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# ON SOME MISCONCEPTIONS AND CHATTERJEE-TYPE $G$-CONTRACTION

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**Abstract:** Let $(X, d)$ be a $G$-metric space, $f$, a self-map on $X$ and $x_0 \in X$. Some misconceptions are brought about in findings of Mustafa et al [2], and a fixed point theorem for a Chatterjee-type $G$-contraction on a complete $G$-metric space is proved. More over, the unique fixed point p will be its contractive fixed point, in the sense that for each each $x_0 \in X$, the $f$-iterates $x_0, fx_0, ..., f^n x_0, ...$ converge to $p$.

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**Key Words:** $G$-metric space, Chatterjee-type $G$-contraction, fixed point, $G$-contractive fixed point

## 1. Introduction

Let $X$ be a nonempty set and $G : X \times X \times X \to \mathbb{R}$ such that

(G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

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(G4) $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(z,x,y)$
$= G(y,z,x) = G(z,y,x)$ for all $x,y,z \in X$

(G5) $G(x,y,z) \leq G(x,w,w) + G(w,y,z)$ for all $x,y,z,w \in X$

Then $G$ is called a $G$-metric on $X$ and the pair $(X,G)$, a $G$-metric space. Axiom (G4) reveals that $G$ is symmetric in the three variables $x$, $y$ and $z$, and Axiom (G5) is referred to as the rectangle inequality (of $G$). This notion was introduced by Mustafa and Sims [3] in 2006.

From the definition of $G$-metric space, it immediately follows that

$$G(x,y,y) \leq 2G(x,x,y) \text{ for all } x,y \in X. \tag{1}$$

We use the following notions, developed in [3]:

**Definition 1.1.** Let $(X,G)$ be a $G$-metric space. A $G$-ball in $X$ is defined by

$$B_G(x,r) = \big\{y \in X : G(x,y,y) < r\big\}.$$

It is easy to see that the family of all $G$-balls forms a base topology, called the $G$-metric topology $\tau(G)$ on $X$.

Also

$$\rho_G(x,y) = G(x,y,y) + G(x,x,y) \text{ for all } x,y \in X. \tag{2}$$

induces a metric on $X$, and the $G$-metric topology coincides with the metric topology induced by the metric $\rho_G$. This allows us to readily transform many concepts from metric space into the setting of $G$-metric space.

**Definition 1.2.** A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a $G$-metric space $(X,G)$ is said to be *$G$-convergent* with limit $p \in X$ if it converges to $p$ in the $G$-metric topology $\tau(G)$.

**Lemma 1.1.** *The following statements are equivalent in a $G$-metric space $(X,G)$:*

*(a)* $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *is $G$-convergent with limit* $p \in X$,

*(b)* $\lim\limits_{n\to\infty} G(x_n, x_n, p) = 0$,

*(c)* $\lim\limits_{n\to\infty} G(x_n, p, p) = 0$.

**Definition 1.3.** A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a $G$-metric space $(X,G)$ is said to be $G$-Cauchy if $\lim\limits_{n,m\to\infty} G(x_n, x_m, x_m) = 0$.

**Definition 1.4.** A $G$-metric space $(X, G)$ is said to be $G$-complete if every $G$-Cauchy sequence in $X$ converges in it.

**Definition 1.5.** Let $(X, G)$ be a $G$-metric space. A set $S \subset X$ is said to be $G$-bounded or simply bounded if there exists a positive number $M$ such that $G(x, y, z) < M$ for all $x, y, z \in S$.

**Definition 1.6.** Let $(X, G)$ be a $G$-metric space. We define the diameter of $S \subset X$ by diam $S = \delta(S) = \sup\{G(x, y, z) : x, y, z \in S\}$. The set $S$ is $G$-bounded if and only if $\delta(S) < \infty$.

As a part of an extensive research in $G$-metric spaces, we refer to a couple of interesting results from [2]. The first of them is:

**Theorem 1.1.** *Let $(X, G)$ be a complete $G$-metric space and $f : X \to X$ satisfying one of the following conditions:*

$$G(fx, fy, fy) \leq k \max\{G(x, fy, fy), G(y, fx, fx), G(y, fy, fy)\} \tag{3}$$

*or*

$$G(fx, fy, fy) \leq k \max\{G(x, x, fy), G(y, y, fx), G(y, y, fy)\} \tag{4}$$

*for all $x, y, z \in X$, where $0 \leq k < 1$. Then $f$ has a unique fixed point $p$ and $f$ is $G$-continuous at $p$.*

In the proof of Theorem 1.1, the authors used the following notations:

$$\Gamma_n = \max\{G(x_i, x_j, x_j) : \ i, j \in \{0, 1, ..., n+1\}\}, \ n \in \mathbb{N};$$
$$\Gamma = \max\{\Gamma_k : \ k = n, ..., m-1\} \text{ for } m > n,$$

where $x_n = f^n x_0$ for each $x_0 \in X$. The authors used the induction to show that

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n \Gamma_n \cdot \tag{5}$$

Then with $\Gamma = \max\{\Gamma_k : \ k = n, ..., m-1\}$ for $m > n$, and the rectangle inequality, they established

$$G(x_n, x_m, x_m) \leq \left(\tfrac{k^n}{1-k}\right) \Gamma \cdot \tag{6}$$

Further, the authors employed the limit as $n \to \infty$ in (6) to see that

$$G(x_n, x_m, x_m) \to 0. \tag{7}$$

That is $\langle x_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence.

We have two observations in these arguments. Firstly, the sets

$$A_n = \{G(x_i, x_j, x_j) : \ i, j \in \{0, 1, ..., n+1\}\}, \ n = 1, 2, 3, ... \tag{8}$$

constitute an expanding sequence of sets of nonnegative real numbers. Hence

$$\max A_n \leq \max A_{n+1} \text{ or } \Gamma_n \leq \Gamma_{n+1} \text{ for all } n. \tag{9}$$

Therefore, $\Gamma = \max\{\Gamma_n, \Gamma_{n+1}, ..., \Gamma_{m-1}\} = \Gamma_{m-1}$.

Then for $m > n$, from the repeated application of the rectangle inequality and (5), it follows that

$$\begin{aligned} G(x_n, &x_m, x_m) \\ &\leq \underbrace{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + ... + G(x_{m-1}, x_m, x_m)}_{m-n \text{ terms}} \\ &\leq k^n \left(1 + k + \cdots + k^{m-n-1}\right) \Gamma_{m-1} \leq \left(\tfrac{k^n}{1-k}\right) \Gamma_{m-1}, \end{aligned}$$

which gives (6).

**Definition 1.7.** Given $x_0 \in X$, the orbit at $x_0$ is the the sequence of iterates

$$O_f(x_0) = \{x_0, x_1, ..., x_n, ...\}, \text{ where } x_n = f x_{n-1} \text{ for } n \geq 1. \tag{10}$$

**Remark 1.1.** We now claim that the above proof of (7) requires that $O_f(x_0)$ at each $x_0 \in X$ is bounded so that $\sup[O_f(x_0)] = \delta < \infty$.

If possible, suppose that $O_f(x_0)$ is unbounded. Then there exists a positive integer $n$ such that

$$G(x_1, x_n, x_n) \geq \mu \max\{G(x_1, x_r, x_r) : 0 \leq r \leq n-1\}, \tag{11}$$

where

$$\mu = \max\left\{\tfrac{2k}{1-k}, 2k\right\}. \tag{12}$$

Now from the inequality (3), we have

$$G(x_1, x_n, x_n) \leq k \max\{G(x_0, x_n, x_n), G(x_{n-1}, x_1, x_1), G(x_{n-1}, x_n, x_n)\}. \tag{13}$$

Here $M = \max\{G(x_0, x_n, x_n), G(x_{n-1}, x_1, x_1), G(x_{n-1}, x_n, x_n)\}$.

**Case(a):** Suppose that $M = G(x_0, x_n, x_n)$. Then (13) gives

$$\begin{aligned} G(x_1, x_n, x_n) &\le kG(x_0, x_n, x_n) \\ &\le k[G(x_0, x_1, x_1) + G(x_1, x_n, x_n)] \\ &\le \left(\tfrac{k}{1-k}\right) G(x_0, x_1, x_1) \\ &\le \mu \max\{G(x_0, x_r, x_r) : 0 \le r \le n\} \\ &< G(x_1, x_n, x_n), \end{aligned}$$

which is a contradiction.

**Case(b):** Suppose that $M = G(x_{n-1}, x_1, x_1)$. Then (13) gives

$$\begin{aligned} G(x_1, x_n, x_n) &\le kG(x_{n-1}, x_1, x_1) \le 2kG(x_1, x_{n-1}, x_{n-1}) \\ &\le \mu \max\{G(x_0, x_r, x_r) : 0 \le r \le n\} < G(x_1, x_n, x_n) \end{aligned}$$

which is again a contradiction.

**Case(c):** Finally, suppose that $M = G(x_{n-1}, x_n, x_n)$. Then (13) gives

$$\begin{aligned} G(x_1, x_n, x_n) &\le kG(x_{n-1}, x_n, x_n) \\ &\le k[G(x_{n-1}, x_1, x_1) + G(x_1, x_n, x_n) \\ &\le \left(\tfrac{2k}{1-k}\right) G(x_1, x_{n-1}, x_{n-1}) \\ &\le \mu \max\{G(x_0, x_r, x_r) : 0 \le r \le n\} \\ &< G(x_1, x_n, x_n) \end{aligned}$$

which is also a contradiction.

These three contradictions prove that $O_f(x_0)$ is bounded and $\sup[O_f(x_0)] = \delta < \infty$.

Then from (6), it follows that

$$G(x_n, x_m, x_m) \le \frac{k^n \delta}{1-k}. \tag{14}$$

Applying the limit as $n \to \infty$ in (14), we get (7). This proves that $\langle x_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence.

The second result of [2] is:

**Theorem 1.2.** *Let* $(X,G)$ *be a complete* $G$*-metric space and* $f : X \to X$ *satisfying one of the following conditions:*

$$\begin{aligned} G(fx,fy,fz) \leq & k \max\{G(x,fy,fy), G(x,fz,fz), G(y,fx,fx) \\ & G(y,fz,fz), G(z,fx,fx), G(z,fy,fy)\} \end{aligned} \tag{15}$$

*or*

$$\begin{aligned} G(fx,fy,fz) \leq & k \max\{G(x,x,fy), G(x,x,fz), G(y,y,fx) \\ & G(y,y,fz), G(z,z,fx), G(z,z,fy)\} \end{aligned} \tag{16}$$

*for all* $x, y, z \in X$*, where* $0 \leq k < 1$*. Then* $f$ *has a unique fixed point* $p$ *and* $f$ *is* $G$*-continuous at* $p$*.*

**Remark 1.2.** It was claimed in [2] that Theorem 1.2 is a Corollary to Theorem 1.1, which is a misconception. In fact, the conditions (3) and (4) follow as particular cases of conditions (15) and (16) with $y = z$ respectively. Therefore, it is appropriate to assert that Theorem 1.1 is a Corollary to Theorem 1.2. The proof for Theorem 1.2 is just similar to the above proof and is omitted here.

**Remark 1.3.** Also if $k = 0$, writing $z = y = fx$ in (15) or (16), we see that $G(fx, f^2x, f^2x) = 0$ so that $f^2x = fx$ for each $x \in X$. That is, every $fx$ is a fixed point of $f$ Theorem 1.2. In other words, the fixed point is not unique in Theorem 1.2. Therefore, Theorem 1.2 in its revised form is stated as follows:

**Theorem 1.3.** *Let* $(X,G)$ *be a* $G$*-metric space and* $f : X \to X$ *satisfying either* (15) *or* (16)*, where* $0 < k < 1$*. If* $X$ *is* $G$*-complete, then* $f$ *has a unique fixed point* $p$*.*

Omitting the terms $G(x,fz,fz)$, $G(y,fx,fx)$ and $G(z,fy,fy)$ in (15), and restricting $k$ to $(0,1/3)$, say $0 \leq \gamma < 1/3$, we get

**Corollary 1.1.** *Let* $(X,G)$ *be a complete* $G$*-metric space and* $f : X \to X$ *satisfying one of the following conditions:*

$$\begin{aligned} G(fx,fy,fz) \leq & \gamma \max\{G(x,fy,fy), G(y,fz,fz), G(z,fx,fx)\} \\ & \text{for all } x,y,z \in X, \end{aligned} \tag{17}$$

*where* $0 \leq \gamma < 1/3$*. Then* $f$ *has a unique fixed point.*

Since the maximum of three nonnegative numbers cannot exceed their sum, (17) is weakened as

$$G(fx,fy,fz) \leq \gamma[G(x,fy,fy) + G(y,fz,fz) + G(z,fx,fx)],$$

$$\text{for all } x, y, z \in X, \tag{18}$$

where $0 < \gamma < 1/3$. This is analogous to Chatterjee's contraction [1] in metric space with the choice

$$\rho(fx, fy) \leq c[G(x, fy) + G(y, fx)] \text{ for all } x, y \in X, \tag{19}$$

where $0 < c < 1/2$. We therefore call $f$ satisfying (18), a Chatterjee-type $G$-contraction.

In the next section, we shall obtain a fixed point for a Chatterjee-type $G$-contraction on a complete $G$-metric space.

## 2. Main Result

The notion of $G$-contractive fixed point was introduced by Phaneendra with Kumara Swamy in [4]. In fact

**Definition 2.1.** A fixed point $p$ of $f$ on a $G$-metric space $(X, G)$ is a $G$-contractive fixed point of it if the orbit $O_f(x_0) = \langle x_0, fx_0, ..., f^n x_0, ...\rangle$ at each $x_0 \in X$ is $G$-convergent with limit $p$.

It was shown that the unique fixed point of the self-map $f$ with the following choices is a $G$-contractive fixed point.

(a) $G(fx, fy, fz) \leq qG(x, y, z)$ for all $x, y, z \in X$, where $0 \leq q < 1$,

(b) $G(fx, fy, fz) \leq aG(x, fx, fx) + bG(y, fy, fy) + cG(z, fz, fz) + eG(x, y, z)$ for all $x, y, z \in X$, where $a, b, c$ and $e$ are nonnegative real numbers with $a + b + c + e < 1$.

We now prove

**Theorem 2.1.** *Let $f$ be a Chatterjee-type contraction on a complete $G$-metric space $(X, G)$ with the choice (18). Then $f$ has a unique fixed point $p$, which will be its contractive fixed point as well.*

*Proof.* Let $x_0 \in X$ be arbitrary. Define $\langle x_n\rangle_{n=1}^{\infty} \subset X$ by

$$x_n = fx_{n-1} \text{ for } n \geq 1. \tag{20}$$

Writing $x = x_{n-1}$ and $y = z = x_n$ in (18) and then using (20) and (G5), we get

$$G(fx_{n-1}, fx_n, fx_n) = G(x_n, x_{n+1}, x_{n+1})$$

$$\begin{aligned}
&\le \gamma\big[G(x_{n-1}, fx_n, fx_n) + G(x_n, fx_n, fx_n) + G(x_n, fx_{n-1}, fx_{n-1})\big]\\
&\le \gamma\big[G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\big]\\
&\le \gamma\big[G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})\big]\\
&\le kG(x_{n-1}, x_n, x_n),
\end{aligned}$$

where $k = \frac{\gamma}{1-2\gamma}\cdot$ By induction, we have

$$G(x_n, x_{n+1}, x_{n+1}) \le k^n G(x_0, x_1, x_1) \text{ for } n \ge 1. \tag{21}$$

Now for all $n, m \in N$ with $m > n$, by $(G5)$ and (21), we obtain

$$\begin{aligned}
G(x_n, x_m, x_m) &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})\\
&\qquad + \cdots + G(x_{m-1}, x_m, x_m) \ (m-n \text{ terms})\\
&\le (\underbrace{k^n + k^{n+1} + k^{n+2} + \cdots + k^{n+(m-n-1)}}_{m-n \text{ terms}})G(x_0, x_1, x_1)\\
&= k^n(\underbrace{1 + k + k^2 + \cdots + k^{m-n-1}}_{m-n \text{ terms}})G(x_0, x_1, x_1)\\
&\le k^n \cdot \tfrac{1-k^{m-n}}{1-k} \cdot G(x_0, x_1, x_1)\\
&\le \tfrac{k^n}{1-k} \cdot G(x_0, x_1, x_1).
\end{aligned}$$

Since $k = \frac{\gamma}{1-2\gamma} < 1$, applying the limit as $n \to \infty$ in this, we find that $\lim\limits_{n,m\to\infty} G(x_n, x_m, x_m) = 0$.

Thus $\langle x_n\rangle_{n=1}^{\infty}$ is $G$-Cauchy sequence in $X$. Since $X$ is $G$-Complete, there exists a point $p \in X$ such that $\langle x_n\rangle_{n=1}^{\infty}$ is $G$-convergent to $p$. That is

$$\lim_{n\to\infty} x_{n-1} = \lim_{n\to\infty} x_n = p \tag{22}$$

Now writing $x = x_{n-1}$ and $y = z = p$ in (18),

$$\begin{aligned}
G(fx_{n-1}, fp, fp) &= G(x_n, fp, fp)\\
&\le \gamma\big[G(x_{n-1}, fp, fp) + G(p, fp, fp) + G(p, fx_{n-1}, fx_{n-1})\big]\\
&\le \gamma\big[G(x_{n-1}, fp, fp) + G(p, fp, fp) + G(p, x_n, x_n)\big]
\end{aligned}$$

Proceeding the limit as $n \to \infty$ in this and using (22), and then simplifying, we get

$$G(p, fp, fp) \le 2\gamma G(p, fp, fp). \tag{23}$$

If $fp \neq p$, (23) would imply that

$$0 < G(p, fp, fp) \leq 2\gamma G(p, fp, fp) < G(p, fp, fp),$$

which is a contradiction. Therefore, $fp = p$. That is $p$ is a fixed point of $f$. The uniqueness of the fixed point follows easily from (18).

We finally prove that $p$ is a G-Contractive fixed point of $f$. In fact, let $x_0 \in X$ be arbitrary. Writing $x = f^{n-1}x_0$ and $y = z = p$ in (18) and using (G5), we get

$$\begin{aligned} G(f^n x_0, p, p) &= G(f^n x_0, fp, fp) \\ &\leq \gamma\big[G(f^{n-1}x_0, fp, fp) + G(p, fp, fp) + G(p, f^n x_0, f^n x)\big] \\ &\leq \gamma\big[G(f^{n-1}x_0, p, p) + 2G(f^n x_0, p, p)\big] \\ &\leq \tfrac{\gamma}{1-2\gamma} G(f^{n-1}x_0, p, p). \end{aligned}$$

Since $\frac{\gamma}{1-2\gamma} < 1$, we see that $G(f^n x_0, p, p) \to 0$ as $n \to \infty$ for each $x_0 \in X$. Thus $p$ is a $G$-Contractive fixed point of $f$. □

## 3. Conclusion

Let $(X, d)$ be a $G$-metric space, $f$, a self-map on $X$ and $x_0 \in X$. Misconceptions regarding two fixed point theorems of Mustafa et al [2] have been discussed. Then a fixed point theorem for a Chatterjee-type $G$-contraction on a complete $G$-metric space has been proved. The unique fixed point will be its contractive fixed point, to which the $f$-iterates $x_0, fx_0, ..., f^n x_0, ...$ converge, for each each $x_0 \in X$.

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