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ORIGINAL ARTICLE

On a class of analytic functions related to Robertson's formula and subordination

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Abstract

In this paper, we define and study a class of analytic functions in the unit disc by modification of the well-known Robertson's analytic formula for starlike functions with respect to a boundary point combined with subordination. An integral representation and growth theorem are proved. Early coefficients and the Fekete–Szegö functional are also estimated.

Keywords Univalent function · Starlike function of order α · Starlike function with respect to a boundary point · Coefficient estimates

Mathematics Subject Classification 30C45 · 30C50 · 30C80

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1 Introduction

Let $\mathcal{H}$ denote the class of all holomorphic functions in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. By $\mathcal{A}$ we denote the subclass of $\mathcal{H}$ of all functions h normalized by $h(0) = 0$ and $h'(0) = 1$, i.e., of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

and by $\mathcal{S}$ the subclass of $\mathcal{A}$ of univalent functions. We say that a function $f \in \mathcal{H}$ is subordinate to a function $g \in \mathcal{H}$ and write $f \prec g$ if there exists a function $\omega \in \mathcal{H}$ such that $\omega(0) = 0$, $\omega(\mathbb{D}) \subset \mathbb{D}$ and $f(z) = g(\omega(z))$ for every $z \in \mathbb{D}$. In case when g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Let us denote by $\mathcal{P}$ the class of functions $p \in \mathcal{H}$ normalized by $p(0) = 1$ and such that $\Re\, \mathfrak{p}(\mathfrak{z}) >$ for $z \in \mathbb{D}$, which is known as the Carathéodory class. By $\mathcal{P}^*(1)$ we denote the subclass of $\mathcal{P}$ of all ϕ such that $\phi(0) = 1$, $\phi'(0) > 0$, ϕ is univalent in $\mathbb{D}$ and $\phi(\mathbb{D})$ is a set symmetric with respect to the real axis and starlike with respect to 1. Thus, every $\phi \in \mathcal{P}^*(1)$ can be represented as

$$\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \quad z \in \mathbb{D}, \tag{1}$$

with $B_1 > 0$.

Given $\phi \in \mathcal{P}^*(1)$, let $\mathcal{P}(\phi) := \{p \in \mathcal{P} : p \prec \phi\}$. The class $\mathcal{P}^*(1)$ plays a fundamental role in defining suitable classes of analytic functions as was proposed first by Ma and Minda [14]. As an example, given $\phi \in \mathcal{P}^*(1)$, let $\mathcal{S}^*(\phi)$ denote the class of all $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec \phi(z)$ for $z \in \mathbb{D}$. Such defined classes are called of Ma and Minda type.

The two well-known subclasses of $\mathcal{A}$, are namely the class of starlike and convex functions of order α $(0 \le \alpha < 1)$ introduced by Robertson [18] given, respectively, by

$$\mathcal{S}^*(\alpha) := \left\{ h \in \mathcal{A} : \Re \frac{\mathfrak{z}\mathfrak{h}'(\mathfrak{z})}{\mathfrak{h}(\mathfrak{z})} > \alpha, \ \mathfrak{z} \in \mathbb{D} \right\}$$

and

$$\mathcal{K}(\alpha) := \left\{ h \in \mathcal{A} : \Re\left(+ \frac{\mathfrak{z}\mathfrak{h}''(\mathfrak{z})}{\mathfrak{h}'(\mathfrak{z})} \right) > \alpha, \ \mathfrak{z} \in \mathbb{D} \right\}.$$

It is well known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}$ and $\mathcal{K}(\alpha) \subset \mathcal{S}$. By virtue of the well-known Alexander's relation, we see that $h \in \mathcal{K}(\alpha)$ in $\mathbb{D}$ if and only if the function $\mathbb{D} \ni z \mapsto zh'(z) \in \mathcal{S}^*(\alpha)$ for each $0 \le \alpha < 1$. It is clear that $\mathcal{S}^*(\alpha) = \mathcal{S}^*(\phi)$ with $\phi(z) = (1+z)/(1-z)$, $z \in \mathbb{D}$.

We say that $h \in \mathcal{H}$ is close-to-convex if and only if there exists a function $\Phi \in \mathcal{K} := \mathcal{K}(0)$ such that

$$\Re \frac{\mathfrak{h}'(\mathfrak{z})}{\Phi'(\mathfrak{z})} > , \quad \mathfrak{z} \in \mathbb{D}.$$

The class of close-to-convex functions was introduced by Kaplan [7].

Even though starlikeness of order α has been explored extensively by many authors over a long period of time, not much was known about the class of univalent functions g in $\mathcal{H}$ that map the disc $\mathbb{D}$ onto domains Ω which are starlike with respect to a boundary point. This breakthrough concept was introduced by Robertson [19] who defined the subclass $\mathcal{G}^*$ of $\mathcal{H}$ of functions g such that $g(0) = 1$, $g(1) := \lim_{r \to 1^-} g(r) = 0$, g maps univalently $\mathbb{D}$ onto a domain starlike with respect to the origin and $\Re(\mathrm{e}^{\mathrm{i}\delta}\mathfrak{g}(\mathfrak{z})) >$ for some real δ and all $z \in \mathbb{D}$. Assume that the constant function $g \equiv 1$ also belongs to $\mathcal{G}^*$. Robertson conjectured that the class $\mathcal{G}^*$ coincides with the class $\mathcal{G}$ of all $g \in \mathcal{H}$ of the form

$$g(z) = 1 + \sum_{n=1}^{\infty} g_n z^n, \quad z \in \mathbb{D}, \tag{2}$$

such that

$$\Re\left(\frac{\mathfrak{z}\mathfrak{g}'(\mathfrak{z})}{\mathfrak{g}(\mathfrak{z})} + \frac{+\mathfrak{z}}{-\mathfrak{z}}\right) > , \quad \mathfrak{z} \in \mathbb{D}, \tag{3}$$

proving that $\mathcal{G} \subset \mathcal{G}^*$. Robertson's conjecture was confirmed by Lyzzaik [13] in 1984, who proved that $\mathcal{G}^* \subset \mathcal{G}$. In [19], Robertson proved that if $g \in \mathcal{G}$ and $g \neq 1$, then g is close-to-convex and univalent in $\mathbb{D}$. It is worth mentioning that the analytic condition (3) was known much earlier to Styer [22].

Lecko [9] proposed an alternative analytic characterization of starlike functions with respect to a boundary point proving the necessity, while Lecko and Lyzzaik [12] showing the sufficiency confirmed this new analytic characterizations (see also [10], Chapter VII]). Inspired by Robertson's paper, a class of functions based on the concept of spiral-like domains with respect to a boundary point was introduced by Aharanov et al. [2] (see also [11]).

A closely related class to the class $\mathcal{G}$ is the family $\mathcal{G}(M)$, $M > 1$, consisting of all $g \in \mathcal{H}$ of the form (2) such that

$$\Re\left(\frac{\mathfrak{z}\mathfrak{g}'(\mathfrak{z})}{\mathfrak{g}(\mathfrak{z})} + \frac{\mathfrak{z}\mathrm{P}'(\mathfrak{z}\,\mathfrak{M})}{\mathrm{P}(\mathfrak{z}\,\mathfrak{M})}\right) > , \quad \mathfrak{z} \in \mathbb{D},$$

introduced by Jakubowski [5]. Here

$$\mathrm{P}(z; M) := \frac{4z}{\left(\sqrt{(1-z)^2 + 4z/M} + 1 - z\right)^2}, \quad z \in \mathbb{D}$$

denotes the Pick function. The class

$$\mathcal{G}(1) := \left\{ g \in \mathcal{H} : g(0) = 1,\ \Re\left(\frac{\mathfrak{z}\mathfrak{g}'(\mathfrak{z})}{\mathfrak{g}(\mathfrak{z})} + \right) > ,\ \mathfrak{z} \in \mathbb{D} \right\}$$

was considered in [5] also. Todorov [23] associated the class $\mathcal{G}$ with the functional $f(z)/(1-z)$ for $z \in \mathbb{D}$, and obtained a structured formula and coefficient estimates. Obradovič and Owa [16], and Silverman and Silvia [21] introduced independently the classes $\mathcal{G}_\alpha$, where $\alpha \in [0, 1)$, of all $g \in \mathcal{H}$ of the form (2) such that

$$\mathfrak{R}\left(\frac{\mathfrak{z}\mathfrak{g}'(\mathfrak{z})}{\mathfrak{g}(\mathfrak{z})} + (- \alpha)\frac{+\mathfrak{z}}{-\mathfrak{z}}\right) > , \quad \mathfrak{z} \in \mathbb{D}.$$

Silverman and Silvia [21] observed that for each $\alpha \in [0, 1)$ the class $\mathcal{G}_\alpha$ is a subclass of $\mathcal{G}^*$. Clearly, $\mathcal{G}_{1/2} = \mathcal{G}$. Abdullah et al. [1] obtained a few properties and some inequalities related to the functional coefficient associated with the class $\mathcal{G}$. In [6], Jakubowski and Włodarczyk defined the class $\mathcal{G}(A, B)$ for $-1 < A \leq 1$ and $-A < B \leq 1$, of all $g \in \mathcal{H}$ of the form (2) such that

$$\mathfrak{R}\left(\frac{\mathfrak{z}\mathfrak{g}'(\mathfrak{z})}{\mathfrak{g}(\mathfrak{z})} + \mathfrak{Q}(\mathfrak{z}\,\mathfrak{A}, \mathfrak{B})\right) > , \quad \mathfrak{z} \in \mathbb{D},$$

where

$$Q(z; A, B) := \frac{1 + Az}{1 - Bz}, \quad z \in \mathbb{D}. \tag{4}$$

Using Ma and Minda idea, Mohd and Darus [15] introduced the class $\mathcal{S}^*_b(\phi)$, where $\phi \in \mathcal{P}^*(1)$, of all $g \in \mathcal{H}$ of the form (2) such that

$$\frac{2zg'(z)}{g(z)} + \frac{1 + z}{1 - z} \prec \phi(z), \quad z \in \mathbb{D}. \tag{5}$$

The main goal of this paper is to define and study the following class of functions.

Definition 1 Let $\phi \in \mathcal{P}^*(1)$ and $-1 < A \leq 1$, $-A < B \leq 1$. By $\mathcal{G}(\phi; A, B)$ we denote the class of all $g \in \mathcal{H}$ of the form (2) such that

$$\frac{2zg'(z)}{g(z)} + Q(z; A, B) \prec \phi(z), \quad z \in \mathbb{D}, \tag{6}$$

where Q is given by (4).

Remark 1 1. Notice that for formula (6) to be well defined, the function

$$p(z) := \frac{2zg'(z)}{g(z)} + \frac{1 + Az}{1 - Bz}, \quad z \in \mathbb{D} \tag{7}$$

should be holomorphic in $\mathbb{D}$. We easily that g does not vanish in $\mathbb{D}$. Indeed, suppose that $g(z_0) = 0$ for some $z_0 \in \mathbb{D}$. Since $g(0) = 1$, $z_0 \neq 0$. Therefore, there exist $r \in (0, 1 - |z_0|)$ and $m \in \mathbb{N}$ such that

$$g(z) = (z - z_0)^m h(z), \quad z \in \mathbb{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\},$$

where $h \in \mathcal{H}$ and $h(z_0) \neq 0$. Since

$$\frac{zg'(z)}{g(z)} = \frac{z[m(z-z_0)^{m-1}h(z) + (z-z_0)^m h'(z)]}{(z-z_0)^m h(z)}$$
$$= \frac{mz}{z-z_0} + \frac{zh'(z)}{h(z)}, \quad z \in \mathbb{D}(z_0, r) \setminus \{z_0\},$$

we see that p has a simple pole at z_0 which is impossible. Thus $g(z) \neq 0$ for $z \in \mathbb{D}$.

Note that the class $\mathcal{G}(\phi; 1, 1)$ coincides with the class $\mathcal{S}_b^*(\phi)$ due to Mohd and Darus [15], the class $\mathcal{G}(Q(\cdot; 1, 1); 1, 1)$ is identical to the class $\mathcal{G}$ and $\mathcal{G}(Q(\cdot; 1, 1); 0, 0) = \mathcal{G}(1)$. If $B = -A$, then $Q(\cdot; A, -A) \equiv 1$ and, therefore, again $\mathcal{G}(Q(\cdot; A, -A); 1, 1) = G(1)$.

It is worth reminding in this place that the function (4) was used in many papers, where different classes generated by the appropriate Carathéodory functions were considered. It is to be observed that for $B < 1$ the function Q maps univalently the disc $\mathbb{D}$ onto a disc lying in the right half-plane. However if $B = 1$, then $Q(\mathbb{D}; A, B) = \{w : \Re(\mathfrak{w}) > (- \mathfrak{A})/\}$.

2 Representation and growth theorems

Let $\phi \in \mathcal{P}^*(1)$. Since $1 \in \phi(\mathbb{D})$, it follows that $\mathbb{D}(1, a) \subset \phi(\mathbb{D})$ for some $a > 0$. Therefore, for each $n \in \mathbb{N}$ we see that $1 + az^n \prec \phi(z)$, $z \in \mathbb{D}$. Given $n \in \mathbb{N}$, consider now $g_n \in \mathcal{G}(\phi; A, B)$ defined by

$$\frac{2zg_n'(z)}{g_n(z)} + \frac{1 + Az}{1 - Bz} = 1 + az^n, \quad z \in \mathbb{D},$$

i.e., for $B \neq 0$,

$$g_n(z) = (1 - Bz)^{\frac{A+B}{2B}} \exp\left(\frac{a}{2n} z^n\right), \quad z \in \mathbb{D},$$

and for $B = 0$,

$$g_n(z) = \exp\left(-\frac{1}{2}Az + \frac{a}{2n} z^n\right), \quad z \in \mathbb{D}.$$

Hence, for $B \neq 0$,

$$|g_n(z)| \geq (1 - |B|r)^{\frac{A+B}{2B}} \exp\left(-\frac{a}{2n} r^n\right), \quad |z| = r < 1.$$

Thus, for $B < 1$, $B \neq 0$,

$$|g_n(z)| \geq (1 - |B|)^{\frac{A+B}{2B}} \exp\left(-\frac{a}{2n}\right) > 0 \quad z \in \mathbb{D}.$$

Thus, $g_n(\mathbb{D})$ is not a set starlike with respect to a boundary point at the origin, so $g_n \notin \mathcal{G}^*$ and therefore $g_n \notin \mathcal{G}$. In the same way, we deduce that $g_n \notin \mathcal{G}$ when $B = 0$. Thus, we have

Theorem 1 *Let $\phi \in \mathcal{P}^*(1)$ and $-1 < A \le 1$, $-A < B < 1$. Then*

$$\mathcal{G}(\phi; A, B) \not\subset \mathcal{G}.$$

Now we prove the representation theorem which indeed offers a useful technique to construct functions in the class $\mathcal{G}(\phi; A, B)$.

Theorem 2 *Let $\phi \in \mathcal{P}^*(1)$ and $-1 < A \le 1$, $-A < B \le 1$. Then $g \in \mathcal{G}(\phi; A, B)$ if and only if there exists a function $p \in \mathcal{H}$ such that $p \prec \phi$ and for $z \in \mathbb{D}$,*

$$g(z) = \begin{cases} (1-Bz)^{\frac{A+B}{2B}} \exp\left(\dfrac{1}{2}\displaystyle\int_0^z \frac{p(\zeta)-1}{\zeta} d\zeta\right), & B \neq 0, \\ \exp\left(-\dfrac{A}{2}z + \dfrac{1}{2}\displaystyle\int_0^z \frac{p(\zeta)-1}{\zeta} d\zeta\right), & B = 0. \end{cases} \tag{8}$$

Proof Assume that $g \in \mathcal{G}(\phi; A, B)$. Consider the function p defined by (7) which is equivalent to

$$\frac{2g'(z)}{g(z)} + \frac{A+B}{1-Bz} = \frac{p(z)-1}{z}, \quad z \in \mathbb{D}. \tag{9}$$

Clearly, $p \prec \phi$.

When $B \neq 0$, then by integration (9), one can easily get

$$\log \frac{(g(z))^2}{(1-Bz)^{\frac{A+B}{B}}} = \int_0^z \frac{p(\zeta)-1}{\zeta} \mathrm{d}\zeta, \quad z \in \mathbb{D}, \ \log 1 := 0.$$

Hence, we obtain

$$(g(z))^2 = (1-Bz)^{\frac{A+B}{B}} \exp\left(\int_0^z \frac{p(\zeta)-1}{\zeta} \mathrm{d}\zeta\right), \quad z \in \mathbb{D},$$

which yields the first formula in (8).

For $B = 0$, by virtue of (9), we obtain

$$\log(g(z))^2 + Az = \int_0^z \frac{p(\zeta)-1}{\zeta} \mathrm{d}\zeta, \quad z \in \mathbb{D}, \ \log 1 := 0.$$

Hence,

$$(g(z))^2 = \exp\left(-Az + \int_0^z \frac{p(\zeta)-1}{\zeta} d\zeta\right), \quad z \in \mathbb{D},$$

which yields the second formula in (8).

Assume now that $p \in \mathcal{H}$ is such that $p \prec \phi$ and a function g is defined by (8). Since $\phi(0) = 1$, $p(0) = 1$ and, therefore, g is holomorphic in $\mathcal{D}$. As we see, (8) can transformed to the formula (7). Hence, we deduce that $g \in \mathcal{G}(\phi; A, B)$ which completes the proof.

Define h_ϕ as a holomorphic solution of the differential equation

$$\frac{zh'_\phi(z)}{h_\phi(z)} = \phi(z), \quad z \in \mathbb{D},\ h_\phi(0) = 0,\ h'_\phi(0) = 1,$$

i.e.,

$$h_\phi(z) = z\exp\left(\int_0^z \frac{\phi(\zeta) - 1}{\zeta} d\zeta\right), \quad z \in \mathbb{D}. \tag{10}$$

Theorem 3 *Let $\phi \in \mathcal{P}^*(1)$ and $-1 < A \le 1$, $-A < B \le 1$ and let $0 < r < 1$. If $g \in \mathcal{G}(\phi; A, B)$, then for $B \ne 0$,*

$$\sqrt{\frac{-h_\phi(-r)}{r}}(1 - |B|r)^{\frac{A+B}{2B}} \le |g(z)| \le \sqrt{\frac{h_\phi(-r)}{r}}(1 + |B|r)^{\frac{A+B}{B}}, \quad |z| = r, \tag{11}$$

and for $B = 0$,

$$\sqrt{\frac{-h_\phi(-r)}{r}}\exp\left(-\frac{1}{2}|A|r\right) \le |g(z)| \le \sqrt{\frac{h_\phi(-r)}{r}}\exp\left(\frac{1}{2}|A|r\right), \quad |z| = r. \tag{12}$$

Proof Consider $B \ne 0$. Define

$$h(z) := \frac{z(g(z))^2}{(1 - Bz)^{\frac{A+B}{B}}}, \quad z \in \mathbb{D}. \tag{13}$$

By Remark 1, the function g is non-vanishing in $\mathbb{D}$. Therefore, h is holomorphic in $\mathbb{D}$. A simple calculation shows that

$$\frac{zh'(z)}{h(z)} = \frac{2zg'(z)}{g(z)} + \frac{1 + Az}{1 - Bz}, \quad z \in \mathbb{D}. \tag{14}$$

One can see from the above relation that $g \in \mathcal{G}(\phi; A, B)$ if and only if $h \in \mathcal{S}^*(\phi)$. Using the result of Ma and Minda ([14], Corollary 1') we deduce that

$$-h_\phi(-r) \le |h(z)| \le h_\phi(r), \quad |z| = r, \tag{15}$$

i.e., by (13,

$$-h_\phi(-r) \le \left|\frac{z(g(z))^2}{(1 - Bz)^{\frac{A+B}{B}}}\right| \le h_\phi(r), \quad |z| = r,$$

which yields (11).

Consider $B = 0$. Define

$$h(z) := z\exp(Az)(g(z))^2, \quad z \in \mathbb{D}. \tag{16}$$

Clearly, h is holomorphic in $\mathbb{D}$ and

$$\frac{zh'(z)}{h(z)} = \frac{2zg'(z)}{g(z)} + 1 + Az, \quad z \in \mathbb{D}.$$

Arguing as above, we see that the condition (15) holds and consequently by (16),

$$-h_\phi(-r) \le \left|z\exp(Az)(g(z))^2\right| \le h_\phi(r), \quad |z| = r,$$

which yields (12).

Remark 2 As Ma an Minda noted ([14], p. 161) a function $r \mapsto -h_\phi(-r)$ for $r \in (0,1)$, is increasing and bounded above by 1.

Theorem 4 *Let $\phi \in \mathcal{P}^*(1)$ and $-1 < A \le 1$, $-A < B < 1$ and let $0 < r < 1$. If $g \in \mathcal{G}(\phi; A, B)$, then for $B \ne 0$,*

$$\left|\arg\frac{g(z_0)}{(1 - Bz_0)^{\frac{A+B}{2B}}}\right| \le \frac{1}{2}\max_{|z|=r}\arg\frac{h_\phi(z)}{z}, \quad |z_0| = r, \tag{17}$$

and for $B = 0$,

$$\left|\arg\left(g(z_0)\exp\left(\frac{Az_0}{2}\right)\right)\right| \le \frac{1}{2}\max_{|z|=r}\arg\frac{h_\phi(z)}{z}, \quad |z_0| = r, \tag{18}$$

where $\arg 1 := 0$.

Proof Define a function h by (13) in case when $B \ne 0$, and by (16) in case when $B = 0$. Clearly, in both cases, $h \in \mathcal{S}^*(\phi)$. Thus, in view of a result due to Ma and Minda ([14], Corollary 3'), the following inequality holds

$$\left|\arg\frac{h(z_0)}{z_0}\right| \le \max_{|z|=r}\arg\frac{h_\phi(z)}{z}, \quad |z_0| = r,$$

which by substituting (13) and (16) yields (17) and (18), respectively.

3 Necessary and sufficient condition for the class $\mathcal{G}(\phi;A,B)$

We state the following result due to Ruscheweyh ([20], Theorem 1) for proving our main theorem in this section. Recall that h_ϕ which appears below is defined by (10).

Lemma 1 *Let $\phi \in \mathcal{P}^*(1)$ be a convex function and $h \in \mathcal{A}$. Then $h \in \mathcal{S}^*(\phi)$ if and only if for all $|s| \le 1$ and $|t| \le 1$,*

$$\frac{h(sz)}{h(tz)} \prec \frac{h_\phi(sz)}{h_\phi(tz)}, \quad z \in \mathbb{D}. \tag{19}$$

Theorem 5 *Let* $\phi \in \mathcal{P}^*(1)$ *be a convex function and* $-1 < A \leq 1$, $-A < B \leq 1$. *Then* $g \in \mathcal{G}(\phi; A, B)$ *if and only if for all* $|s| \leq 1$ *and* $|t| \leq 1$,

$$\frac{s}{t}\left(\frac{1-Btz}{1-Bsz}\right)^{\frac{A+B}{B}}\left(\frac{g(sz)}{g(tz)}\right)^2 \prec \frac{h_\phi(sz)}{h_\phi(tz)}, \quad z \in \mathbb{D},\ B \neq 0, \tag{20}$$

and

$$\frac{s}{t}\exp(A(s-t)z)\left(\frac{g(sz)}{g(tz)}\right)^2 \prec \frac{h_\phi(sz)}{h_\phi(tz)}, \quad z \in \mathbb{D},\ B = 0. \tag{21}$$

Proof Consider the function h defined by (13) in case when $B \neq 0$, and by (16) in case when $B = 0$. As we know, in both cases $h \in \mathcal{S}^*(\phi)$. Thus, in case $B \neq 0$ by substituting (13) into (19), we get at once (20). Similarly, when $B = 0$ by substituting (16) into (19), we obtain (21).

In the previous theorem, we assumed that the function $\phi \in \mathcal{P}^*(1)$ is convex univalent. Now, we drop the assumption of convexity on ϕ and we return to the whole class $\mathcal{P}^*(1)$.

Theorem 6 *Let* $\phi \in \mathcal{P}^*(1)$ *and* $-1 < A \leq 1$, $-A < B \leq 1$. *If* $g \in \mathcal{G}(\phi; A, B)$, *then*

$$\frac{(g(z))^2}{(1-z)^{\frac{A+B}{B}}} \prec \frac{h_\phi(z)}{z}, \quad z \in \mathbb{D},\ B \neq 0, \tag{22}$$

and

$$\exp(Az)(g(z))^2 \prec \frac{h_\phi(z)}{z}, \quad z \in \mathbb{D},\ B = 0. \tag{23}$$

Proof Consider the function h defined by (13) in case when $B \neq 0$, and by (16) in case when $B = 0$. In view of (14), we have

$$\frac{zh'(z)}{h(z)} \prec \frac{zh'_\phi(z)}{h_\phi(z)} = \phi(z), \quad z \in \mathbb{D}.$$

Since $h \in \mathcal{S}^*(\phi)$, from Theorem 1' of [14] it follows that

$$\frac{h(z)}{z} \prec \frac{h_\phi(z)}{z}, \quad z \in \mathbb{D},$$

i.e., (22) and (23) by substituting (13) and (16), respectively.

4 Initial coefficient bounds for the class $\mathcal{G}(\phi, A, B)$

In this section, making use of the following Lemmas, we obtain a few coefficient estimates for $g \in \mathcal{G}(\phi; A, B)$. Let $\mathcal{B} := \{\omega \in \mathcal{H} : |\omega(z)| < 1,\ z \in \mathbb{D}\}$ and $\mathcal{B}_0$ be the subclass of $\mathcal{B}$ of all ω such that $\omega(0) = 0$. Elements of $\mathcal{B}_0$ are known as Schwarz functions.

To prove the main theorem of this section, we will use the next two lemmas.

Lemma 2 ([8]) *If $\omega \in \mathcal{B}_0$ is of the form*

$$\omega(z) = \sum_{n=1}^{\infty} w_n z^n, \quad z \in \mathbb{D}, \tag{24}$$

then for $v \in \mathbb{C}$,

$$|w_2 - v w_1^2| \le \max\{1, |v|\}. \tag{25}$$

The following lemma was shown by Prokhorov and Szynal [17].

Lemma 3 ([17]) *If $\omega \in \mathcal{B}$, then for any real numbers q_1 and q_2, the following sharp estimate holds*:

$$|w_3 + q_1 w_1 w_2 + q_2 w_1^3| \le H(q_1, q_2), \tag{26}$$

where

$$H(q_1,q_2) := \begin{cases} 1 & \text{for } (q_1,q_2) \in D_1 \cup D_2, \\ |q_2| & \text{for } (q_1,q_2) \in \cup_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{12}, \end{cases}$$

and the sets $D_k,\ k = 1, 2, \ldots, 12$, are defined in [17].

Now we present some upper bounds for early coefficients and for the Fekete–Szegö functional in the class $\mathcal{G}(\phi; A, B)$.

Theorem 7 *Let $\phi \in \mathcal{P}^*(1)$ be of the form (1) and $-1 < A \le 1$, $-A < B \le 1$. If $g \in \mathcal{G}(\phi; A, B)$ is of the form (2), then*

$$|2d_1 + A + B| \le B_1, \tag{27}$$

$$|d_1| \le \frac{1}{2}(B_1 + A + B), \tag{28}$$

$$\left|4d_2 - 2d_1^2 + (A+B)B\right| \leq \max\{B_1, B_2\}, \tag{29}$$

$$|d_2| \leq \frac{1}{8}\left[|A^2 - B^2| + 2(A+B)B_1 + \max\left\{2, \left|2B_2 + B_1^2\right|\right\}\right], \tag{30}$$

$$\left|6d_3 - 6d_1d_2 + 2d_1^3 + (A+B)B^2\right| \leq B_1 H\left(\frac{2B_2}{B_1}, \frac{B_3}{B_1}\right) \tag{31}$$

and

$$\begin{aligned}|d_3| \leq \frac{1}{6}\Bigg[&B_1 H\left(2\frac{B_2}{B_1} + \frac{3}{4}B_1, \frac{B_3}{B_1} + \frac{3}{4}B_2 + \frac{1}{8}B_1^2\right)\\ &+ \frac{3}{8}(A+B)\max\left\{2, \left|2B_2 + B_1^2\right|\right\}\\ &+ \frac{3}{8}|A^2 - B^2|B_1 + \frac{1}{2}(A+B)\left|A^2 - 4AB + 3B^2\right|\Bigg]\end{aligned} \tag{32}$$

and for $\delta \in \mathbb{R}$,

$$\begin{aligned}|d_2 - \delta d_1^2| \leq \frac{1}{4}\Bigg[&B_1 \max\left\{1, \left|\frac{B_2}{B_1} + \left(\frac{1}{2} - \delta\right)B_1\right|\right\} + (A+B)|1 - 2\delta|B_1\\ &+ \frac{1}{2}(A+B)|A - B - 2\delta(A+B)|\Bigg].\end{aligned} \tag{33}$$

Proof By (5), there exists $w \in \mathcal{B}_0$ of the form (24) such that

$$\frac{2zg'(z)}{g(z)} + \frac{1+Az}{1-Bz} = \phi(w(z)), \quad z \in \mathbb{D}. \tag{34}$$

In view of (2), we obtain

$$\begin{aligned}\frac{2zg'(z)}{g(z)} + \frac{1+Az}{1-Bz} = &1 + (2d_1 + A + B)z + [4d_2 - 2d_1^2 + (A+B)B]z^2\\ &+ \left[6d_3 - 6d_1d_2 + 2d_1^3 + (A+B)B^2\right] + \cdots, \quad z \in \mathbb{D}.\end{aligned} \tag{35}$$

By (1) and (24) for $z \in \mathbb{D}$, we have

$$\phi(w(z)) = 1 + B_1w_1z + (B_1w_2 + B_2w_1^2)z^2 + (B_1w_3 + 2B_2w_1w_2 + B_3w_1^3)z^3 + \cdots \tag{36}$$

Using now (35), (36) and (34) by comparing corresponding coefficients, we get

$$\begin{aligned} &2d_1 + A + B = B_1 w_1, \\ &4d_2 - 2d_1^2 + (A+B)B = B_1 w_2 + B_2 w_1^2, \\ &6d_3 - 6d_1 d_2 + 2d_1^3 + (A+B)B^2 = B_1 w_3 + 2B_2 w_1 w_2 + B_3 w_1^3. \end{aligned} \tag{37}$$

Recall that $B_1 = \phi'(0) > 0$. Since

$$|w_1| \le 1, \tag{38}$$

(e.g., [4], Vol. I, p. 85]) from the first equation in (37) it follows (27) and hence (28).

The second equation in (37) together with (25) yields

$$\begin{aligned} \left|4d_2 - 2d_1^2 + (A+B)B\right| &= \left|B_1 w_2 + B_2 w_1^2\right| \\ &= B_1 \left|w_2 + \frac{B_2}{B_1} w_1\right| \le B_1 \max\left\{1, \frac{B_2}{B_1}\right\} = \max\{B_1, B_2\}, \end{aligned}$$

i.e., the inequality (29).

Substituting the first formula in (37) for d_1 into the second formula in (37), we obtain

$$4d_2 = B_1 \left[w_2 + \left(\frac{B_2}{B_1} + \frac{1}{2} B_1\right) w_1^2\right] - (A+B)B_1 w_1 + \frac{1}{2}(A^2 - B^2). \tag{39}$$

Hence, using (25) and (38), we get

$$4|d_2| \le B_1 \max\left\{1, \left|\frac{B_2}{B_1} + \frac{1}{2} B_1\right|\right\} + (A+B)B_1 + \frac{1}{2}\left|A^2 - B^2\right|,$$

which yields (30).

The third equation in (37) by applying (26) yield

$$\begin{aligned} \left|6d_3 - 6d_1 d_2 + 2d_1^3 + (A+B)B^2\right| &= \left|B_1 w_3 + 2B_2 w_1 w_2 + B_3 w_1^3\right| \\ &= B_1 \left|w_3 + 2\frac{B_2}{B_1} w_1 w_2 + \frac{B_3}{B_1} w_1^3\right| \le B_1 H\left(\frac{2B_2}{B_1}, \frac{B_3}{B_1}\right), \end{aligned}$$

i.e., the inequality (31).

Substituting formulas for d_1 and d_2 as in (39) into the third formula in (37), we obtain

$$\begin{aligned} 6d_3 = {} & B_1 w_3 + \left(2B_2 + \frac{3}{4} B_1^2\right) w_1 w_2 + \left(B_3 + \frac{3}{4} B_1 B_2 + \frac{1}{8} B_1^3\right) w_1^3 \\ & - \frac{3}{4}(A+B) B_1 w_2 - \frac{3}{8}(A+B)(2B_2 + B_1^2) w_1^2 + \frac{3}{8}(A^2 - B^2) B_1 w_1 \\ & - \frac{1}{2}(A+B)(A^2 - 4AB + 3B^2). \end{aligned}$$

Now applying (25), (26) and (38), we get

$$
\begin{aligned}
6|d_3| \le & B_1\left|w_3+\left(2\frac{B_2}{B_1}+\frac{3}{4}B_1\right)w_1w_2+\left(\frac{B_3}{B_1}+\frac{3}{4}B_2+\frac{1}{8}B_1^2\right)w_1^3\right| \\
& +\frac{3}{4}(A+B)B_1\left|w_2+\left(\frac{B_2}{B_1}+\frac{1}{2}B_1\right)w_1^2\right|+\frac{3}{8}|A^2-B^2|B_1|w_1| \\
& +\frac{1}{2}(A+B)|A^2-4AB+3B^2| \\
\le & B_1H\left(2\frac{B_2}{B_1}+\frac{3}{4}B_1,\frac{B_3}{B_1}+\frac{3}{4}B_2+\frac{1}{8}B_1^2\right) \\
& +\frac{3}{4}(A+B)\max\left\{1,\left|B_2+\frac{1}{2}B_1^2\right|\right\} \\
& +\frac{3}{8}|A^2-B^2|B_1+\frac{1}{2}(A+B)|A^2-4AB+3B^2|,
\end{aligned}
$$

which yields (32).

Using (39), the formula for d_1 by applying the inequalities (25) and (38), for $\delta \in \mathbb{R}$, we get

$$
\begin{aligned}
|d_2-\delta d_1^2| \le & \frac{1}{4}\left[B_1\left|w_2+\left(\frac{B_2}{B_1}+\left(\frac{1}{2}-\delta\right)B_1\right)w_1^2\right|\right. \\
& \left.+(A+B)|1-2\delta|B_1+\frac{1}{2}(A+B)|A-B-2\delta(A+B)|\right] \\
\le & \frac{1}{4}\left[B_1\max\left\{1,\left|\frac{B_2}{B_1}+\left(\frac{1}{2}-\delta\right)B_1\right|\right\}+(A+B)|1-2\delta|B_1\right. \\
& \left.+\frac{1}{2}(A+B)|A-B-2\delta(A+B)|\right],
\end{aligned}
$$

i.e., the inequality (33).

In case when v is a real number result of Lemma 2 can be improved in the following way.

Lemma 4 ([3]) *If $\omega \in \mathcal{B}_0$ is of the form (24), then*

$$
|w_2-vw_1^2| \le \begin{cases} -v, & v \le -1, \\ 1, & -1 \le v \le 1, \\ v, & v \ge 1. \end{cases} \tag{40}
$$

For $v<-1$ or $v>1$, equality holds if and only if $\omega(z)=z,\ z\in\mathbb{D}$, or one of its rotations. For $-1<v<1$, equality holds if and only if $\omega(z)=z^2,\ z\in\mathbb{D}$, or one of its rotations. For $v=-1$, equality holds if and only if $\omega(z)=z(\lambda+z)/(1+\lambda z),\ 0\le\lambda\le 1,\ z\in\mathbb{D}$, or one of its rotations, while for $v=1$, equality holds if and only if $\omega(z)=-z(\lambda+z)/(1+\lambda z),\ 0\le\lambda\le 1,\ z\in\mathbb{D}$, or one of its rotations.

Taking into account the above lemma, we can improve the results of Theorem 7. From (39), we have

$$4|d_2| \leq \begin{cases} B_2 + \frac{1}{2}B_1^2 + \gamma(A,B), & \frac{B_2}{B_1} + \frac{1}{2}B_1 \geq 1, \\ \gamma(A,B), & -1 \leq \frac{B_2}{B_1} + \frac{1}{2}B_1 \leq 1, \\ -B_2 - \frac{1}{2}B_1^2 + \gamma(A,B), & \frac{B_2}{B_1} + \frac{1}{2}B_1 \leq -1, \end{cases}$$

where

$$\gamma(A,B) := (A+B)B_1 + \frac{1}{2}\left|A^2 - B^2\right|.$$

Analogously, by applying (40) for $\delta \in \mathbb{R}$, we obtain

$$|d_2 - \delta d_1^2| \leq \begin{cases} \frac{B_1}{4}\left(\frac{B_2}{B_1} + \frac{B_1}{2}(1-2\delta)\right) + \Psi(A,B,\delta), & \delta \leq \sigma_1 \\ \frac{B_1}{4} + \Psi(A,B,\delta), & \sigma_1 \leq \delta \leq \sigma_2 \\ \frac{B_1}{4}\left((2\delta-1)\frac{B_1}{2} - \frac{B_2}{B_1}\right) + \Psi(A,B,\delta), & \delta \geq \sigma_2, \end{cases} \tag{41}$$

where

$$\sigma_1 := \frac{1}{B_1}\left(\frac{B_2}{B_1} - 1\right) + \frac{1}{2}, \quad \sigma_2 := \frac{1}{B_1}\left(\frac{B_2}{B_1} + 1\right) + \frac{1}{2},$$

and

$$\Psi(A,B,\delta) := |A(1-2\delta) - B(1+2\delta)|\frac{(A+B)}{8} + |1-2\delta|\frac{(A+B)B_1}{4}.$$

Since

$$6d_3 = B_1\left(w_3 + \frac{B_2}{B_1}w_1w_2 + \frac{B_3}{B_1}w_1^3\right) + 6d_1\left(d_2 - \frac{d_1^2}{3}\right) - (A+B)B^2,$$

using (40) and (41) with $\delta = 1/3$, we get

$$|d_3| \leq \frac{B_1}{6}H\left(\frac{B_2}{B_1}, \frac{B_3}{B_1}\right) + \left(\frac{|A-5B|}{3} + \frac{(A+B)B_1}{6}\right) + \frac{1}{6}(A+B)B^2$$
$$+ B_1\left(\frac{(1+B_1)(A+B)}{8}\right)\max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{6}\right|\right\}.$$

Remark 3 For the choice of $\phi(z) = (1+z)/(1-z),\ z \in \mathbb{D}$, with $A = 1$ and $B = 1$, inequalities (28, (29) and (31) reduce to the results of Abdullah et al. [1]. However, the technique adopted in this paper is different.

Compliance with ethical standards

Conflict of interest The authors declare that they do not have conflict of interests.

Ethical standard This research complies with ethical standards

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