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On excessive index of certain networks

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ABSTRACT

Matching is one of the most extensively studied areas in Computer Science and is interesting from the combinatorial point of view as well. A matching in a graph $G = (V, E)$ is a subset M of edges, no two of which have a vertex in common. A matching M is said to be *perfect* if every vertex in G is an endpoint of one of the edges in M. The excessive index of a graph G is the minimum number of perfect matchings to cover the edge set of G. The study of excessive index has a number of applications particularly in scheduling theory. In this paper we determine the excessive index for certain classes of graphs including augmented butterfly network and honeycomb network. We also prove that the excessive index does not exist for butterfly and Benes networks.

1. Introduction

Parallel processing is expected to bring a breakthrough in the increase of computing speed. But the new parallel computer systems are often expensive assets that must be shared by many users. To take full advantage of high performance computing, parallel applications must be carefully managed to guarantee quality of service and fairness in using shared resources. Thus a good scheduling strategy is indispensable in an efficient parallel computer system [7,8]. The study of excessive index has a number of applications in scheduling theory [6]. Further, in [19] Mazzuoccolo proves that the well known Berge–Fulkerson conjecture [10–12] can be stated in terms of the excessive index of cubic graphs.

A matching in a graph $G = (V, E)$ is a subset M of edges, no two of which have a vertex in common. A matching M is said to be *perfect* if every vertex in G is an endpoint of one of the edges in M. Thus a perfect matching in G is a 1-regular spanning subgraph of G. In the literature it is also known as a 1-*factor* of G. The perfect matching problem is known to be in randomized NC. Finding a perfect matching has received considerable attention in the field of parallel algorithms. Though deterministic parallel algorithms are known for planar bipartite graphs, no deterministic algorithm exists for the non-bipartite case.

A graph G is 1-*extendable* if every edge of G belongs to at least one 1-factor of G. A 1-*factor cover* of G is a set $\mathcal{F}$ of 1-factors of G such that $\bigcup_{F \in \mathcal{F}} F = E(G)$. A 1-factor cover of minimum cardinality is called an *excessive factorization* [4]. The *excessive index* of G, denoted $\chi'_e(G)$, is the size of an excessive factorization of G. We define $\chi'_e(G) = \infty$ if G is not 1-extendable. A graph G is 1-*factorizable* if its edge set $E(G)$ can be partitioned into edge-disjoint 1-factors. The problem of determining whether a regular graph G is 1-factorizable is *NP*-complete [13].

Bonisoli and Cariolaro [4] observed that the problem of determining the excessive index for regular graphs is *NP*-hard. Cariolaro and Fu [5] determined the excessive index of complete multipartite graphs, which proved to be a challenging task.

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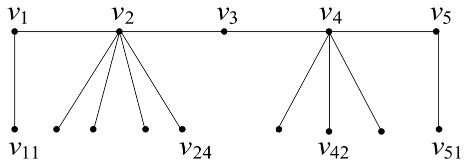

Fig. 1. $CAT(1,4,0,3,1)$.

The excessive index of a bridgeless cubic graph has been studied by Fouquet et al. [9]. Furthermore, the excessive index is being investigated for regular graphs in [1,20].

If k is a non-negative integer, a *k-edge coloring* of a graph G is a map $\Phi : E(G) \to C$, where C is a set of colors of cardinality k, such that adjacent edges of G are mapped into distinct colors. The minimum k for which a k-edge coloring of G exists is called the *chromatic index* of G and is denoted by $\chi'(G)$. In general, it is proved that $\chi'_e(G) \geqslant \chi'(G)$ and that the difference between $\chi'_e(G)$ and $\chi'(G)$ can be arbitrarily large [4]. In this paper we obtain forbidden subgraphs for a graph to be 1-extendable. We identify forbidden subgraphs in butterfly and Benes networks and thus prove that their excessive index is infinite. We also obtain the excessive index for augmented butterfly networks and honeycomb networks.

2. General results on excessive index

In the sequel let C_n be a cycle of length n. Let δ and $\triangle$ denote the minimum and maximum degree of G respectively. The vertices belonging to the edges of a matching are said to be *saturated* by the matching. The remaining vertices are *unsaturated*. A perfect matching or 1-factor is a matching that saturates every vertex of G.

Lemma 1. *(See [4].) Let G be a graph. Then $\chi'_e(G) \geqslant \triangle$.*

Lemma 2. *(See [4].) If G is regular and has even order, then $\chi'_e(G) = \triangle$ if and only if G is 1-factorizable.*

A tree is called a caterpillar if the deletion of vertices of degree one leaves a path.

Definition 1. (See [16].) Let $m \geqslant 1$, and k_i, $1 \leqslant i \leqslant m$, be non-negative integers such that $m + k_1 + \cdots + k_m \geqslant 3$. A tree which is obtained from a path $P : v_1 v_2 \cdots v_m$ by joining v_i to new vertices v_{ij}, $1 \leqslant j \leqslant k_i$, is called a caterpillar $CAT(k_1, k_2, \ldots, k_m)$. It has $m + k_1 + \cdots + k_m$ vertices. Path P is called the spine of the caterpillar. The vertices and the edges on the spine are called spine vertices and spine edges respectively. See Fig. 1.

Let $H(CAT(k_1, k_2, \ldots, k_m))$ denote a graph that contains $CAT(k_1, k_2, \ldots, k_m)$ as an induced subgraph with $\deg_H v_i$ equal to $\deg v_i$ in $CAT(k_1, k_2, \ldots, k_m)$, $1 \leqslant i \leqslant m$ and $\deg_H v_{ij} > 1$, $1 \leqslant j \leqslant k_i$, $1 \leqslant i \leqslant m$. An edge $e = (u, v)$ is a pendent edge in G if $\deg_G u > 1$ and $\deg_G v = 1$ or vice versa.

Lemma 3. *Let m be odd and let G be a graph $H(CAT(k_1, 0, k_3, 0, \ldots, k_{m-2}, 0, k_m))$. Then $\chi'_e(G) \geqslant \sum_{1 \leqslant j \leqslant m} k_j$.*

Proof. We claim that pendent edges $v_i v_{is}$ and $v_j v_{jt}$, $1 \leqslant s \leqslant k_i$, $1 \leqslant t \leqslant k_j$, $i < j$ of $CAT(k_1, 0, k_3, 0, \ldots, k_{m-1}, 0, k_m)$ belong to distinct perfect matchings of G. Suppose not, without loss of generality let $v_i v_{i1}$ and $v_j v_{j1}$, $i < j$ be edges in a perfect matching M of G such that for no vertex v_l, $i < l < j$, l odd, an edge $v_l v_{lr}$, $1 \leqslant r \leqslant k_l$ is in M. The vertices v_{i+1}, $v_{i+2}, \ldots, v_{j-1}$ are odd in number and hence cannot be saturated by M, a contradiction, thus proving our claim. $\square$

Lemma 4. *Let G be a graph $H(CAT(1, k_2, 0, k_4, 0, \ldots, k_{m-3}, 0, k_{m-1}, 1))$. Then $\chi'_e(G) \geqslant \sum_{1<j<m} k_j + 2$.*

Proof. Clearly $m - 1$ is even and hence m is odd. We claim that pendent edges $v_i v_{is}$ and $v_j v_{jt}$, $1 \leqslant s \leqslant k_i$, $1 \leqslant t \leqslant k_j$, $1 \neq i < j \neq m$ of $CAT(1, k_2, 0, k_4, 0, \ldots, k_{m-3},\ 0,\ k_{m-1},\ 1)$ belong to distinct perfect matchings of G. Suppose not, without loss of generality let $v_i v_{i1}$ and $v_j v_{j1}$, $1 \neq i < j \neq m$ be edges in a perfect matching M of G such that for no vertex v_l, $i < l < j$, l even, an edge $v_l v_{lr}$, $1 \leqslant r \leqslant k_l$ is in M. The vertices v_{i+1}, $v_{i+2}, \ldots, v_{j-1}$ are odd in number and hence cannot be saturated by M, a contradiction, thus proving our claim.

Again the spine edge $v_1 v_2$ and the pendent edge $v_i v_{is}$, i even, $3 \leqslant i \leqslant m-1$, $1 \leqslant s \leqslant k_i$ belong to distinct perfect matchings of G since the vertices v_3, $v_4, \ldots, v_{i-1}$ are odd in number. Similarly the spine edge $v_m v_{m-1}$ and the pendent edge $v_i v_{it}$, i even, $1 \leqslant i \leqslant m-2$, $1 \leqslant t \leqslant k_i$ belong to distinct perfect matchings of G. Further, the spine edges $v_1 v_2$ and $v_m v_{m-1}$ belong to distinct perfect matchings of G, since the vertices v_3, $v_4, \ldots, v_{m-2}$ are odd in number. Therefore $\chi'_e(G) \geqslant \sum_{1<j<m} k_j + 2$. $\square$

The following results exhibit forbidden induced subgraphs for a graph G to be 1-extendable.

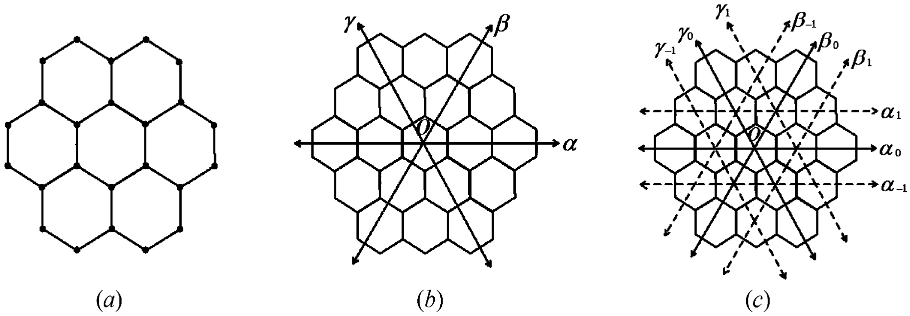

Fig. 2. (a) $HC(2)$, (b) α, β, γ axes, (c) α, β, γ lines.

Lemma 5. *Let G be a graph with a pendent edge. Then* $\chi'_e(G) = \infty$.

Proof. There does not exist any perfect matching containing an edge adjacent to a pendent edge in G. $\square$

Lemma 6. *Let G be a graph on n vertices with an even cycle* $C_m : v_1v_2 \cdots v_mv_1$, $m < n$, *as an induced subgraph. Let* $\deg_G v_1$ *and* $\deg_G v_3$ *be greater than 2 and* $\deg_G v_i = 2$, *for every* $i \neq 1, 3$. *Then* $\chi'_e(G) = \infty$.

Proof. Suppose not, let $E = \{M_1, M_2, \ldots, M_k\}$ be a 1-factor cover of G. Since $\deg_G(v_1) \geqslant 3$ there exists at least one vertex x not on C_m such that xv_1 is in at least one member of E. Without loss of generality, let $xv_1 \in M_1$. Then $v_2v_3, v_4v_5, \ldots, v_{\frac{m}{2}}v_{\frac{m}{2}+1}, \ldots, v_{m-2}v_{m-1} \in M_1$ and so the vertex v_m is not saturated by M_1, a contradiction. $\square$

Lemma 7. *Let G be a graph on n vertices with an odd cycle* $C_m : v_1v_2 \cdots v_mv_1$, $m < n$ *with* $\deg_G v_1 > 2$ *as an induced subgraph. Suppose either* $\deg_G v_2 > 2$ *or* $\deg_G v_3 > 2$ *and* $\deg_G v_i = 2$, *for* $i \neq 1, j$, $j = 2$ *or* 3. *Then* $\chi'_e(G) = \infty$.

Proof. Suppose not, let $\{M_1, M_2, \ldots, M_k\}$ be a 1-factor cover of G. If $\deg_G v_2 > 2$, without loss of generality let $v_1v_2 \in M_1$. Then $v_3v_4, v_5v_6, \ldots, v_{m-2}v_{m-1} \in M_1$ and so v_m is not saturated by M_1. Similarly if $\deg_G v_3 > 2$ and $v_3v_4 \in M_1$. Then $v_5v_6, \ldots, v_{m-2}v_{m-1}, v_mv_1 \in M_1$ and so the vertex v_2 is not saturated by M_1. In either case, M_1 is not a 1-factor, a contradiction. $\square$

In a similar way, one can prove the following slightly more general result.

Lemma 8. *Let G be a graph on n vertices with an even cycle* $C_m : v_1v_2 \cdots v_mv_1$, $m < n$, *as an induced subgraph. Let* $\deg_G v_1$ *and* $\deg_G v_i$, *i odd, be of degree greater than 2 and all other* v_j*'s of degree 2,* $j \neq 1, i$. *Then* $\chi'_e(G) = \infty$.

Lemma 9. *Let G be a graph on n vertices with an odd cycle* $C_m : v_1v_2 \cdots v_mv_1$, $m < n$, *as a subgraph. Let* $v_1, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ *be vertices of* C_m *of degree greater than* 2, *where* $i_1, i_2, \ldots, i_k$ *are all odd or all even and all other* v_j*'s be of degree 2,* $j \notin \{1, i_1, \ldots, i_k\}$. *Then* $\chi'_e(G) = \infty$.

3. Honeycomb networks

A honeycomb network can be built in various ways. The honeycomb network $HC(1)$ is a hexagon. The honeycomb network $HC(2)$ is obtained by adding a layer of six hexagons to the boundary edges of $HC(1)$ as shown in Fig. 2(a). Inductively honeycomb network $HC(n)$ is obtained from $HC(n-1)$ by adding a layer of hexagons around the boundary of $HC(n-1)$. The number of vertices and edges of $HC(n)$ are $6n^2$ and $9n^2 - 3n$ respectively [21].

Ben-Natan et al. [3] mentioned the honeycomb array as a processor network. The demand for high speed computation models has motivated the research in many domains of parallel computing. Honeycomb is an interconnection network belonging to the family of planar graphs which is not yet intensively studied as meshes for example. Honeycomb network has interesting properties: Honeycomb architecture is suitable in various applications: in wireless networks, its dual is used for cellular phone station placement [22], image processing [2], and in Chemistry as the representation of benzenoid hydrocarbons [21]. Honeycomb networks bear resemblance to atomic or molecular lattice structures of chemical compounds.

The edges of $HC(1)$ are in 3 different directions. If the perpendicular bisectors of these edges meet at point O, then O is called the center of the honeycomb network $HC(1)$. O is also considered to be the center of $HC(n)$. Through O draw three oriented lines perpendicular to the three edge directions and name them as α, β, γ axes. See Fig. 2(b). The α line through O, denoted by α_o, passes through $2n - 1$ hexagons. Any line parallel to α_o and passing through $2n - 1 - i$ hexagons is denoted by α_i, $1 \leqslant i \leqslant n - 1$ if the hexagons are above α_o and by α_{-i}, $1 \leqslant i \leqslant n - 1$ if the hexagons are below α_o. Further

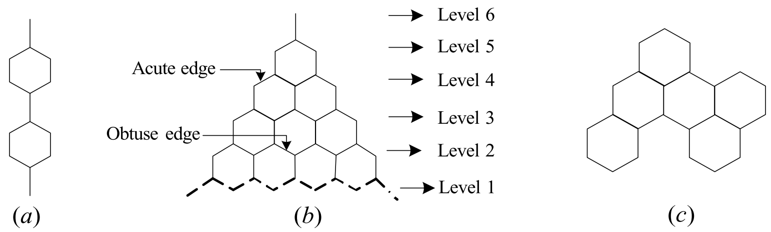

Fig. 3. (a) Hexagonal chain of dimension 3, (b) triangulene graph $T(4)$ with level 1 vertices shown as dotted line, (c) a benzenoid system.

β_j, β_{-j}, $1 \leqslant j \leqslant n-1$ are defined as lines parallel to β_0 lying to the right and left of β_0 respectively. In the same way γ_k, γ_{-k}, $1 \leqslant k \leqslant n-1$ are defined. See Fig. 2(c).

Theorem 1. *Let G be the honeycomb network HC(n), $n \geqslant 1$. Then $\chi_e'(G) \geqslant n+1$.*

Proof. We find that $CAT(1, 0, 1, 0, 1, \ldots, 0, 1)$ with the $2n+1$ vertices above α_{n-1} as spine vertices, is an induced subgraph of G. By Lemma 3, $\chi_e'(HC(n)) \geqslant n+1$. $\square$

Definition 2. Let P be a path of length $2k-1$, $k \geqslant 1$. A graph obtained from P by replacing alternate edges of P beginning from the second edge by a hexagon with a pair of diametrically opposite vertices identified with the vertices of P is called a hexagonal chain of dimension k. See Fig. 3(a).

A benzenoid system is a finite connected plane graph in which every interior face is a regular hexagon with each side of length 1. See Fig. 3(c).

Definition 3. A regular triangulene graph $T(n)$ is a benzenoid system with $\frac{n(n+1)}{2}$ regular hexagons and these hexagons are arranged in the shape of an equilateral triangle with each side having the same number of hexagons and pendent edges being added at the three corners of $T(n)$ as shown in Fig. 3(b).

For our convenience we define levels for triangulene graph. The triangulene graph $T(n)$ consists of $n+2$ levels as shown in Fig. 3(b). Edges with one end at level i and the other end at level $i+1$ are vertical edges. Other edges of $T(n)$ are called acute edges or obtuse edges according as these edges make an acute angle or an obtuse angle with an imaginary line drawn perpendicular to vertical edges. Consider the three sides of the triangulene, call one of the sides as base, one as left step path and the other as right step path.

Theorem 2. *Let G be the honeycomb network HC(n), $n \geqslant 1$. Then $\chi_e'(G) = n+1$.*

Proof. Removal of the edges perpendicular to and crossing α_0, β_0 and γ_0 disconnects the graph $HC(n)$ into 6 components such that each component is isomorphic to a triangulene graph of dimension $n-2$ and are named as $T_1(n-2)$, $T_2(n-2)$, $T_3(n-2)$, $T_4(n-2)$, $T_5(n-2)$ and $T_6(n-2)$ in the counterclockwise order around O. Thus in $HC(n)$ the six triangulene graphs $T_i(n-2)$, $1 \leqslant i \leqslant 6$ are joined in such a way that the vertices in the right step path of $T_i(n-2)$ are adjacent to the corresponding vertices (same level) of the left step path of $T_{i+1}(n-2)$, i mod 6.

Let M_{i0} and M_{i0}' be the matchings in $T_i(n-2)$, $1 \leqslant i \leqslant 6$ such that M_{i0}, i odd, consists of acute edges and M_{i0}, i even, consists of obtuse edges in level l, $1 \leqslant l \leqslant n$. Similarly M_{i0}', i odd, consists of obtuse edges and M_{i0}', i even, consists of acute edges in level l, $1 \leqslant l \leqslant n$. Thus the obtuse edges in each level l, $1 \leqslant l \leqslant n$ leave a vertex x unsaturated in the left step path and the acute edges in each level l, $1 \leqslant l \leqslant n$ leave a vertex y unsaturated in the right step path such that $xy \in E(HC(n))$. Include xy in M_{i0} and M_{i0}', $1 \leqslant i \leqslant 6$. Therefore $\bigcup_{1 \leqslant i \leqslant 6} M_{i0}$ and $\bigcup_{1 \leqslant i \leqslant 6} M_{i0}'$ form two perfect matchings and cover all the acute edges, obtuse edges and the edges perpendicular to α_0, β_0, γ_0.

Let $M_{i1}, M_{i2}, \ldots, M_{i\lfloor \frac{n}{2} \rfloor}$, $M_{i1}', M_{i2}', \ldots, M_{i\lfloor \frac{n}{2} \rfloor}'$, $1 \leqslant i \leqslant 6$ be the matchings such that if n is even, $M_{i\lfloor \frac{n}{2} \rfloor}$ is nothing but $M_{i\lfloor \frac{n}{2} \rfloor}'$ in $T_i(n-2)$, $1 \leqslant i \leqslant 6$. The hexagonal chain of dimension l, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$ are induced subgraphs of $T_i(n-2)$, $1 \leqslant i \leqslant 6$ with a pendent edge uv in the left step path or right step path whose pendent vertex u is at the level $2l$. See Fig. 4. Now M_{il} (M_{il}'), $1 \leqslant i \leqslant 6$, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$ contains the vertical edge in the left step path (right step path) of the hexagonal chain l, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$ of the triangulene graph $T_i(n-2)$, $1 \leqslant i \leqslant 6$. Thus all vertical edges in the triangulene graph are covered by M_{il} and M_{il}', $1 \leqslant i \leqslant 6$, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$ if n is even and all the vertical edges except the pendent edge with the pendent vertex at level n are covered if n is odd. In order to extend M_{il} and M_{il}', $1 \leqslant i \leqslant 6$, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$ into perfect matchings of G, some edges from M_{i0} and M_{i0}', $1 \leqslant i \leqslant 6$ are included as follows:

Note that each hexagonal chain l, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$ in $T_i(n-2)$, $1 \leqslant i \leqslant 6$ lies between level 1 and $2l$. From the level 1 edges, include acute edges to the right and obtuse edges to the left of the pendent edge in M_{il} and M_{il}', $1 \leqslant i \leqslant 6$, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$.

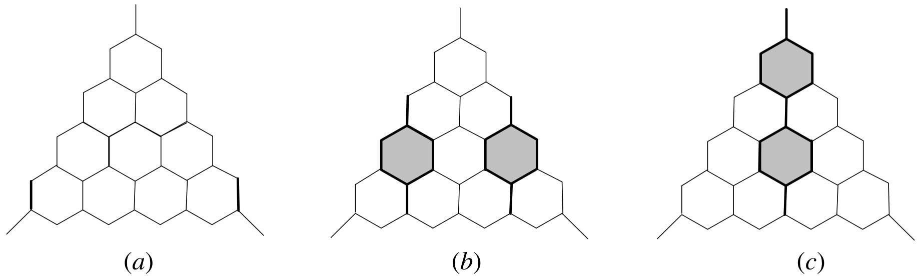

Fig. 4. Bold lines and shaded hexagons show hexagonal chains in $T(4)$ of (a) dimension 1, (b) dimension 2 and (c) dimension 3.

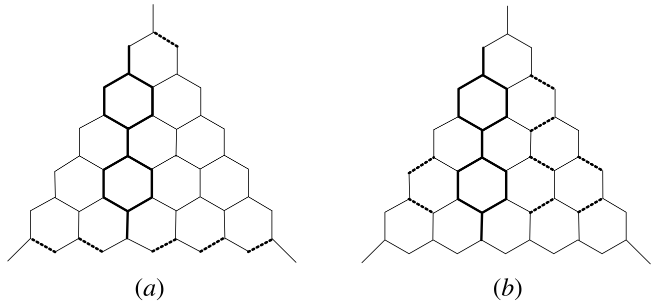

Fig. 5. Edges included in M_{il} and M'_{il} are the dotted lines selected on either side of (a) pendent edges at levels 1 and 6 (b) hexagons at levels 2, 3, 4 and 5.

Similarly from the edges of level $2l$, include obtuse edges to the right and acute edges to the left of the pendent edge in M_{il} and M'_{il}, $1 \leqslant i \leqslant 6$, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$. For edges in the levels 2 to $2l-1$, starting with level 2, choose acute edges to the right and obtuse edges to the left of the hexagons of the hexagonal chain from alternate levels and choose obtuse edges to the right and acute edges to the left of the hexagons from the remaining levels in M_{il} and M'_{il}, $1 \leqslant i \leqslant 6$, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$. See Fig. 5. Choosing acute edges to the right and the obtuse edges to the left as defined above leaves a vertex x unsaturated in the right step path and a vertex y unsaturated in the left step path of $T_i(n-2)$, $1 \leqslant i \leqslant 6$ such that $xy \in E(HC(n))$. Include xy in M_{il} and M'_{il}, $1 \leqslant i \leqslant 6$, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$.

Now consider the levels $2l+1$ to n, n even. Include the acute edges, i odd, $1 \leqslant i \leqslant 6$ and the obtuse edges, i even, $1 \leqslant i \leqslant 6$ in M_{il} and M'_{il}, $1 \leqslant i \leqslant 6$, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$. Repeat the same procedure when n is odd for the levels $2l+1$ to n in M_{il} and M'_{il}, $1 \leqslant i \leqslant 6$, $2 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$ and for the level $2l+1$ to $n-2$ in M_{i1} and M'_{i1}. The inclusion of these edges leaves a vertex x unsaturated in the right step path and the vertex y unsaturated in the left step path such that $xy \in E(HC(n))$. Include xy in M_{il} and M'_{il}, $1 \leqslant i \leqslant 6$, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$. Add the vertical edge with pendent vertex at level n in M_{i1} and M'_{i1}. Thus $\bigcup_{1 \leqslant i \leqslant 6} M_{il}$ and $\bigcup_{1 \leqslant i \leqslant 6} M'_{il}$, $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$ form a perfect matchings in $HC(n)$. $\square$

4. Butterfly and Benes networks

A multistage network consists of a series of switch stages and interconnection patterns, which allows N inputs to be connected to N outputs. The butterfly and Benes networks are important multistage interconnection networks, which possess attractive topologies for communication networks. They have been used in parallel computing systems such as IBM SP1/SP2, MIT Transit Project, and NEC Cenju-3, and used as well in the internal structures of optical couplers, e.g., star couplers [17]. Butterfly graphs were originally defined as the underlying graphs of fast Fourier transform (FFT) networks, which can perform the FFT very efficiently [14]. Manuel et al. [17] have identified new topological representations for butterfly and Benes networks as diamond representations and proved that the normal and diamond representations of butterfly and Benes networks are isomorphic.

The set V of nodes of an r-dimensional butterfly correspond to pairs $[w, i]$, where i, $0 \leqslant i \leqslant r$, is the level of a node and w is an r-bit binary number that denotes the column of the node. Two nodes $[w, i]$ and $[w', i']$ are linked by an edge if and only if $i' = i + 1$ and either:

1. w and w' are identical, or
2. w and w' differ in precisely the i'th bit.

The edges in the network are undirected. An r-dimensional butterfly is denoted by $BF(r)$. See Fig. 6(a). An r-dimensional Benes network has $2r+1$ levels, each level with $2r$ nodes. The level 0 to level r nodes in the network form an r-dimensional butterfly. The middle level of the Benes network is shared by two butterflies [15]. An r-dimensional Benes is denoted by $B(r)$. Fig. 6(b) shows a $B(2)$ network.

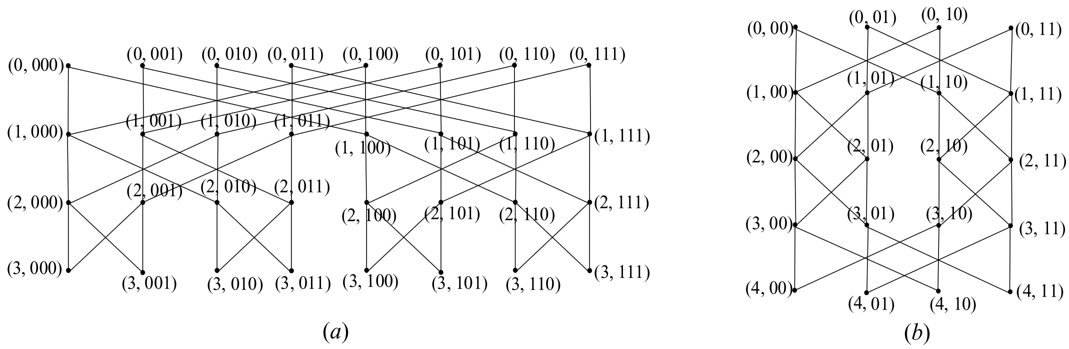

Fig. 6. (a) Butterfly $BF(3)$. (b) Benes $B(2)$.

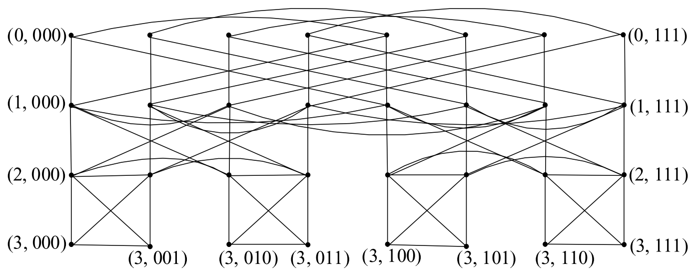

Fig. 7. Augmented butterfly network of dimension 3.

Definition 4. (See [18].) Let $n \geqslant 1$ be an integer. The vertices of the n-dimensional augmented butterfly network are the pairs (r, x) where r is a non-negative integer $0 \leqslant r \leqslant n$ called the level and $x = (x_1x_2x_3 \dots x_n)$ is a binary string of length n. In AB_n the vertex (r, x), $0 \leqslant r \leqslant n-1$, is adjacent to the vertices $(r+1,\, x)$, $(r+1,\, x_1x_2 \dots x_r\overline{x_{r+1}}x_{r+2} \dots x_n)$, $(r,\, x_1x_2 \dots \overline{x_r}x_{r+1} \dots x_n)$ and $(r,\, x_1x_2 \dots x_r\overline{x_{r+1}}x_{r+2} \dots x_n)$. Further the vertex $(n,\, x_1x_2x_3 \dots x_n)$ is adjacent to $(n,\, x_1x_2x_3 \dots \overline{x_n})$. See Fig. 7.

In particular, when $r = 0$, the vertex $(0,\, x_1x_2x_3 \dots x_n)$ is adjacent to the vertices $(1,\, x_1x_2x_3 \dots x_n)$, $(1,\, \overline{x_1}x_2x_3 \dots x_n)$ and $(0,\, \overline{x_1}x_2x_3 \dots x_n)$. Also when $r = n$, $(n,\, x_1x_2x_3 \dots x_n)$ is adjacent to the vertices $(n,\, x_1x_2x_3 \dots \overline{x_n})$, $(n-1,\, x_1x_2x_3 \dots x_n)$ and $(n-1,\, x_1x_2x_3 \dots \overline{x_n})$. Clearly AB_n has $(n+1)2^n$ vertices and $3n \times 2^n$ edges.

In the sequel, an edge joining (r, x) and $(r+1,\, x)$ is denoted by $[(r, x)(r+1,\, x)]$.

Definition 5. The edges $e = [(r, x)(r', y)]$ and $e' = [(r, u)(r', v)]$, $1 \leqslant r \leqslant n-1$ in AB_n are called mirror images if

(i) $r' = r$ or $r' = r+1$, and
(ii) $x \,\&\, u$ and $y \,\&\, v$ differ in exactly the first bit.

In particular, when $r = n$, the edge $[(n, x_1x_2 \dots x_n)(n, x_1x_2 \dots \overline{x_n})]$ is the mirror image of the edge $[(n, \overline{x_1}x_2 \dots x_n)(n, \overline{x_1}x_2 \dots \overline{x_n})]$.

Remark 1. (See [18].) The edges between (r, x) and $(r, x_1x_2 \dots x_{r-1}\overline{x_r}x_{r+1} \dots x_n)$ and between (r, x) and $(r, x_1x_2 \dots x_r\overline{x_{r+1}}x_{r+2} \dots x_n)$ are called level edges. The edges between (r, x) and $(r+1, x)$ are called straight edges while the edges between (r, x) and $(r+1,\, x_1x_2 \dots x_r\overline{x_{r+1}}x_{r+2} \dots x_n)$ are called cross edges.

Theorem 3. *Let G be a butterfly network $BF(r)$ or Benes network $B(r)$, $r \geqslant 2$. Then $\chi'_e(G) = \infty$.*

Proof. Both $BF(r)$ and $B(r)$ contain an induced 4-cycle namely $[(0, x_1x_2 \dots x_r)(1, x_1x_2 \dots x_r)]$ $[(1, x_1x_2 \dots x_r)(0, \overline{x_1}x_2 \dots x_r)]$ $[(0, \overline{x_1}x_2 \dots x_r)(1, \overline{x_1}x_2 \dots x_r)]$ $[(1, \overline{x_1}x_2 \dots x_r)(0, x_1x_2 \dots x_r)]$ with alternate vertices of degree 4 and remaining vertices of degree 2. By Lemma 6, $\chi'_e(G) = \infty$. $\square$

Theorem 4. *Let G be an augmented butterfly network AB_n, $n \geqslant 2$. Then $\chi'_e(G) = 6$.*

Proof. By Lemma 1, $\chi'_e(G) \geqslant 6$. We now construct six perfect matchings M_1, M_2, M_3, M_4, M_5 and M_6 covering all the edges of AB_n. We prove something more that all 0 level edges in AB_n belong to each of four perfect matchings, say M_1, M_2,

M_3 and M_4. We prove the result by induction on the dimension n of the augmented butterfly AB_n. When $n=2$ it can be manually checked that the following six perfect matchings cover the edge set of AB_2.

$$M_1 = \{[(1,00)(2,00)], [(1,01)(2,01)], [(1,10)(2,10)], [(1,11)(2,11)], [(0,00)(0,10)], [(0,01)(0,11)]\},$$
$$M_2 = \{[(1,00)(2,01)], [(1,01)(2,00)], [(1,10)(2,11)], [(1,11)(2,10)], [(0,00)(0,10)], [(0,01)(0,11)]\},$$
$$M_3 = \{[(1,00)(1,01)], [(1,10)(1,11)], [(2,10)(2,11)], [(2,00)(2,01)], [(0,00)(0,10)], [(0,01)(0,11)]\},$$
$$M_4 = \{[(1,00)(1,10)], [(1,01)(1,11)], [(2,10)(2,11)], [(2,00)(2,01)], [(0,00)(0,10)], [(0,01)(0,11)]\},$$
$$M_5 = \{[(0,00)(1,00)], [(1,10)(0,10)], [(2,10)(2,11)], [(2,00)(2,01)], [(1,01)(0,01)], [(0,11)(1,11)]\},$$
$$M_6 = \{[(0,10)(1,00)], [(1,10)(0,00)], [(2,10)(2,11)], [(2,00)(2,01)], [(1,01)(0,11)], [(0,01)(1,11)]\}.$$

In these perfect matchings M_i, $1 \leqslant i \leqslant 6$, the edges $[(0,00)(0,10)]$ and $[(0,01)(0,11)]$ appear in M_1, M_2, M_3 and M_4. Thus the induction hypothesis is true for $k=2$.

We assume the result for $n=k$ and prove the result for $n=k+1$. Deletion of the vertices at level 0 and the edges $[(1, x_1x_2\dots x_{k+1})(1, \overline{x_1}x_2\dots x_{k+1})]$, $x_i \in \{0,1\}$, yield two copies of augmented butterfly network of dimension k, say AB_k and AB_k'. By induction hypothesis let M_1', M_2', M_3', M_4', M_5' and M_6' be six perfect matchings that cover the edge set of AB_k and M_1'', M_2'', M_3'', M_4'', M_5'' and M_6'' be six perfect matchings that cover the edge set of AB_k' in such a way that the edges in M_i'', $1 \leqslant i \leqslant 6$ are the mirror images of the edges in M_i', $1 \leqslant i \leqslant 6$.

Now let $M_i = M_i' \cup M_i''$, $1 \leqslant i \leqslant 6$. By induction hypothesis let M_1, M_2, M_3 and M_4 be four matchings that contain the edges $[(1, x_1x_2\dots x_{k+1})\ (1, x_1\overline{x_2}\dots x_{k+1})]$, $x_i \in \{0,1\}$. Since $\deg(2, x_1x_2\dots x_{k+1}) = 6$ there exists exactly one perfect matching, say M_1, in which the edges $[(2, x_1x_2\dots x_{k+1})\ (2, x_1\overline{x_2}\dots x_{k+1})]$ and $[(1, x_1x_2\dots x_{k+1})\ (1, x_1\overline{x_2}\dots x_{k+1})]$ appear.

Delete the edges $[(1, x_1x_2\dots x_{k+1})\ (1, x_1\overline{x_2}\dots x_{k+1})]$ from M_2, M_3 and M_4 and include the edges $[(1, x_1x_2\dots x_{k+1})\ (1, \overline{x_1}x_2\dots x_{k+1})]$ and $[(0, x_1x_2\dots x_{k+1})\ (0, \overline{x_1}x_2\dots x_{k+1})]$ in M_2; $[(0, x_1x_2\dots x_{k+1})\ (1, x_1x_2\dots x_{k+1})]$ in M_3; $[(0, x_1x_2\dots x_{k+1})\ (1, \overline{x_1}x_2\dots x_{k+1})]$ in M_4 and the edges $[(0, x_1x_2\dots x_{k+1})\ (0, \overline{x_1}x_2\dots x_{k+1})]$ in M_1, M_5 and M_6. Thus the straight edges between level 0 and level 1 are covered by M_3 and the cross edges between level 0 and 1 are covered by M_4. The edges in level 0 are in $M_1 \cap M_2 \cap M_5 \cap M_6$. The edges between AB_k and AB_k' in level 1 of AB_{k+1} are covered by M_2. □

5. Conclusion

The excessive index has been determined only for few networks. It would be an interesting line of research to determine excessive index for circulant networks, torii, hexagonal networks and so on.

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