatis Mariae Curie-Skłodowska. Sectio A*, vol. 53, pp. 95–105, 1991.

[2] A. W. Goodman, "On uniformly starlike functions," *Journal of Mathematical Analysis and Applications*, vol. 155, no. 2, pp. 364–370, 1991.

[3] A. W. Goodman, "On uniformly convex functions," *Annales Polonici Mathematici*, vol. 56, no. 1, pp. 87–92, 1991.

[4] S. D. Purohit and R. K. Raina, "Certain subclasses of analytic functions associated with fractional q-calculus operators," *Mathematica Scandinavica*, vol. 109, no. 1, pp. 55–70, 2011.

[5] A. Huda and M. Darus, "Integral operator defined by q-analogue of Liu-Srivastava operator," *Studia Universitatis Babeş-Bolyai—Series Mathematica*, vol. 58, no. 4, pp. 529–537, 2013.

[6] M. K. Aouf, "On a certain class of meromorphic univalent functions with positive coeffcients," *Rendiconti di Matematica e delle sue Applicazioni*, vol. 7, no. 11, pp. 209–219, 1991.

[7] M. K. Aouf and G. Murugusundaramoorthy, "On a subclass of uniformly convex functions defined by the Dziok-Srivastava operator," *Australian Journal of Mathematical Analysis and Applications*, vol. 5, no. 1, article 3, pp. 1–17, 2008.

[8] J. Dziok, G. Murugusundaramoorthy, and J. Sokół, "On certain class of meromorphic functions with positive coeffcients," *Acta Mathematica Scientia B*, vol. 32, no. 4, pp. 1–16, 2012.

[9] J.-L. Liu and H. M. Srivastava, "Subclasses of meromorphically multivalent functions associated with a certain linear operator," *Mathematical and Computer Modelling*, vol. 39, no. 1, pp. 35–44, 2004.

[10] J.-L. Liu and H. M. Srivastava, "Classes of meromorphically multivalent functions associated with the generalized hypergeometric function," *Mathematical and Computer Modelling*, vol. 39, no. 1, pp. 21–34, 2004.

[11] M. L. Mogra, T. R. Reddy, and O. P. Juneja, "Meromorphic univalent functions with positive coeffcients," *Bulletin of the Australian Mathematical Society*, vol. 32, pp. 161–176, 1985.

[12] S. Owa and N. N. Pascu, "Coeffcient inequalities for certain classes of meromorphically starlike and meromorphically convex functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 17, pp. 1–6, 2003.

[13] C. Pommerenke, "On meromorphic starlike functions," *Pacific Journal of Mathematics*, vol. 13, pp. 221–235, 1963.

[14] B. A. Uralegaddi and C. Somanatha, "Certain differential operators for meromorphic functions," *Houston Journal of Mathematics*, vol. 17, no. 2, pp. 279–284, 1991.

[15] H. Silverman, "Partial sums of starlike and convex functions," *Journal of Mathematical Analysis and Applications*, vol. 209, no. 1, pp. 221–227, 1997.

[16] N. E. Cho and S. Owa, "Partial sums of certain meromorphic functions," *Journal of Inequalities in Pure & Applied Mathematics*, vol. 5, no. 2, article 30, 2004.

[17] M. R. Ganigi and B. A. Uralegaddi, "New criteria for meromorphic univalent functions," *Bulletin Mathematique de la Societe des Sciences Mathematiques de la Republique Socialiste de Roumanie*, vol. 33, no. 1, pp. 9–13, 1989.