EVOLUTION EQUATIONS AND
CONTROL THEORY
Volume 10, Number 2, June 2021

doi:10.3934/eect.2020066

pp. 271–296

# ON THE APPROXIMATE CONTROLLABILITY OF NEUTRAL INTEGRO-DIFFERENTIAL INCLUSIONS OF SOBOLEV-TYPE WITH INFINITE DELAY

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(Communicated by Cristina Pignotti)

Abstract. In our manuscript, we organize a group of sufficient conditions of neutral integro-differential inclusions of Sobolev-type with infinite delay via resolvent operators. By applying Bohnenblust-Karlin's fixed point theorem for multivalued maps, we proved our results. Lastly, we present an application to support the validity of the study.

1. **Introduction.** As well known, mathematical control theory has many fundamental perceptions, mainly controllability is one among them. Roughly speaking, controllability has the meaning that be capable of steer the state of the dynamical system to a suitable state using the control function involving in the system. A detailed discussion about theory and applications related to controllability, one can verify the research articles [1, 6, 7, 17, 21, 22, 23, 24, 25, 27, 30, 31, 32, 33, 34, 35, 36, 37, 38, 41, 43]. Contingent upon the idea of the issues, these equations may take different structures, for example, ordinary and partial differential equations and a few times a mix of associating frameworks of both types. It should emphasize that the thought of "aftereffect" presented in material science is significant. It isn't adequate to utilize customary or partial differential equations. A way to deal with determination this issue is to use integro-differential equations. Detailed subtleties on theoretical results related to integro-differential systems, one can view [1, 2, 6, 9, 10, 18, 25, 28, 37, 39, 40].

Neutral differential equations emerge in a lot of fields related to applied mathematics, so only neutral systems acquired much attention in the current generation. Mainly, neutral systems with or without delay help as an ideal arrangement of several partial neutral systems that appear in issues associated with heat flow in materials, visco-elasticity, propagation of waves, and several natural developments. Very useful discussion about neutral systems involving in differential equations, one can refer [10, 13, 14, 17, 27, 29, 35, 38]. Differential systems of Sobolev-type appear commonly in mathematical forms of much physical development, for example, in

2020 *Mathematics Subject Classification.* Primary: 34G10, 34G25; Secondary: 93B05.

*Key words and phrases.* Approximate controllability, Neutral system, Sobolev-type equations, Integro-differential system.

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the fluid flow through fissured rocks, thermodynamics, shear in second-order fluids, etc., one can check [1, 2, 6, 18, 23, 27, 29, 41].

This article mainly focusing on the approximate controllability results for integro-differential inclusions of Sobolev-type has the following form

$$\begin{aligned} \big(Kz(\alpha)\big)' \in \mathscr{A}\Big[z(\alpha) + \int_0^\alpha F(\alpha-\xi)z(\xi)d\xi\Big] \\ + E_2(\alpha, z_\alpha) + \mathscr{B}x(\alpha),\ \alpha \in V = [0,c], \end{aligned} \tag{1}$$

$$z(\alpha) = \psi(\alpha) \in \mathcal{P}_g, \quad \alpha \in (-\infty, 0], \tag{2}$$

and the neutral integro-differential inclusions of Sobolev-type has the following form

$$\begin{aligned} \frac{d}{d\alpha}\big(Kz(\alpha) - E_1(\alpha, z_\alpha)\big) \in \mathscr{A}\Big[z(\alpha) + \int_0^\alpha F(\alpha-\xi)z(\xi)d\xi\Big] \\ + E_2(\alpha, z_\alpha) + \mathscr{B}x(\alpha),\ \alpha \in V = [0,c], \end{aligned} \tag{3}$$

$$z(\alpha) = \psi(\alpha) \in \mathcal{P}_g, \quad \alpha \in (-\infty, 0], \tag{4}$$

where the operator $F(\alpha)$, $\alpha \in V$ is bounded on Hilbert space $\mathscr{Z}$, the state variable $z(\cdot)$ takes values in $\mathscr{Z}$ with $|\cdot|$. The operators $\mathscr{A}$ and $K$ are linear in $\mathscr{H}$. The linear operator $\mathscr{B}$ is bounded from $\mathscr{V}$ into $\mathscr{Z}$. The control function $x(\cdot)$ is presented in $L^2(V, \mathscr{V})$, a Hilbert space of admissible control functions, $E_2 : V \times \mathcal{P}_g \to BCC(\mathscr{Z})$ is a nonempty, bounded, closed and convex multivalued map, $E_1 : V \times \mathcal{P}_g \to \mathscr{Z}$. The histories $z_\alpha : (-\infty, 0] \to \mathcal{P}_g$, $z_\alpha(\theta) = z(\alpha+\theta)$, $\theta \le 0 \in \mathcal{P}_g$, where $\mathcal{P}_g$ is phase space defined later.

Our contributions are: (i) A new set of sufficient conditions are formulated and proved for the approximate controllability of neutral integro-differential inclusions of Sobolev-type with infinite delay under fundamental and straightforward assumptions on the system operators, in particular, that the corresponding linear system is approximately controllable. (ii) Further, we extend the result to obtain the conditions for the solvability of controllability results for neutral integro-differential inclusions of Sobolev-type with the infinite delay with nonlocal conditions. (iii) We show that our achievement has no analog for the concept of complete controllability. Finally, we give an example of the system which is not entirely controllable, but approximately controllable. (iv) More precisely, the controllability problem can be converted into the solvability problem of a functional operator equation in appropriate Hilbert spaces, and Bohnenblust-Karlin's fixed point theorem used to solve the problem.

We now subdivide our article into the following Sections. Section 2, we introduce a few essential facts and definitions associated with our study that is employed, which utilizes throughout the discussion of this article. The Section 3 is reserved for discussion about the approximate controllability of for integro-differential inclusions of Sobolev-type. Section 4 is reserved for discussion about the approximate controllability of neutral integro-differential inclusions. An example is given in Section 5, which verifies our theoretical results.

## 2. **Preliminaries.**

We present essential facts, ideas and lemmas desired to organize the main results of our paper. $B_p(z, \mathscr{Z})$ signifies the closed ball having center and radius $z$ and $p > 0$ respectively in $\mathscr{Z}$.

We now present $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{Z} \to \mathscr{Z}$ and $K : D(\mathscr{A}) \subset \mathscr{Z} \to \mathscr{Z}$ fulfill the being next conditions studied in [20]:

($\mathbf{E_1}$) The linear operators $\mathscr{A}$ and $K$ are closed.

($\mathbf{E_2}$) $D(K) \subset D(\mathscr{A})$ and $K$ is bijective.
($\mathbf{E_3}$) $K^{-1} : \mathscr{Z} \to D(K)$ is continuous.

Additionally, in view of ($\mathbf{E_1}$) and ($\mathbf{E_2}$) $K^{-1}$ is closed, by ($\mathbf{E_3}$) and applying closed graph theorem, one can get boundedness of $\mathscr{A}K^{-1} : \mathscr{Z} \to \mathscr{Z}$. Designate $\|K^{-1}\| = \widetilde{P}_K$ and $\|K\| = P_K$.

Presently we characterize abstract phase space $\mathcal{P}_g$ and one can refer [7, 42] for more details. Consider $g : (-\infty, 0] \to (0, +\infty)$ is a continuous function along $j = \int_{-\infty}^{0} E_1(\alpha)d\alpha < +\infty$. For any $c > 0$,

$$\mathcal{P} = \{\psi : [-c, 0] \to \mathscr{Z} \text{ such that } \psi(\alpha) \text{ is bounded and measurable}\},$$

along

$$\|\psi\|_{[-c,0]} = \sup_{\xi\in[-c,0]} ||\psi(\xi)||, \quad \forall\ \psi \in \mathcal{P}.$$

Now we characterize

$$\begin{aligned}\mathcal{P}_g =&\{\psi : (-\infty, 0] \to \mathscr{Z} \text{ such that for any } \ b > 0, \psi|_{[-b,0]} \in \mathcal{P}\\ &\text{and} \int_{-\infty}^{0} g(\xi)\|\psi\|_{[\xi,0]}d\xi < +\infty\}.\end{aligned}$$

Provided that $\mathcal{P}_g$ is endowed along

$$\|\psi\|_{\mathcal{P}_g} = \int_{-\infty}^{0} g(\xi)\|\psi\|_{[\xi,0]}d\xi,\ \forall\ \psi \in \mathcal{P}_g,$$

therefore $(\mathcal{P}_g, \|\cdot\|_{\mathcal{P}_g})$ is a Banach space.

Presently we discuss

$$\mathcal{P}'_g = \{z : (-\infty, b] \to \mathscr{H} \ \text{ such that } z|_V \in \mathcal{C}(V, \mathscr{H}),\ z_0 = \psi \in \mathcal{P}_g\}.$$

Fix $\|\cdot\|'_g$ be a seminorm in $\mathcal{P}'_g$ characterize by

$$\|z\|'_g = \|\psi\|_{\mathcal{P}_g} + \sup\{||z(\xi)|| : \xi \in [0, c]\},\ z \in \mathcal{P}'_g.$$

**Lemma 2.1.** [7] *Assume $z \in \mathcal{P}'_g$, then for $\alpha \in V,\ z_\alpha \in \mathcal{P}_g$. Moreover,*

$$j|z(\alpha)| \le \|z_\alpha\|_{\mathcal{P}_g} \le \|\psi\|_{\mathcal{P}_g} + j \sup_{\xi\in[0,\alpha]} |z(\xi)|,$$

*where $j = \int_{-\infty}^{0} E_1(\alpha)d\alpha < +\infty$.*

Consider the linear differential equation

$$z'(\alpha) = \mathscr{A}\Big(z(\alpha) + \int_0^\alpha F(\alpha-\xi)z(\xi)d\xi\Big) \tag{5}$$

which obtains a resolvent operator.

**Definition 2.2.** [15] A family of bounded linear operators $M(\alpha) \in K(\mathscr{Z}), \alpha \in V$ is called a resolvent operator for (5) provided that

(**a**) $M(0) = I$ (the identity operator on $\mathscr{Z}$),
(**b**) for all $z \in \mathscr{Z}$, $M(\alpha)z$ is continuous for $\alpha \in V$,
(**c**) $M(\alpha) \in K(Y)$, $\alpha \in V$. For $y \in Y$, $M(\alpha)y \in C^1([0, c], \mathscr{Z}) \cap C([0, c], Y)$ and

$$\begin{aligned}M'(\alpha)y &= \mathscr{A}K^{-1}\Big[M(\alpha)y + \int_0^\alpha F(\alpha-\xi)M(\xi)yd\xi\Big]\\ &= M(\alpha)\mathscr{A}K^{-1}y + \int_0^\alpha M(\alpha-\xi)\mathscr{A}K^{-1}F(\xi)yd\xi,\ \alpha \in V.\end{aligned}$$

$M(\alpha)$ can be obtained from $\mathscr{A}K^{-1}$. More details regarding this, one can view [9, 26, 12, 28].

Additionally, we consider the following assumptions given in [15]:

($\mathbf{E_4}$) $M(\alpha) \in K(\mathscr{Z})$, $\alpha \in V$. Also, $M(\alpha) : Y \to Y$ and for $z(\cdot)$ continuous in $Y$, $\mathscr{A}M(\cdot)z(\cdot) \in L^1([0,c],\mathscr{Z})$. For $z \in \mathscr{Z}$, $M'(\alpha)z$ is continuous in $\alpha \in V$, where $K(M)$ is the space of all bounded linear operators on $\mathscr{Z}$ and $Y$ is the Hilbert space formed from $D(\mathscr{A})$, the domain of $\mathscr{A}$, endowed with the graph norm and $\mathscr{A}K^{-1}M = M\mathscr{A}K^{-1}$.

**Theorem 2.3.** *If* ($\mathbf{E_4}$) *is satisfied, then the system* (5) *permits* $(M(\alpha))_{\alpha\geq 0}$.

In view of [8, 16], we present some fundamental ideas and facts related to multimaps.

A multimap $\mathcal{K} : \mathscr{Z} \to 2^{\mathscr{Z}} \setminus \{\emptyset\}$ is convex (closed) valued provided that $\mathcal{K}(z)$ is convex (closed) for every $z \in \mathscr{Z}$. $\mathcal{K}$ is bounded on bounded sets provided that $\mathcal{K}(H) = \bigcup_{z\in H}\mathcal{K}(z)$ is bounded in $\mathscr{Z}$ for any bounded set $H$ of $\mathscr{Z}$, i.e., $\sup_{z\in H}\Big\{\sup\{\|z\| : z \in \mathcal{K}(z)\}\Big\} < \infty$.

**Definition 2.4.** The multimap $\mathcal{K}$ is said to be upper semicontinuous on $\mathscr{Z}$ provided that for every $z_0 \in \mathscr{Z}$, $\mathcal{K}(z_0)$ is a nonempty closed subset of $\mathscr{Z}$ and provided that for each open set $H$ of $\mathscr{Z}$ including $\mathcal{K}(z_0)$, there exists an open neighborhood $V$ of $z_0$ such that $\mathcal{K}(V) \subseteq H$.

**Definition 2.5.** The multimap $\mathcal{K}$ is said to be completely continuous provided that $\mathcal{K}(H)$ is relatively compact for every bounded subset $H$ of $\mathscr{Z}$.

Provided that $\mathcal{K}$ is completely continuous with nonempty values, then $\mathcal{K}$ is upper semicontinuous, if and only if $\mathcal{K}$ has a closed graph, that is, $z_n \to z_*$, $v_n \to v_*$, $v_n \in \mathcal{K}z_n$ imply $z_* \in \mathcal{K}z_*$. The multimap $\mathcal{K}$ has a fixed point provided that there is a $z \in \mathscr{Z}$ such that $z \in \mathcal{K}(z)$.

We present two appropriate operators and fundamental assumptions about the operators as follows:

$$\aleph_0^c = \int_0^\alpha K^{-1}M(c-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)d\xi : \mathscr{Z} \to \mathscr{Z},$$
$$R(\beta,\aleph_0^c) = (\beta I + \aleph_0^c)^{-1} : \mathscr{Z} \to \mathscr{Z}.$$

In the above, $\mathscr{B}^*$ stands for adjoint of $\mathscr{B}$ and $M^*(c)$ stands for adjoint of $M(c)$. We can easily conclude the linear operator $\aleph_0^c$ is bounded.

For examining approximate controllability of (3)-(4), we establish the being next assumption:

$\mathbf{H_0}$ $\alpha R(\beta,\aleph_0^c) \to 0$ as $\beta \to 0^+$ in the strong operator topology.

By referring [22], Hypothesis $\mathbf{H_0}$ holds if and only if linear system

$$(Kz(\alpha))' = \mathscr{A}\Big[z(\alpha) + \int_0^\alpha F(\alpha-\xi)z(\xi)d\xi\Big] + (\mathscr{B}x)(\alpha), \quad \alpha \in [0,c], \tag{6}$$
$$z(0) = z_0 \tag{7}$$

is approximately controllable on $V$.

**Lemma 2.6.** [19] *Assume that* $V$ *be a compact real interval, The nonempty set* $BCC(\mathscr{Z})$ *be bounded, closed and convex subset of* $\mathscr{Z}$ *and the multimap* $\mathscr{H}$ *fulfilling* $\mathscr{H} : V \times \mathscr{Z} \to BCC(\mathscr{Z})$ *is measurable to* $\alpha$ *for each fixed* $z \in \mathscr{Z}$, *upper* *semicontinuous to* $z$ *for each* $\alpha \in V, z \in \mathcal{C}$ *the set*

$$S_{E_2,z} = \{h \in L^1(V, \mathscr{Z}) : h(\alpha) \in \mathscr{H}(\alpha, z(\alpha)),\ \alpha \in V\}$$

*is nonempty. Assume that the linear operator* $\mathcal{H}$ *is continuous from* $L^1(V, \mathscr{Z})$ *to* $\mathcal{C}$*, then*

$$\mathcal{H} \circ K_{\mathcal{H}} : \mathcal{C} \to BCC(\mathcal{C}),\ z \to (\mathcal{H} \circ K_{\mathscr{H}})(z) = \mathcal{H}(S_{E_2,z}),$$

*is closed in* $\mathcal{C} \times \mathcal{C}$*.*

**Lemma 2.7.** [3, Bohnenblust-Karlin's fixed point theorem]. *Assume that the nonempty set* $B$ *is a subset of* $\mathscr{Z}$*, which is bounded, closed and convex. Assume* $F : B \to 2^{\mathscr{Z}} \setminus \{\emptyset\}$ *is upper semicontinuous with closed, convex values and such that* $F(B) \subseteq B$ *and* $F(B)$ *is compact, then* $F$ *has a fixed point.*

3. **Approximate controllability.** This section mainly focusing approximate controllability of (1)-(2). To begin with, we characterize the mild solution of (1)-(2).

**Definition 3.1.** A function $z : (-\infty, c] \to \mathscr{Z}$ is called a mild solution of (1)-(2) provided that $z_0 = \psi \in \mathcal{P}_g$ on $(-\infty, 0]$ and

$$\begin{aligned} z(\alpha) =& K^{-1}M(\alpha)K\psi(0) + \int_0^\alpha K^{-1}M(\alpha - \xi)h(\xi)d\xi \\ &+ \int_0^\alpha K^{-1}M(\alpha - \xi)\mathscr{B}x(\xi)d\xi, \quad \alpha \in V, \end{aligned}$$

is fulfilled.

We introduce the being next assumptions to discuss our main results of this section:

**$\mathbf{H_1}$** $M(\alpha)$, $\alpha > 0$ is compact.

**$\mathbf{H_2}$** $E_2 : V \times \mathcal{P}_g \to BCC(\mathscr{Z})$ is $L^1$-Caratheodory and which satisfies:
For every $\alpha \in V$, $E_2(\alpha, \cdot)$ is u.s.c; for every $z \in \mathcal{P}_g$, $E_2(\cdot, z)$ is measurable and $z \in \mathcal{P}_g$,

$$S_{E_2,z} = \Big\{h \in L^1(V, \mathscr{Z}) : h(\alpha) \in E_2(\alpha, z_\alpha),\ \text{for almost everywhere } \alpha \in V\Big\},$$

is nonempty.

**$\mathbf{H_3}$** For $p > 0$, there exists $\beta_p : V \to \mathbb{R}^+$ such that

$$\sup\Big\{\|h\| : h(\alpha) \in E_2(\alpha, z_\alpha)\Big\} \le \beta_{p'}(\alpha),$$

for a.e. $\alpha \in V$.

**$\mathbf{H_4}$** $\xi \to \beta_r(\xi) \in L^1(V, \mathbb{R}^+)$ and there exists $\gamma > 0$ such that

$$\lim_{p\to\infty} \frac{\int_0^\alpha \beta_p(\xi)d\xi}{p} = \gamma < \infty.$$

**$\mathbf{H_5}$** $\mathscr{A}K^{-1}$ is the infinitesimal generator of $M(\alpha)$ in $\mathscr{Z}$ and $P > 0$ and $P_F > 0$ such that

$$\|M(\alpha)\| \le P, \quad \|F(\alpha)\| \le P_F,\ \forall\ \alpha \in V.$$

To demonstrate (1)-(2) is approximately controllable, provided that for all $\beta > 0$, there exists a function $x(\cdot)$ which is continuous such that

$$z(\alpha) = K^{-1}M(\alpha)K\psi(0) + \int_0^\alpha K^{-1}M(\alpha - \xi)h(\xi)d\xi$$

$$+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}x_\beta(\xi,z)d\xi,\ h\in S_{E_2,z}, \tag{8}$$

$$x_\beta(\alpha,z)=\mathscr{B}^*K^{-1}\mathscr{H}(c-\alpha)\mathcal{R}(\beta,\aleph_0^c)q(z(\cdot)), \tag{9}$$

where

$$q(z(\cdot))=z_c-K^{-1}M(\alpha)K\psi(0)-\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi.$$

**Theorem 3.2.** *If* $\mathbf{H_0}$*-*$\mathbf{H_5}$ *are fulfilled, then* (1)*-*(2) *has a mild solution on* $V$*, given that*

$$\widetilde{P}_K P\Big(1+\frac{1}{\alpha}\widetilde{P}_K P^2 P_{\mathscr{B}}^2 c\Big)\gamma j<1. \tag{10}$$

*In the above,* $P_{\mathscr{B}}=\|\mathscr{B}\|$*.*

*Proof.* For any $\varrho>0$, we look at the operator $\bigwedge^\varrho:\mathcal{P}_g'\to 2^{\mathcal{P}_g'}$ described by $\bigwedge^\varepsilon x$ the set of $z\in\mathcal{P}_g'$ such that

$$z(\alpha)=\begin{cases}\psi(\alpha), \quad \alpha\in(-\infty,0],\\ K^{-1}M(\alpha)K\psi(0)+\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi\\ +\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}x_\beta(\xi,z)d\xi,\ \alpha\in V,\end{cases}$$

where $h\in S_{E_2,z}$. To demonstrate $\bigwedge^\varrho$ has a fixed point and we conclude it is the solution of (3)-(4). Obviously, $z_1=z(c)\in(\bigwedge^\varrho z)(c)$, which means that $x_\varrho(z,\alpha)$ drives (1)-(2) from $z_0\to z_c$ in finite time $c$.

For $\psi\in\mathcal{P}_g$, we now characterize $\widehat{\psi}$ as

$$\widehat{\psi}(\alpha)=\begin{cases}\psi(\alpha), \quad \alpha\in(-\infty,0],\\ K^{-1}M(\alpha)K\psi(0), \quad \alpha\in V,\end{cases}$$

then $\widehat{\psi}\in\mathcal{P}_g'$. Let $z(\alpha)=y(\alpha)+\widehat{\psi}(\alpha)$, $-\infty<\alpha\le c$. We now conclude that $y$ fulfills $y_0=0$ and

$$\begin{aligned}y(\alpha)=&\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi\\ &+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\Upsilon_0^c)\Big[z_c-K^{-1}\mathscr{H}(c)K\psi(0)\\ &-\int_0^\alpha K^{-1}M(c-\eta)h(\eta)d\eta\Big](\xi)d\xi, \quad \alpha\in V.\end{aligned}$$

if and only if $x$ satisfies

$$\begin{aligned}z(\alpha)=&K^{-1}M(\alpha)K\psi(0)+\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi\\ &+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\Upsilon_0^c)\Big[z_c-K^{-1}\mathscr{H}(c)K\psi(0)\\ &-\int_0^\alpha K^{-1}M(c-\xi)h(\eta)d\eta\Big](\xi)d\xi, \quad \alpha\in V.\end{aligned}$$

and $z(\alpha)=\psi(\alpha)$, $\alpha\in(-\infty,0]$.

Let $\mathcal{P}''_g = \{y \in \mathcal{P}'_g : y_0 = 0 \in \mathcal{P}_g\}$. For any $y \in \mathcal{P}''_g$,

$$\begin{aligned}\|y\|_c &= \|y_0\|_{\mathcal{P}_g} + \sup\{\|y(\xi)\| : 0 \le \xi \le c\}\\ &= \sup\{\|y(\xi)\| : 0 \le \xi \le c\},\end{aligned}$$

therefore $(\mathcal{P}''_g, \|\cdot\|_b)$ is a Banach space. Fix $B_p = \{y \in \mathcal{P}''_g : \|y\|_c \le p\}$ for $p > 0$, then $B_p \subseteq \mathcal{B}''_h$ is uniformly bounded, and for $y \in B_p$, in view of Lemma 2.1, one can get

$$\begin{aligned}\|y_\alpha + \hat{\psi}_\alpha\|_{\mathcal{P}_g} &\le \|y_\alpha\|_{\mathcal{P}_g} + \|\hat{\psi}_\alpha\|_{\mathcal{P}_g}\\ &\le j(p + M|\psi(0)|) + \|\psi\|_{\mathcal{P}_g} = p'. \end{aligned} \tag{11}$$

Define $\Psi : \mathcal{P}''_g \to \mathcal{P}''_g$ provided by $\Psi y$ the set of $\overline{z} \in \mathcal{P}''_g$ such that

$$\overline{z}(\alpha) = \begin{cases} 0, \quad \alpha \in (-\infty, 0],\\ K^{-1}M(\alpha)K\psi(0) + \int_0^\alpha K^{-1}M(\alpha-\eta)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta, \Upsilon_0^c)\Big[z_c\\ -K^{-1}M(c)K\psi(0) - \int_0^\alpha K^{-1}M(c-\eta)h(\eta)d\eta\Big](\xi)d\xi\\ + \int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi,\ \alpha \in V. \end{cases}$$

Clearly, a fixed point of $\Psi^\varrho$ exists if and only if a fixed point of $\Pi$ exists. So, our goal is to show a fixed point of $\Pi$ exists. We now split our proof into five steps for comfort.

**Step 1.** $\Psi$ is convex for all $z \in B_p$. Actually, if $\phi_1$, $\phi_2$ then there exists $h_1$, $h_2 \in S_{E_2,z}$ such that for each $\alpha \in V$, we have

$$\begin{aligned}\phi_i(\alpha) =& \int_0^\alpha K^{-1}M(\alpha-\xi)h_i(\xi)d\xi\\ &+ \int_0^\alpha K^{-1}M(\alpha-\eta)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta, \Upsilon_0^c)\Bigg[z_c - K^{-1}M(c)K\psi(0)\\ &- \int_0^\alpha K^{-1}M(c-\eta)h_i(\eta)d\eta\Bigg](\xi)d\xi, \quad i = 1, 2.\end{aligned}$$

Assume $\delta \in [0,1]$. Then for each $\alpha \in V$, one can get

$$\begin{aligned}(\delta\phi_1 + (1-\delta)\phi_2)(\alpha) =& \int_0^\alpha K^{-1}M(\alpha-\xi)[\delta h_1(\xi) + (1-\delta)h_2(\xi)]d\xi\\ &+ \int_0^\alpha K^{-1}M(\alpha-\eta)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta, \Upsilon_0^c)\Bigg[z_c - K^{-1}M(c)K\psi(0)\\ &- \int_0^\alpha K^{-1}M(c-\xi)[\delta h_1(\xi) + (1-\delta)h_2(\eta)]d\eta\Bigg](\xi)d\xi.\end{aligned}$$

We can easily prove $S_{E_2,z}$ is convex since $E_2$ has convex values. Therefore, $\delta h_1 + (1-\delta)h_2 \in S_{E_2,z}$. Consequently, $\delta\psi_1 + (1-\delta)\psi_2 \in \Pi(z)$.

**Step 2.** To prove $p > 0$ such that $\Pi(B_p) \subseteq B_p$. Otherwise, there exists $\varrho > 0$ such that for all $p > 0$ and $\alpha \in V$, there exists $y_p \in B_p$, but $\Pi(y_p) \notin B_p$, that is, $|\Pi(y_p)(\alpha)| > p$ for some $\alpha \in V$. For such $\varrho > 0$,

$$\begin{aligned}p <& |(\Psi y^p)(\alpha)|\\ \le& \Big|\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi\Big| + \Big|\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}x_\beta(\xi, y+\widehat{\psi})d\xi\Big|\end{aligned}$$

$$\begin{aligned}
&\leq K^{-1}P\int_0^{\alpha}\beta_{p'}(\xi)d\xi+\frac{1}{\alpha}K^{-1}P^2P_{\mathscr{B}}^2c\Big[K^{-1}P\|\psi(0)\|+K^{-1}P\int_0^{\alpha}\beta_{p'}(\xi)d\xi\Big]\\
&\leq \widetilde{P}_KP\int_0^{\alpha}\beta_{p'}(\xi)d\xi+\frac{1}{\alpha}\widetilde{P}_KP^2P_{\mathscr{B}}^2c\Big[|x_1|+\widetilde{P}_KP\|(0)\|+\widetilde{P}_KP\int_0^{\alpha}\beta_{p'}(\xi)d\xi\Big]\\
&\leq \widetilde{P}_KP\Big(1+\frac{1}{\alpha}\widetilde{P}_KP^2P_{\mathscr{B}}^2c\Big)\Big[\int_0^{\alpha}\beta_{p'}(\xi)d\xi\Big]+\widehat{M},
\end{aligned}$$

where $\widehat{P_c}$ is independent of $p$. Separating the two sides of the above mentioned inequality by $p$ and perceiving that $p'=j(p+K^{-1}M|\psi(0)|)+\|\psi\|_{\mathcal{P}_g}$ as $p\to\infty$, we acquire that

$$\liminf_{p\to+\infty}\frac{\int_0^{\alpha}\beta_{p'}(\xi)d\xi}{p}=\liminf_{p\to+\infty}\left(\frac{\int_0^{\alpha}\beta_{p'}(\xi)d\xi}{p'}\cdot\frac{p'}{p}\right)=\gamma j,$$

Thus, we have

$$\widetilde{P}_KP\Big(1+\frac{1}{\alpha}\widetilde{P}_KP^2P_{\mathscr{B}}^2c\Big)\gamma j\geq 1$$

and contradicts to (19). So, $p>0$ and some $h\in S_{E_2,z}$, $\Pi(B_p)\subseteq B_p$.

**Step 3.** $\Psi(B_p)$ is equicontinuous. In fact, assume $\varrho>0$ be small, $0<\alpha_1<\alpha_2\leq c$. For each $y\in B_p$ and $\overline{z}\in\Psi_1y$, there exists $h\in S_{E_2,z}$ such that for every $\alpha\in V$, one can get

$$\begin{aligned}
|\overline{z}(\alpha_2)-\overline{z}(\alpha_1)|=&\Big|\int_{\alpha_1}^{\alpha_2}K^{-1}M(\alpha_2-\xi)h(\xi)d\xi\Big|\\
&+\Big|\int_{\alpha_1-\varepsilon}^{\alpha_1}K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]h(\xi)d\xi\Big|\\
&+\Big|\int_0^{\alpha_1-\varepsilon}K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]h(\xi)d\xi\Big|\\
&+\Big|\int_0^{\alpha_1-\varepsilon}K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]\mathscr{B}x_{\beta}^{p}(\eta,x)d\eta\Big|\\
&+\Big|\int_{\alpha_1-\varepsilon}^{\alpha_1}K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]\mathscr{B}x_{\beta}^{p}(\eta,x)d\eta\Big|\\
&+\Big|\int_{\alpha_1}^{\alpha_2}K^{-1}M(\alpha_2-\xi)\mathscr{B}x_{\beta}^{p}(\eta,x)d\eta\Big|\\
\leq&\widetilde{P}_KP\int_{\alpha_1}^{\alpha_2}\beta_{p'}(\xi)d\xi+\widetilde{P}_K\int_{\alpha_1-\varepsilon}^{\alpha_1}\|M(\alpha_2-\xi)-M(\alpha_1-\xi)\|\beta_{p'}(\xi)d\xi\\
&+\widetilde{P}_K\int_0^{\alpha_1-\varepsilon}\|M(\alpha_2-\xi)-M(\alpha_1-\xi)\|\beta_{p'}(\xi)d\xi\\
&+\widetilde{P}_KP_B\int_0^{\alpha_1-\varepsilon}\|M(\alpha_2-\eta)-M(\alpha_1-\eta)\|\Big[|x_1|+\widetilde{P}_KP\|\psi(0)\|\\
&+\widetilde{P}_KP\int_0^{\alpha}\beta_{p'}(\xi)d\xi\Big](\xi)d\xi\\
&+\widetilde{P}_KP_B\int_{\alpha_1-\varepsilon}^{\alpha_1}\|M(\alpha_2-\eta)-M(\alpha_1-\eta)\|\Big[|x_1|+\widetilde{P}_KP\|\psi(0)\|
\end{aligned}$$

$$+ \widetilde{P}_K P \int_0^{\alpha} \beta_{p'}(\xi)d\xi \Bigg](\xi)d\xi$$

$$+ \widetilde{P}_K P M_B \int_{\alpha_1}^{\alpha_2} \Bigg[|x_1| + \widetilde{P}_K P\|\psi(0)\| + \widetilde{P}_K P \int_0^{\alpha} \beta_{p'}(\xi)d\xi\Bigg](\xi)d\xi. \tag{12}$$

Hence, for $\varrho > 0$, one can confirm that (12) tends to zero as $\alpha_2 \to \alpha_1$. Then again, the compactness of $\mathscr{M}(\alpha)$ for $\alpha > 0$ gives continuity in uniform operator topology. Hence $\Pi$ maps $B_p$ into an equicontinuous family of functions.

**Step 4.** $\Pi(\alpha) = \{\phi(\alpha) : \phi \in \Psi(B_p)\}$ is relatively compact in $\mathscr{X}$.

Assume $\alpha \in (0, c]$, $\varrho > 0$, $0 < \varrho < \alpha$. Now $z \in B_p$, we provide

$$\phi_\varrho(\alpha) = \int_0^{\alpha-\varrho} K^{-1}M(\alpha-\xi)h(\xi)d\xi$$
$$+ \int_0^{\alpha-\varrho} K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^* K^{-1}M^*(c-\alpha)\mathcal{R}(\beta, \aleph_0^c)\Bigg[z_c$$
$$- K^{-1}M(c)K\psi(0) - \int_0^{\alpha} K^{-1}M(c-\eta)h(\eta)d\eta\Bigg](\xi)d\xi.$$

Because $M(\alpha)$ is compact, $\Pi_\varrho(\alpha) = \{\phi_\varrho(\alpha) : \phi_\varrho \in \Psi(B_p)\}$ is relatively compact in $\mathscr{X}$ for each $\varrho$, $0 < \varrho < \alpha$. Furthermore, for all $0 < \varrho < \alpha$, one can get

$$|\phi(\alpha) - \phi_\varrho(\alpha)| \le \int_{\alpha-\varrho}^{\alpha} K^{-1}M(\alpha-\xi)h(\xi)d\xi$$
$$+ \int_{\alpha-\varrho}^{\alpha} K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^* K^{-1}M^*(c-\alpha)\mathcal{R}(\beta, \aleph_0^c)\Bigg[z_c$$
$$- K^{-1}M(c)K\psi(0) - \int_0^{\alpha} K^{-1}M(c-\eta)h(\eta)d\eta\Bigg](\xi)d\xi.$$

Thus, there exist relatively compact sets arbitrarily close $\bigwedge(\alpha) = \{\psi(\alpha) : \psi \in \Pi(B_p)\}$, $\widetilde{\bigwedge}(\alpha)$ is relatively compact in $\mathscr{X}$ for all $\alpha \in [0, c]$. Because the compactness at $\alpha = 0$, therefore $\bigwedge(\alpha)$ is relatively compact in $\mathscr{X}$, for all $\alpha \in [0, c]$.

**Step 5.** $\Psi$ has a closed graph.

Assume that $y_n \to y_*$ as $n \to \infty$, $\overline{z}_n \in \Pi y_n$, for all $y_n \in B_p$, and $\overline{z}_n \to \overline{z}_*$ as $n \to \infty$. Now, we demonstrate $\overline{z}_* \in \Pi y_*$. Because $\overline{z}_n \in \Pi y_n$, there exists $h_n \in S_{E_2, y_n}$ such that

$$\overline{z}_n(\alpha) = \int_0^{\alpha} K^{-1}M(\alpha-\xi)h_n(\xi)d\xi$$
$$+ \int_0^{\alpha} K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^* K^{-1}M^*(c-\alpha)\mathcal{R}(\beta, \aleph_0^c)\Bigg[z_c - K^{-1}M(c)K\psi(0)$$
$$- \int_0^{\alpha} K^{-1}M(c-\xi)h_n(\eta)d\eta\Bigg](\xi)d\xi,\ \alpha \in V.$$

We must demonstrate that there exists $h_* \in S_{E_2, y_*}$ such that

$$\overline{z}_*(\alpha) = \int_0^{\alpha} K^{-1}M(\alpha-\xi)h_*(\xi)d\xi$$

$$+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\aleph_0^c)\Big[z_c-K^{-1}M(c)K\psi(0)$$
$$-\int_0^\alpha K^{-1}M(c-\xi)h_*(\eta)d\eta\Big](\xi)d\xi,\ \alpha\in V.$$

Now, for each $\alpha \in V$, because $E_1$ is continuous and from $x^\varrho$, one can get

$$\Bigg\|\Bigg(\overline{z}_n(\alpha)-\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\aleph_0^c)\Big[z_c$$
$$-K^{-1}M(c)K\psi(0)\Big](\xi)d\xi\Bigg)-\Bigg(\overline{z}_*(\alpha)$$
$$-\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\aleph_0^c)\Big[z_c$$
$$-K^{-1}M(c)K\psi(0)\Big](\xi)d\xi\Bigg)\Bigg\|\to 0 \text{ as } n\to\infty.$$

Assume the linear operator $\Theta : L^1(V,\mathscr{X}) \to C(V,\mathscr{X})$ which is continuous,

$$(\Theta f)(\alpha)=\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi$$
$$-\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta,\aleph_0^c)\Big(\int_0^\alpha K^{-1}M(c-\tau)h(\tau)d\tau\Big)d\xi.$$

Thus, in view of Lemma 2.7, $\Theta \circ S_{E_2}$ is a closed graph operator. In addition, from $\Theta$, one can get that

$$\overline{z}_n(\alpha)-K^{-1}M(\alpha)K\psi(0)-\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\aleph_0^c)\Big[z_c$$
$$-K^{-1}M(c)K\psi(0)\Big](\xi)d\xi\in\Theta(S_{E_2,y_n}).$$

Because $y_n \to y_*$, $y_* \in S_{E_2,y_*}$, from Lemma 2.7,

$$\overline{z}_*(\alpha)-K^{-1}M(\alpha)K\psi(0)-\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\aleph_0^c)\Big[z_c$$
$$-K^{-1}M(c)K\psi(0)\Big](\xi)d\xi=\int_0^\alpha K^{-1}M(\alpha-\xi)\Big[h_*(\xi)$$
$$+\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\aleph_0^c)\Big(\int_0^\alpha K^{-1}M(c-\tau)h_*(\tau)d\tau\Big)\Big](\xi)d\xi$$

for some $h_* \in (S_{E_2,y_*})$. Thus, $\Pi$ has a closed graph.

From **Step 1-5** in conjunction with the Arzela-Ascoli theorem, one can come to an end that $\Pi$ is a compact multivalued map, upper semi-continuous with convex closed values. From Lemma 2.7, one can assume $\Pi$ has a fixed point $z$ and that is a mild solution of (1)-(2). □

**Definition 3.3.** The differential system (1)-(2) is called approximately controllable on $V$ provided that $\overline{R(c,z_0)} = \mathscr{X}$, where $R(c,z_0) = \{z_c(z_0;x) : x(\cdot) \in L^2(V,\mathscr{V})\}$ is a mild solution of (1)-(2).

**Theorem 3.4.** *Assume* $\mathbf{H_0}$-$\mathbf{H_5}$ *and* $\mathbf{H_7}$ *hold. In addition* $N \in L^1(J,[0,\infty))$ *such that*
$\sup_{x\in\mathcal{P}_g}\|E_2(\alpha,z)\| \leq N(\alpha)$ *for a.e.* $\alpha \in V$, *then* (1)-(2) *is approximately controllable on* $V$.

*Proof.* Suppose $\widehat{x}^{\alpha}(\cdot)$ be a fixed point of $\Gamma$ in $\mathfrak{B}_p$. In view of Theorem 3.2, any fixed point of $\psi^{\varrho}$ is a mild solution of (1)-(2) under

$$\widehat{x}^{\alpha}(\alpha) = \mathscr{B}^{*}K^{-1}M^{*}(c-\alpha)\mathcal{R}(\beta,\aleph_0^c)p(\widehat{z}^{\beta})$$

and fulfills

$$\widehat{x}^{\alpha}(c) = z_c + \beta\mathcal{R}(\beta,\aleph_0^c)p(\widehat{x}^{\beta}). \tag{13}$$

Further, in view of the assumption on $E_2$ and Dunford-Pettis Theorem, one can get $\{h^{\alpha}(\xi)\}$ is weakly compact in $L^1(V,\mathscr{Z})$, accordingly there is a subsequence $\{h^{\alpha}(\xi)\}$, which converges weakly to say $h(\xi)$ in $L^1(V,\mathscr{Z})$. Characterize

$$w = z_c - K^{-1}M(\alpha)K\psi(0) - \int_0^{\alpha} K^{-1}M(\alpha-\xi)h(\xi)d\xi.$$

Now, we have

$$\begin{aligned}\|p(\widehat{x}^{\beta}) - w\| =& \Big\|\int_0^{\alpha} K^{-1}M(c-\xi)[h(\xi,\widehat{x}^{\alpha}(\xi)) - h(\xi)]d\xi\Big\| \\ \leq& \sup_{\alpha\in V}\Big\|\int_0^{\alpha} K^{-1}M(\alpha-\xi)[h(\xi,\widehat{x}^{\alpha}(\xi)) - h(\xi)]d\xi\Big\|. \end{aligned}\tag{14}$$

From Ascoli-Arzela theorem of infinite-dimensional version, we demonstrate $l(\cdot) \to \int_0^{\cdot}\mathscr{M}(\cdot-\xi)l(\xi)d\xi : L^1(V,\mathscr{Z}) \to C(V,\mathscr{Z})$ is compact. Hence, $\|q(\widehat{z}^{\beta}) - w\| \to 0$ as $\beta \to 0^{+}$. Furthermore, in view of (21),

$$\begin{aligned}\|\widehat{x}^{\alpha}(c) - z_c\| \leq& \|\beta\mathcal{R}(\beta,\aleph_0^c)(w)\| + \|\beta\mathcal{R}(\beta,\aleph_0^c)\|\|p(\widehat{x}^{\beta}) - w\| \\ \leq& \|\beta\mathcal{R}(\beta,\aleph_0^c)(w)\| + \|p(\widehat{x}^{\beta}) - w\|.\end{aligned}$$

In view of $\mathbf{H_0}$ and from (22), $\|\widehat{z}^{\beta}(c) - z_c\| \to 0$ as $\beta \to 0^{+}$ and which shows the approximate controllability of (1)-(2). □

Inspired by [4, 5, 27, 24, 32, 35], we study the approximate controllability of our system (1)-(2) with nonlocal conditions has the form

$$\begin{aligned}\big(Kz(\alpha)\big)' \in& \mathscr{A}\Big[z(\alpha) + \int_0^{\alpha} F(\alpha-\xi)z(\xi)d\xi\Big] \\ &+ E_2(\alpha,z_{\alpha}) + \mathscr{B}x(\alpha),\ \alpha \in V = [0,c],\end{aligned}\tag{15}$$

$$z(\alpha) = \psi(\alpha) + q(z_{\alpha_1}, z_{\alpha_2}, z_{\alpha_3}, \cdots, z_{\alpha_n}) \in \mathcal{P}_g, \quad \alpha \in (-\infty,0], \tag{16}$$

where $0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n \leq c$, $q : \mathcal{P}_g^n \to \mathcal{P}_g$ and which satisfies:

$\mathbf{H_6}$ $q : \mathcal{P}^n \to \mathcal{P}$ is continuous and $L_i(q) > 0$ such that

$$\|q(x_1,x_2,x_3,\cdots,x_n) - q(y_1,y_2,y_3,\cdots,y_n)\| \leq \sum_{i=1}^{n} L_i(q)\|x-y\|_{\mathcal{B}},$$

for each $x, y \in \mathcal{P}_g$ and $N_q = \sup\{\|q(x_{\alpha_1},x_{\alpha_2},x_{\alpha_3},\cdots,x_{\alpha_n})\| : x \in \mathcal{P}_g\}$.

**Definition 3.5.** A function $z : (-\infty,c] \to \mathscr{Z}$ is called a mild solution of (15)-(16) if $z_0 = \psi \in \mathcal{P}_g$ on $(-\infty,0]$ and

$$z(\alpha) = K^{-1}M(\alpha)E[\psi(0) + q(z_{\alpha_1}, z_{\alpha_2}, z_{\alpha_3}, \cdots, z_{\alpha_n})(0)]$$

$$+\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}x(\xi)d\xi,\quad \alpha\in V,$$

is fulfilled.

**Theorem 3.6.** *If* $\mathbf{H_0}$-$\mathbf{H_6}$ *are fulfilled, then* (15)-(16) *is approximately controllable on* $V$ *if*

$$\widetilde{P}_K P\Big(1+\frac{1}{\alpha}\widetilde{P}_K P^2 P_{\mathscr{B}}^2 c\Big)\gamma j<1.$$

4. **Neutral systems.** This section mainly focusing approximate controllability of (3)-(4). To begin with, we characterize the mild solution of (3)-(4).

**Definition 4.1.** A function $z:(-\infty,c]\to\mathscr{X}$ is called a mild solution of (3)-(4) provided that $z_0=\psi\in\mathcal{P}_g$ on $(-\infty,0]$ and

$$\begin{aligned}z(\alpha)&=K^{-1}M(\alpha)[K\psi(0)-E_1(0,\psi)]+K^{-1}E_1(\alpha,z_\alpha)\\&+\int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi,z_\xi)d\xi\\&+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau,z_\tau)d\tau d\xi\\&+\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}x(\xi)d\xi,\quad \alpha\in V,\end{aligned}$$

is fulfilled.

**H₇** $E_1:V\times\mathcal{P}_g$ is continuous and

- **(i)** There exists $L_1>0$, $\widetilde{L_1}>0$ for $\alpha\in V$ and $y,z\in\mathcal{P}_g$ such that $\mathscr{A}K^{-1}E_1$ fulfills

  $$\|\mathscr{A}K^{-1}E_1(\alpha,y)-\mathscr{A}K^{-1}E_1(\alpha,z)\|\le\widetilde{L_g}\|y-z\|_{\mathcal{P}_g},\ y,z\in\mathcal{P}_g,$$

  and $L_1=\sup_{\alpha\in V}\|\mathscr{A}^{-1}E_1(\alpha,0)\|$.
- **(ii)** There exists $l_g>0$, $\widetilde{l_g}>0$ such that

  $$\|E_1(\alpha,y)-E_1(\alpha,z)\|\le\widetilde{l_g}\|y-z\|_{\mathcal{P}_g},\ y,z\in\mathcal{P}_g,$$

  and $l_g=\sup_{\alpha\in V}\|E_1(\alpha,0)\|$.

To demonstrate (3)-(4) is approximately controllable, provided that for all $\beta>0$, there exists $x(\cdot)$ such that

$$\begin{aligned}z(\alpha)=&K^{-1}M(\alpha)[K\psi(0)-E_1(0,\psi)]+K^{-1}E_1(\alpha,z_\alpha)\\&+\int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi,z_\xi)d\xi\\&+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau,z_\tau)d\tau d\xi\\&+\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi\\&+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}x_\beta(\xi,z)d\xi,\quad h\in S_{E_2,z},\end{aligned}\tag{17}$$

$$x_\beta(\alpha,z)=\mathscr{B}^*K^{-1}M(c-\alpha)\mathcal{R}(\beta,\aleph_0^c)q(z(\cdot)),\tag{18}$$

where

$$\begin{aligned}q(z(\cdot)) =& z_c - K^{-1}M(\alpha)[K\psi(0) - E_1(0,\psi)] - K^{-1}E_1(\alpha, z_\alpha)\\&+ \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi, z_\xi)d\xi\\&- \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau, z_\tau)d\tau d\xi\\&- \int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi.\end{aligned}$$

**Theorem 4.2.** *If* $\mathbf{H_1}$-$\mathbf{H_5}$ *are fulfilled, then* (3)-(4) *has a mild solution on* $V$*, given that*

$$\widetilde{P}_K P\Big(1+\frac{1}{\alpha}\widetilde{P}_K P^2 P_{\mathscr{B}}^2 c\Big)\Big[j\big(\widetilde{l_g} + P\widetilde{L_g}(1+P_F)\big)\Big] < 1. \tag{19}$$

*In the above* $P_{\mathscr{B}} = \|\mathscr{B}\|$.

*Proof.* For any $\varepsilon > 0$, $\psi^\varepsilon : \mathcal{P}'_g \to 2^{\mathcal{P}'_g}$ defined by $\psi^\varepsilon x$, the set of $z \in \mathcal{P}'_g$ such that

$$z(\alpha) = \begin{cases}\psi(\alpha), \quad \alpha \in (-\infty, 0],\\ K^{-1}M(\alpha)[K\psi(0) - E_1(0,\psi)] + K^{-1}E_1(\alpha, z_\alpha)\\ + \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi, z_\xi)d\xi\\ + \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau, z_\tau)d\tau d\xi\\ + \int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi\\ + \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}x_\beta(\xi, z)d\xi,\ \alpha \in V,\end{cases}$$

where $h \in S_{E_2,z}$. To demonstrate $\Psi^\varrho$ has a fixed point and we conclude it is the solution of (3)-(4). Obviously, $z_1 = z(c) \in (\Psi^\varrho z)(c)$, which means that $x_\varrho(z, \alpha)$ drives (3)-(4) from $z_0 \to z_c$ in finite time $c$.

For $\psi \in \mathcal{P}_g$, we now characterize $\widehat{\psi}$ as

$$\widehat{\psi}(\alpha) = \begin{cases}\psi(\alpha), \quad \alpha \in (-\infty, 0],\\ K^{-1}M(\alpha)K\psi(0), \quad \alpha \in V,\end{cases}$$

then $\widehat{\psi} \in \mathcal{P}'_g$. Let $z(\alpha) = y(\alpha) + \widehat{\psi}(\alpha)$, $-\infty < \alpha \le c$. We conclude that $y$ fulfills $y_0 = 0$ and

$$\begin{aligned}y(\alpha) =& -K^{-1}M(\alpha)E_1(0,\psi) + K^{-1}E_1(\alpha, y_\alpha + \widehat{\psi}_\alpha)\\&+ \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi, y_\xi + \widehat{\psi}_\xi)d\xi\\&+ \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau, y_\tau + \widehat{\psi}_\tau)d\tau d\xi\\&+ \int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi\\&+ \int_0^\alpha K^{-1}M(\alpha-\eta)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta, \aleph_0^c)\Big[x_1 - K^{-1}M(c)[K\psi(0)\\&- E_1(0,\psi)] - K^{-1}E_1(c, y_c + \widehat{\psi}_c) - \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi, y_\xi + \widehat{\psi}_\xi)d\xi\end{aligned}$$

$$- \int_0^\alpha K^{-1}M(c-\eta)h(\eta)d\eta$$
$$- \int_0^\alpha K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau, y_\tau+\widehat{\psi}_\tau)d\tau d\xi\Bigg](\xi)d\xi, \ \alpha \in V.$$

if and only if $x$ satisfies

$$z(\alpha) = K^{-1}M(\alpha)[K\psi(0) - E_1(0,\psi)] + K^{-1}E_1(\alpha, y_\alpha + \widehat{\psi}_\alpha)$$
$$+ \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi, y_\xi+\widehat{\psi}_\xi)d\xi$$
$$+ \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau, y_\tau+\widehat{\psi}_\tau)d\tau d\xi$$
$$+ \int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi$$
$$+ \int_0^\alpha K^{-1}M(\alpha-\eta)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta, \aleph_0^c)\Bigg[x_1 - K^{-1}M(c)[K\psi(0)$$
$$- E_1(0,\psi)] - K^{-1}E_1(c, y_c+\widehat{\psi}_c) - \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi, y_\xi+\widehat{\psi}_\xi)d\xi$$
$$- \int_0^\alpha K^{-1}M(c-\eta)h(\eta)d\eta$$
$$- \int_0^\alpha K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau, y_\tau+\widehat{\psi}_\tau)d\tau d\xi\Bigg](\xi)d\xi, \ \alpha \in V.$$

and $z(\alpha) = \psi(\alpha)$, $\alpha \in (-\infty, 0]$.

Define $\Psi : \mathcal{P}_g'' \to \mathcal{P}_g''$ provided by $\Psi y$ the set of $\overline{z} \in \mathcal{P}_g''$ such that

$$\overline{z}(\alpha) = \begin{cases} 0, \quad \alpha \in (-\infty, 0], \\ -K^{-1}M(\alpha)E_1(0,\psi) + K^{-1}E_1(\alpha, y_\alpha+\widehat{\psi}_\alpha) \\ +\int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi, y_\xi+\widehat{\psi}_\xi)d\xi \\ +\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau, y_\tau+\widehat{\psi}_\tau)d\tau d\xi \\ +\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi \\ +\int_0^\alpha K^{-1}M(\alpha-\eta)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta,\aleph_0^c)\Big[x_1 \\ -K^{-1}M(c)[K\psi(0)-E_1(0,\psi)] - K^{-1}E_1(c, y_c+\widehat{\psi}_c) \\ -\int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi, y_\xi+\widehat{\psi}_\xi)d\xi \\ -\int_0^\alpha K^{-1}M(c-\eta)h(\eta)d\eta \\ -\int_0^\alpha K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau, y_\tau+\widehat{\psi}_\tau)d\tau d\xi\Big](\xi)d\xi, \ \alpha \in V. \end{cases}$$

Clearly, a fixed point of $\Psi^{\varrho}$ exists if and only if a fixed point of $\Pi$ exists. So, our goal is to prove a fixed point of $\Pi$ exists. We now split our proof into five steps for comfort.

**Step 1.** $\Psi$ is convex for all $z \in B_p$. Actually, if $\phi_1$, $\phi_2$ then there exists $h_1$, $h_2 \in S_{E_2,z}$ such that for each $\alpha \in V$, we have

$$\phi_i(\alpha) = -K^{-1}M(\alpha)E_1(0,\psi) + K^{-1}E_1(\alpha, y_\alpha+\widehat{\psi}_\alpha)$$
$$+ \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi, y_\xi+\widehat{\psi}_\xi)d\xi$$

$$+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau,y_\tau+\widehat{\psi}_\tau)d\tau d\xi$$
$$+\int_0^\alpha K^{-1}M(\alpha-\eta)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta,\Upsilon_0^c)\bigg[z_c-K^{-1}M(c)[K\psi(0)$$
$$-E_1(0,\psi)]-K^{-1}E_1(c,y_c+\widehat{\psi}_c)-\int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi,y_\xi+\widehat{\psi}_\xi)d\xi$$
$$+\int_0^\alpha K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau,y_\tau+\widehat{\psi}_\tau)d\tau d\xi$$
$$-\int_0^\alpha K^{-1}M(c-\eta)h_i(\eta)d\eta\bigg](\xi)d\xi+\int_0^\alpha K^{-1}M(\alpha-\xi)h_i(\xi)d\xi,\quad i=1,2.$$

Let $\delta\in[0,1]$. Then for each $\alpha\in V$, we get

$$(\delta\phi_1+(1-\delta)\phi_2)(\alpha)=K^{-1}M(\alpha)[K\psi(0)-E_1(0,\psi)]+K^{-1}E_1(\alpha,y_\alpha+\widehat{\psi}_\alpha)$$
$$+\int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi,y_\xi+\widehat{\psi}_\xi)d\xi$$
$$+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau,y_\tau+\widehat{\psi}_\tau)d\tau d\xi$$
$$+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta,\aleph_0^c)\bigg[z_c-K^{-1}M(c)[K\psi(0)$$
$$-E_1(0,\psi)]-K^{-1}E_1(c,y_c+\widehat{\psi}_c)-\int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi,y_\xi+\widehat{\psi}_\xi)d\xi$$
$$-\int_0^\alpha K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau,y_\tau+\widehat{\psi}_\tau)d\tau d\xi$$
$$-\int_0^\alpha K^{-1}M(c-\tau)[\delta h_1(\tau)+(1-\delta)h_2(\tau)]d\tau\bigg](\xi)d\xi$$
$$+\int_0^\alpha K^{-1}M(\alpha-\xi)[\delta h_1(\xi)+(1-\delta)h_2(\xi)]d\xi.$$

We can easily prove $S_{E_2,z}$ is convex because $F$ has convex values. Therefore, $\gamma h_1+(1-\gamma)h_2\in S_{E_2,z}$. Consequently,

$$\gamma\psi_1+(1-\gamma)\psi_2\in\Pi(x).$$

**Step 2.** To prove $p>0$ such that $\Pi(B_p)\subseteq B_p$. Otherwise, there exists $\varepsilon>0$ such that for all $p>0$ and $\alpha\in V$, there exists $y_p\in B_p$, but $\Pi(y_p)\notin B_p$, that is, $|\Pi(y_p)(\alpha)|>p$ for some $\alpha\in V$. For such $\varrho>0$,

$$r\leq|(\Psi y^p)(\alpha)|$$
$$\leq|K^{-1}M(\alpha)E_1(0,\psi)|+|K^{-1}E_1(\alpha,y_\alpha+\widehat{\psi}_\alpha)|$$
$$+\Big|\int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi,y_\xi+\widehat{\psi}_\xi)d\xi\Big|$$
$$+\Big|\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau,y_\tau+\widehat{\psi}_\tau)d\tau d\xi\Big|$$
$$+\Big|\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi\Big|+\Big|\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}x_\beta(\xi,y+\widehat{\psi})d\xi\Big|$$

$$\begin{aligned}
\leq&\widetilde{P}_K P\|E_1(0,\psi)\| + \widetilde{P}_K\big[\widetilde{l_g}\big(\|y^p_\alpha + \widehat{\psi}_\alpha\|_{\mathcal{P}_g}\big) + l_g\big]\\
&+ \widetilde{P}_K P\int_0^\alpha \big[\widetilde{L_g}\big(\|y^p_\xi + \widehat{\psi}_\xi\|_{\mathcal{P}_g}\big) + L_g\big]d\xi\\
&+ \widetilde{P}_K P\int_0^\alpha P_F\int_0^\xi \big[\widetilde{L_g}\big(\|y^p_\xi + \widehat{\psi}_\xi\|_{\mathcal{P}_g}\big) + L_g\big]d\tau d\xi + \widetilde{P}_K P\int_0^\alpha \beta_{p'}(\xi)d\xi\\
&+ \frac{1}{\alpha}\widetilde{P}_K P^2 P^2_{\mathscr{B}}c\Bigg[\widetilde{P}_K P\|\psi(0)\| + \widetilde{P}_K P\|E_1(0,\psi)\| + \widetilde{P}_K\big[\widetilde{l_g}\big(\|y^r_b + \widehat{\psi}_b\|_{\mathcal{P}_g}\big) + l_g\big]\\
&+ \widetilde{P}_K P\int_0^\alpha \big[\widetilde{L_g}\big(\|y^p_\xi + \widehat{\psi}_\xi\|_{\mathcal{P}_g}\big) + L_g\big]d\xi\\
&+ \widetilde{P}_K P\int_0^\alpha P_F\int_0^\xi \big[\widetilde{L_g}\big(\|y^p_\xi + \widehat{\psi}_\xi\|_{\mathcal{P}_g}\big) + L_g\big]d\tau d\xi + \widetilde{P}_K P\int_0^\alpha \beta_{p'}(\xi)d\xi\Bigg]\\
\leq&\widetilde{P}_K P\|E_1(0,\psi)\| + \widetilde{P}_K\big[\widetilde{l_g}p' + l_g\big] + \widetilde{P}_K P\int_0^\alpha \big[\widetilde{L_g}p' + L_g\big]d\xi\\
&+ \widetilde{P}_K P\int_0^\alpha P_F\int_0^\xi \big[\widetilde{L_g}p' + L_g\big]d\tau d\xi + \widetilde{P}_K P\int_0^\alpha \beta_{p'}(\xi)d\xi\\
&+ \frac{1}{\alpha}\widetilde{P}_K P^2 P^2_{\mathscr{B}}c\Bigg[\widetilde{P}_K P\|\psi(0)\| + \widetilde{P}_K P\|E_1(0,\psi)\| + \widetilde{P}_K\big[\widetilde{l_g}p' + l_g\big]\\
&+ \widetilde{P}_K P\int_0^\alpha P_F\int_0^\xi \big[\widetilde{L_g}p' + L_g\big]d\tau d\xi + \widetilde{P}_K P\int_0^\alpha \big[\widetilde{L_g}p' + L_g\big]d\xi\\
&+ \widetilde{P}_K P\int_0^\alpha \beta_{p'}(\xi)d\xi\Bigg]\\
\leq&\widetilde{P}_K P\Big(1 + \frac{1}{\alpha}\widetilde{P}_K P^2 P^2_{\mathscr{B}}c\Big)\Big[\big(\widetilde{l_g} + P\widetilde{L_g}(1+P_F)\big)p' + \int_0^\alpha \beta_{p'}(\xi)d\xi\Big] + \widehat{P_c},
\end{aligned}$$

where $\widehat{P_c}$ is independent of $p$. Separating the two sides of the above mentioned inequality by $p$ and perceiving that $p' = j(p + P_1|\psi(0)|) + \|\psi\|_{\mathcal{P}_g}$ as $p \to \infty$, we acquire that

$$\liminf_{p\to+\infty}\frac{\int_0^\alpha \beta_{p'}(\xi)d\xi}{p} = \liminf_{p\to+\infty}\left(\frac{\int_0^\alpha \beta_{p'}(\xi)d\xi}{p'}\cdot\frac{p'}{p}\right) = \gamma j.$$

Thus, we have

$$\widetilde{P}_K\Big(1+\frac{1}{\alpha}\widetilde{P}_K P^2 P^2_{\mathscr{B}}c\Big)\Big[j\big(\widetilde{l_g} + P\widetilde{L_g}(1+P_F)\big)\Big] \geq 1$$

and contradicts to (19). So, $p > 0$ and some $h \in S_{E_2,z}$, $\Pi(B_p) \subseteq B_p$.

**Step 3.** $\Pi(B_p)$ is equicontinuous. In fact, assume $\varrho > 0$ be small, $0 < \alpha_1 < \alpha_2 \leq c$. For every $y \in B_p$ and $\overline{z} \in \Pi_1 y$, there exists $h \in S_{E_2,z}$ such that for all $\alpha \in V$, then

$$\begin{aligned}
&|\overline{z}(\alpha_2) - \overline{z}(\alpha_1)| = |K^{-1}M(\alpha_2) - K^{-1}M(\alpha_1)||E_1(0,\psi)|\\
&+ \big|K^{-1}E_1(\alpha_2, y_{\alpha_2} + \widehat{\psi}_{\alpha_2}) - K^{-1}E_1(\alpha_1, y_{\alpha_1} + \widehat{\psi}_{\alpha_1})\big|\\
&+ \Big|\int_{\alpha_1}^{\alpha_2} K^{-1}\mathscr{A}K^{-1}M(\alpha_2-\xi)E_1(\xi, y_\xi + \widehat{\psi}_\xi)d\xi\Big|\\
&+ \Big|\int_{\alpha_1-\varepsilon}^{\alpha_1} K^{-1}\mathscr{A}K^{-1}[M(\alpha_2-\xi) - M(\alpha_1-\xi)]E_1(\xi, y_\xi + \widehat{\psi}_\xi)d\xi\Big|
\end{aligned}$$

$$
\begin{aligned}
&+\Big|\int_0^{\alpha_1-\varepsilon} K^{-1}\mathscr{A}K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]E_1(\xi,y_\xi+\widehat{\psi}_\xi)d\xi\Big|\\
&+\Big|\int_{\alpha_1}^{\alpha_2} K^{-1}\mathscr{A}K^{-1}M(\alpha_2-\xi)\int_0^\xi F(\alpha-\tau)E_1(\xi,y_\xi+\widehat{\psi}_\xi)d\tau d\xi\Big|\\
&+\Big|\int_{\alpha_1-\varepsilon}^{\alpha_1} K^{-1}\mathscr{A}K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]\int_0^\xi F(\alpha-\tau)E_1(\xi,y_\xi+\widehat{\psi}_\xi)d\tau d\xi\Big|\\
&+\Big|\int_0^{\alpha_1-\varepsilon} K^{-1}\mathscr{A}K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]\int_0^\xi F(\alpha-\tau)E_1(\xi,y_\xi+\widehat{\psi}_\xi)d\tau d\xi\Big|\\
&+\Big|\int_0^{\alpha_1-\varepsilon} K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]\mathscr{B}x_\beta^p(\eta,x)d\eta\Big|\\
&+\Big|\int_{\alpha_1-\varepsilon}^{\alpha_1} K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]\mathscr{B}x_\beta^p(\eta,x)d\eta\Big|\\
&+\Big|\int_{\alpha_1}^{\alpha_2} K^{-1}M(\alpha_2-\xi)\mathscr{B}x_\beta^p(\eta,x)d\eta\Big|\\
&+\Big|\int_{\alpha_1}^{\alpha_2} K^{-1}M(\alpha_2-\xi)h(\xi)d\xi\Big|+\Big|\int_{\alpha_1-\varepsilon}^{\alpha_1} K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]h(\xi)d\xi\Big|\\
&+\Big|\int_0^{\alpha_1-\varepsilon} K^{-1}[M(\alpha_2-\xi)-M(\alpha_1-\xi)]h(\xi)d\xi\Big|\\
\le&\widetilde{P}_K|M(\alpha_2)-M(\alpha_1)||E_1(0,\psi)|+\widetilde{P}_K\big|E_1(\alpha_2,y_{\alpha_2}+\widehat{\psi}_{\alpha_2})-E_1(\alpha_1,y_{\alpha_1}+\widehat{\psi}_{\alpha_1})\big|\\
&+\widetilde{P}_KP\int_{\alpha_1}^{\alpha_2}\big[\widetilde{L_g}p'+L_g\big]d\xi+\widetilde{P}_K\int_{\alpha_1-\varepsilon}^{\alpha_1}|M(\alpha_2-\xi)-M(\alpha_1-\xi)|\big[\widetilde{L_g}p'+L_g\big]d\xi\\
&+\widetilde{P}_K\int_0^{\alpha_1-\varepsilon}|M(\alpha_2-\xi)-M(\alpha_1-\xi)|\big[\widetilde{L_g}p'+L_g\big]d\xi\\
&+\widetilde{P}_KP\int_{\alpha_1}^{\alpha_2}P_F\int_0^\xi\big[\widetilde{L_g}p'+L_g\big]d\tau d\xi\\
&+\widetilde{P}_K\int_{\alpha_1-\varepsilon}^{\alpha_1}|M(\alpha_2-\xi)-M(\alpha_1-\xi)|P_F\int_0^\xi\big[\widetilde{L_g}p'+L_g\big]d\tau d\xi\\
&+\widetilde{P}_K\int_0^{\alpha_1-\varepsilon}|M(\alpha_2-\xi)-M(\alpha_1-\xi)|P_F\int_0^\xi\big[\widetilde{L_g}p'+L_g\big]d\tau d\xi\\
&+\widetilde{P}_KP_B\int_0^{\alpha_1-\varepsilon}|M(\alpha_2-\eta)-M(\alpha_1-\eta)|\Bigg[\widetilde{P}_KP\|\psi(0)\|+\widetilde{P}_KP\|E_1(0,\psi)\|\\
&+\widetilde{P}_K\big[\widetilde{l_g}p'+l_g\big]+\widetilde{P}_KP\int_0^\alpha\big[\widetilde{L_g}p'+L_g\big]d\xi+\widetilde{P}_KP\int_0^\alpha P_F\int_0^\xi\big[\widetilde{L_g}p'+L_g\big]d\tau d\xi\\
&+\widetilde{P}_KP\int_0^\alpha\beta_{p'}(\xi)d\xi\Bigg](\xi)d\xi+\widetilde{P}_KP_B\int_{\alpha_1-\varepsilon}^{\alpha_1}|M(\alpha_2-\eta)-M(\alpha_1-\eta)|\Bigg[\widetilde{P}_KP\|\psi(0)\|\\
&+\widetilde{P}_KP\|E_1(0,\psi)\|+\widetilde{P}_K\big[\widetilde{l_g}p'+l_g\big]+\widetilde{P}_KP\int_0^\alpha\big[\widetilde{L_g}p'+L_g\big]d\xi\\
&+\widetilde{P}_KP\int_0^\alpha P_F\int_0^\xi\big[\widetilde{L_g}p'+L_g\big]d\tau d\xi+\widetilde{P}_KP\int_0^\alpha\beta_{p'}(\xi)d\xi\Bigg](\xi)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+ \widetilde{P}_K P M_B \int_{\alpha_1}^{\alpha_2} \Bigg[ \widetilde{P}_K P \|\psi(0)\| + \widetilde{P}_K P \|E_1(0,\psi)\| + \widetilde{P}_K \big[\widetilde{l_g} p' + l_g\big] \\
&+ \widetilde{P}_K P \int_0^{\alpha} \big[\widetilde{L_g} p' + L_g\big] d\xi + \widetilde{P}_K P \int_0^{\alpha} P_F \int_0^{\xi} \big[\widetilde{L_g} p' + L_g\big] d\tau d\xi \\
&+ \widetilde{P}_K P \int_0^{\alpha} \beta_{p'}(\xi) d\xi \Bigg](\xi) d\xi + \widetilde{P}_K P \int_{\alpha_1}^{\alpha_2} \beta_{p'}(\xi) d\xi \\
&+ \widetilde{P}_K \int_{\alpha_1-\varepsilon}^{\alpha_1} |M(\alpha_2-\xi) - M(\alpha_1-\xi)| \beta_{p'}(\xi) d\xi \\
&+ \widetilde{P}_K \int_0^{\alpha_1-\varepsilon} |M(\alpha_2-\xi) - M(\alpha_1-\xi)| \beta_{p'}(\xi) d\xi. \qquad (20)
\end{aligned}
$$

Hence, for $\varrho > 0$, one can confirm that (20) tends to zero as $\alpha_2 \to \alpha_1$. Then again, the compactness of $\mathscr{M}(\alpha)$ for $\alpha > 0$ gives continuity in uniform operator topology. Hence $\Pi$ maps $B_p$ into an equicontinuous family of functions.

**Step 4.** $\Pi(\alpha) = \big\{\phi(\alpha) : \phi \in \Psi(B_p)\big\}$ is relatively compact in $X$.

Assume $\alpha \in (0,c]$, $\varrho > 0$, $0 < \varrho < \alpha$. Now $z \in B_p$, we provide

$$
\begin{aligned}
&\phi_\varrho(\alpha) = -K^{-1} M(\alpha) E_1(0,\psi) + K^{-1} E_1(\alpha, y_\alpha + \widehat{\psi}_\alpha) \\
&+ \int_0^{\alpha-\varrho} K^{-1} \mathscr{A} K^{-1} M(\alpha-\xi) E_1(\xi, y_\xi + \widehat{\psi}_\xi) d\xi \\
&+ \int_0^{\alpha-\varrho} K^{-1} M(\alpha-\xi) \mathscr{A} K^{-1} \int_0^{\xi} F(\xi-\tau) E_1(\tau, y_\tau + \widehat{\psi}_\tau) d\tau d\xi \\
&+ \int_0^{\alpha-\varrho} K^{-1} M(\alpha-\eta) \mathscr{B}\mathscr{B}^* K^{-1} M^*(c-\alpha) \mathcal{R}(\beta, \aleph_0^c) \Bigg[ z_c - K^{-1} M(c)[K\psi(0) \\
&- E_1(0,\psi)] - K^{-1} E_1(c, y_c + \widehat{\psi}_c) - \int_0^{\alpha} K^{-1} \mathscr{A} K^{-1} M(c-\xi) E_1(\xi, y_\xi + \widehat{\psi}_\xi) d\xi \\
&- \int_0^{\alpha} K^{-1} M(c-\xi) \mathscr{A} K^{-1} \int_0^{\xi} F(\xi-\tau) E_1(\tau, y_\tau + \widehat{\psi}_\tau) d\tau d\xi \\
&- \int_0^{\alpha} K^{-1} M(c-\eta) h(\eta) d\eta \Bigg](\tau) d\tau + \int_0^{\alpha} K^{-1} M(\alpha-\xi) h(\xi) d\xi.
\end{aligned}
$$

Because $\mathscr{M}(\alpha)$ is compact, $\bigwedge_\varrho(\alpha) = \{\psi_\varrho(\alpha) : \psi_\varrho \in \Pi(B_p)\}$ is relatively compact in $\mathscr{Z}$ for all $\varrho$, $0 < \varrho < \alpha$. Furthermore, for all $0 < \varrho < \alpha$, we have

$$
\begin{aligned}
|\phi(\alpha) - \phi_\varrho(\alpha)| &\le \Big| \int_{\alpha-\varrho}^{\alpha} K^{-1} \mathscr{A} K^{-1} M(\alpha-\xi) E_1(\xi, y_\xi + \widehat{\psi}_\xi) d\xi \Big| \\
&+ \Big| \int_{\alpha-\varrho}^{\alpha} K^{-1} \mathscr{A} K^{-1} M(\alpha-\xi) \int_0^{\xi} F(\alpha-\xi) E_1(\xi, y_\xi + \widehat{\psi}_\xi) d\xi \Big| \\
&+ \Big| \int_{\alpha-\varrho}^{\alpha} K^{-1} M(\alpha-\xi) h(\xi) d\xi \Big| \\
&+ \Bigg| \int_{\alpha-\varrho}^{\alpha} K^{-1} M(\alpha-\eta) \mathscr{B}\mathscr{B}^* K^{-1} M^*(c-\alpha) \mathcal{R}(\beta, \aleph_0^c) \Bigg[ z_c - K^{-1} M(c)[K\psi(0) \\
&- E_1(0,\psi)] - K^{-1} E_1(c, y_c + \widehat{\psi}_c)
\end{aligned}
$$

$$- \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi, y_\xi + \widehat{\psi}_\xi)d\xi - \int_0^\alpha K^{-1}M(c-\eta)h(\eta)d\eta$$
$$- \int_0^\alpha K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau, y_\tau + \widehat{\psi}_\tau)d\tau d\xi\Bigg](\tau)d\tau\Bigg|.$$

Therefore

$$|\psi(\alpha) - \psi_\varrho(\alpha)| \to 0 \text{ as } \varrho \to 0^+.$$

Thus, there exist relatively compact sets arbitrarily close $\bigwedge(\alpha) = \{\psi(\alpha) : \psi \in \Pi(B_p)\}$, $\widetilde{\bigwedge}(\alpha)$ is relatively compact in $\mathscr{Z}$ for all $\alpha \in [0, c]$. Because the compactness at $\alpha = 0$, therefore $\bigwedge(\alpha)$ is relatively compact in $\mathscr{Z}$, for all $\alpha \in [0, c]$.

**Step 5.** $\Psi$ has a closed graph.

Assume that $y_n \to y_*$ as $n \to \infty$, $\overline{z}_n \in \Pi y_n$, for all $y_n \in B_p$, and $\overline{z}_n \to \overline{z}_*$ as $n \to \infty$. Now, we demonstrate $\overline{z}_* \in \Pi y_*$. Because $\overline{z}_n \in \Pi y_n$, there exists $h_n \in S_{E_2, y_n}$ such that

$$\overline{z}_n(\alpha) = -K^{-1}M(\alpha)E_1(0,\psi) + K^{-1}E_1(\alpha, (y_n)_\alpha + \widehat{\psi}_\alpha)$$
$$+ \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi, (y_n)_\xi + \widehat{\psi}_\xi)d\xi$$
$$+ \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\alpha-\tau)E_1(\tau, (y_n)_\tau + \widehat{\psi}_\tau)d\tau d\xi$$
$$+ \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta, \Upsilon_0^c)\Bigg[z_c - K^{-1}M(c)[K\psi(0)$$
$$- E_1(0,\psi)] - K^{-1}E_1(c, (y_n)_c + \widehat{\psi}_c) - \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi, (y_n)_\xi + \widehat{\psi}_\xi)d\xi$$
$$- \int_0^\alpha K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^\xi F(c-\tau)E_1(\tau, (y_n)_\tau + \widehat{\psi}_\tau)d\tau d\xi$$
$$- \int_0^\alpha K^{-1}M(c-\xi)h_n(\xi)d\xi\Bigg](\xi)d\xi + \int_0^\alpha K^{-1}M(\alpha-\xi)h_n(\xi)d\xi,\ \alpha \in V.$$

We must demonstrate that there exists $h_* \in S_{E_2, y_*}$ such that

$$\overline{z}_*(\alpha) = -K^{-1}M(\alpha)E_1(0,\psi) + K^{-1}E_1(\alpha, (y_*)_\alpha + \widehat{\psi}_\alpha)$$
$$+ \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi, (y_*)_\xi + \widehat{\psi}_\xi)d\xi$$
$$+ \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\alpha-\tau)E_1(\tau, (y_*)_\tau + \widehat{\psi}_\tau)d\tau d\xi$$
$$+ \int_0^\alpha K^{-1}M(\alpha-\eta)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta, \Upsilon_0^c)\Bigg[z_c - K^{-1}M(c)[K\psi(0)$$
$$- E_1(0,\psi)] - K^{-1}E_1(c, (y_*)_c + \widehat{\psi}_c) - \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi, (y_*)_\xi + \widehat{\psi}_\xi)d\xi$$
$$- \int_0^\alpha K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^\xi F(c-\tau)E_1(\tau, (y_*)_\tau + \widehat{\psi}_\tau)d\tau d\xi$$
$$- \int_0^\alpha K^{-1}M(c-\xi)h_*(\xi)d\xi\Bigg](\xi)d\xi + \int_0^\alpha K^{-1}M(\alpha-\xi)h_*(\xi)d\xi,\ \alpha \in V.$$

Now, for all $\alpha \in V$, since $E_1$ is continuous, we get

$$\Bigg\| \Bigg( \overline{z}_n(\alpha) + K^{-1}M(\alpha)E_1(0,\psi) - K^{-1}E_1(\alpha,(y_n)_\alpha + \widehat{\psi}_\alpha)$$

$$- \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi,(y_n)_\xi + \widehat{\psi}_\xi)d\xi$$

$$- \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\alpha-\tau)E_1(\tau,(y_n)_\tau + \widehat{\psi}_\tau)d\tau d\xi$$

$$- \int_0^\alpha K^{-1}M(\alpha-\eta)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\alpha)\mathcal{R}(\beta,\Upsilon_0^c)\Bigg[ z_c - K^{-1}M(c)[K\psi(0)$$

$$- E_1(0,\psi)] - K^{-1}E_1(c,(y_n)_c + \widehat{\psi}_c) - \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi,(y_n)_\xi + \widehat{\psi}_\xi)d\xi$$

$$- \int_0^\alpha K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^\xi F(c-\tau)E_1(\tau,(y_n)_\tau + \widehat{\psi}_\tau)d\tau d\xi$$

$$- \int_0^\alpha K^{-1}M(c-\xi)h_n(\xi)d\xi\Bigg](\xi)d\xi - \int_0^\alpha K^{-1}M(\alpha-\xi)h_n(\xi)d\xi\Bigg)$$

$$- \Bigg( \overline{z}_*(\alpha) + K^{-1}M(\alpha)E_1(0,\psi) - K^{-1}E_1(\alpha,(y_*)_\alpha + \widehat{\psi}_\alpha)$$

$$- \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi,(y_*)_\xi + \widehat{\psi}_\xi)d\xi$$

$$- \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\alpha-\tau)E_1(\tau,(y_*)_\tau + \widehat{\psi}_\tau)d\tau d\xi$$

$$- \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\Upsilon_0^c)\Bigg[ z_c - K^{-1}M(c)[K\psi(0)$$

$$- E_1(0,\psi)] - K^{-1}E_1(c,(y_*)_c + \widehat{\psi}_c) - \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi,(y_*)_\xi + \widehat{\psi}_\xi)d\xi$$

$$- \int_0^\alpha K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^\xi F(c-\tau)E_1(\tau,(y_*)_\tau + \widehat{\psi}_\tau)d\tau d\xi$$

$$- \int_0^\alpha K^{-1}M(c-\xi)h_*(\xi)d\xi\Bigg](\xi)d\xi - \int_0^\alpha K^{-1}M(\alpha-\xi)h_*(\xi)d\xi\Bigg)\Bigg\| \to 0 \text{ as } n\to\infty.$$

Assume the linear operator $\Theta : L^1(V,\mathscr{Z}) \to C(V,\mathscr{Z})$ which is continuous,

$$(\Theta f)(\alpha) = \int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi - \int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)$$
$$(\times)\mathcal{R}(\beta,\aleph_0^c)\Big(\int_0^\alpha K^{-1}M(c-\tau)h(\tau)d\tau\Big)d\xi.$$

Now, for each $\alpha \in V$, because $E_1$ is continuous and from $x^\varrho$, one can get

$$\Bigg( \overline{z}_n(\alpha) + K^{-1}M(\alpha)E_1(0,\psi) - K^{-1}E_1(\alpha,(y_n)_\alpha + \widehat{\psi}_\alpha)$$

$$- \int_0^\alpha K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi,(y_n)_\xi + \widehat{\psi}_\xi)d\xi$$

$$- \int_0^{\alpha} K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^{\xi}F(\alpha-\tau)E_1(\tau,(y_n)_\tau+\widehat{\psi}_\tau)d\tau d\xi$$
$$- \int_0^{\alpha} K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\aleph_0^c)\Big[z_c - K^{-1}M(c)[K\psi(0)$$
$$- E_1(0,\psi)] - K^{-1}E_1(c,(y_n)_c+\widehat{\psi}_c) - \int_0^{\alpha} K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi,(y_n)_\xi+\widehat{\psi}_\xi)d\xi$$
$$- \int_0^{\alpha} K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^{\xi}F(c-\tau)E_1(\tau,(y_n)_\tau+\widehat{\psi}_\tau)d\tau d\xi$$
$$- \int_0^{\alpha} K^{-1}M(c-\xi)h_n(\eta)d\eta\Big](\xi)d\xi - \int_0^{\alpha}K^{-1}M(\alpha-\xi)h_n(\xi)d\xi\Bigg) \in \Theta(S_{E_2,y_n}).$$

Thus, in view of Lemma 2.7, $\Theta \circ S_{E_2}$ is a closed graph operator. In addition, from $\Theta$, one can get that

$$\Bigg(\overline{z}_*(\alpha) + K^{-1}M(\alpha)E_1(0,\psi) - K^{-1}E_1(\alpha,(y_*)_\alpha+\widehat{\psi}_\alpha)$$
$$- \int_0^{\alpha} K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi,(y_*)_\xi+\widehat{\psi}_\xi)d\xi$$
$$- \int_0^{\alpha} K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^{\xi}F(\alpha-\tau)E_1(\tau,(y_*)_\tau+\widehat{\psi}_\tau)d\tau d\xi$$
$$- \int_0^{\alpha} K^{-1}M(\alpha-\xi)\mathscr{B}\mathscr{B}^*K^{-1}M^*(c-\xi)\mathcal{R}(\beta,\aleph_0^c)\Big[z_c - K^{-1}M(c)[K\psi(0)$$
$$- E_1(0,\psi)] - K^{-1}E_1(c,(y_*)_c+\widehat{\psi}_c) - \int_0^{\alpha} K^{-1}\mathscr{A}K^{-1}M(c-\xi)E_1(\xi,(y_*)_\xi+\widehat{\psi}_\xi)d\xi$$
$$- \int_0^{\alpha} K^{-1}M(c-\xi)\mathscr{A}K^{-1}\int_0^{\xi}F(c-\tau)E_1(\tau,(y_*)_\tau+\widehat{\psi}_\tau)d\tau d\xi$$
$$- \int_0^{\alpha} K^{-1}M(c-\xi)h_*(\eta)d\eta\Big](\xi)d\xi - \int_0^{\alpha}K^{-1}M(\alpha-\xi)h_*(\xi)d\xi\Bigg) \in \Theta(S_{E_2,y_*}).$$

for some $h_* \in (S_{E_2,y_*})$. Thus, $\Pi$ has a closed graph.

From **Step 1-5** in conjunction with the Arzela-Ascoli theorem, one can come to an end that $\Pi$ is a compact multivalued map, upper semi-continuous with convex closed values. From Lemma 2.7, one can assume $\Pi$ has a fixed point $z$ and that is a mild solution of (3)-(4). □

**Definition 4.3.** The differential system (3)-(4) is called approximately controllable on $V$ provided that $\overline{R(c,z_0)} = \mathscr{Z}$, where $R(c,z_0) = \{z_c(z_0;x) : x(\cdot) \in L^2(V,\mathscr{V})\}$ is a mild solution of (3)-(4).

**Theorem 4.4.** *If* $\mathbf{H_0}$-$\mathbf{H_5}$ *and* $\mathbf{H_7}$ *are fulfilled and additionally*

(**a**) *$E_1 : [0,c]\times\mathscr{Z}\to\mathscr{Z}$ and $\mathscr{A}E_1(\alpha,\cdot)$ is continuous from weak topology of $\mathscr{Z}$ to strong topology of $\mathscr{Z}$.*

(**b**) *There exists $K \in L^1(V,[0,\infty))$ such that*

$$\sup_{z\in\mathcal{P}_g}\|E_2(\alpha,z)\| + \sup_{y\in\mathcal{P}_g}\|\mathscr{A}E_2(\alpha,y)\| \le K(\alpha),$$

*for a.e. $\alpha \in V$.*

*Then (3)-(4) is approximately controllable on $V$.*

*Proof.* Suppose $\widehat{z}^{\beta}(\cdot)$ has a fixed point of $\psi^{\varrho}$ in $B_p$. In view of Theorem 4.2, any fixed point of $\psi^{\varrho}$ is a mild solution of (3)-(4) under

$$\widehat{z}^{\alpha}(\alpha) = \mathscr{B}^* K^{-1} M^*(c-\alpha)\mathcal{R}(\beta, \aleph_0^c)p(\widehat{x}^{\beta})$$

and fulfills

$$\widehat{x}^{\alpha}(c) = z_c + \beta\mathcal{R}(\beta, \aleph_0^c)p(\widehat{x}^{\beta}). \tag{21}$$

Further, in view of the assumption on $E_2$ and Dunford-Pettis Theorem, one can get $\{h^{\alpha}(\xi)\}$ is weakly compact in $L^1(V, \mathscr{Z})$, accordingly there is a subsequence $\{h^{\alpha}(\xi)\}$, which converges weakly to say $h(\xi)$ in $L^1(V, \mathscr{Z})$. Characterize

$$\begin{aligned} w =& z_c - K^{-1}M(\alpha)[K\psi(0) - E_1(0, \psi)] - K^{-1}E_1(\alpha, z_\alpha) \\ &+ \int_0^{\alpha} K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi, z_\xi)d\xi \\ &- \int_0^{\alpha} K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^{\xi} F(\xi-\tau)E_1(\tau, z_\tau)d\tau d\xi \\ &- \int_0^{\alpha} K^{-1}M(\alpha-\xi)h(\xi)d\xi. \end{aligned}$$

Now, we have

$$\begin{aligned} \|p(\widehat{x}^{\beta}) - w\| =& \Big\| \int_0^{\alpha} K^{-1}M(c-\xi)[h(\xi, \widehat{x}^{\alpha}(\xi)) - h(\xi)]d\xi \Big\| \\ \leq& \sup_{\alpha\in V} \Big\| \int_0^{\alpha} K^{-1}M(\alpha-\xi)[h(\xi, \widehat{x}^{\alpha}(\xi)) - h(\xi)]d\xi \Big\|. \end{aligned} \tag{22}$$

From Ascoli-Arzela theorem of infinite-dimensional version, we demonstrate $l(\cdot) \to \int_0^{\cdot} \mathscr{M}(\cdot-\xi)l(\xi)d\xi : L^1(V, \mathscr{Z}) \to C(V, \mathscr{Z})$ is compact. Hence, $\|q(\widehat{z}^{\beta}) - w\| \to 0$ as $\beta \to 0^+$. Furthermore, in view of (21),

$$\begin{aligned} \|\widehat{x}^{\alpha}(c) - z_c\| \leq& \|a\mathcal{R}(\beta, \aleph_0^c)(w)\| + \|\beta\mathcal{R}(\beta, \aleph_0^c)\|\|p(\widehat{x}^{\beta}) - w\| \\ \leq& \|\beta\mathcal{R}(\beta, \aleph_0^c)(w)\| + \|p(\widehat{x}^{\beta}) - w\|. \end{aligned}$$

In view of $\mathbf{H_0}$ and from (22), $\|\widehat{z}^{\beta}(c) - z_c\| \to 0$ as $\beta \to 0^+$ and which shows the approximate controllability of (3)-(4). □

Inspired by [4, 5, 27, 24, 32, 35], we study the approximate controllability of our system (3)-(4) with nonlocal conditions has the form

$$\begin{aligned} \frac{d}{d\alpha}\big(Kz(\alpha) - E_1(\alpha, z_\alpha)\big) \in& \mathscr{A}\Big[z(\alpha) + \int_0^{\alpha} F(\alpha-\xi)z(\xi)d\xi\Big] \\ &+ E_2(\alpha, z_\alpha) + \mathscr{B}x(\alpha),\ \alpha \in V = [0, c], \end{aligned} \tag{23}$$

$$z(\alpha) = \psi(\alpha) + q(z_{\alpha_1}, z_{\alpha_2}, z_{\alpha_3}, \cdots, z_{\alpha_n}) \in \mathcal{P}_g, \quad \alpha \in (-\infty, 0], \tag{24}$$

where $0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n \leq c$, $q : \mathcal{P}_g^n \to \mathcal{P}_g$ is a given function.

**Definition 4.5.** A function $z : (-\infty, c] \to \mathscr{Z}$ is called a mild solution of (23)-(24) if $z_0 = \psi \in \mathcal{P}_g$ on $(-\infty, 0]$ and

$$\begin{aligned} z(\alpha) =& K^{-1}M(\alpha)E[\psi(0) + q(z_{\alpha_1}, z_{\alpha_2}, z_{\alpha_3}, \cdots, z_{\alpha_n})(0) - E_1(0, \psi)] \\ &+ K^{-1}E_1(\alpha, z_\alpha) + \int_0^{\alpha} K^{-1}\mathscr{A}K^{-1}M(\alpha-\xi)E_1(\xi, z_\xi)d\xi \end{aligned}$$

$$+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{A}K^{-1}\int_0^\xi F(\xi-\tau)E_1(\tau,z_\tau)d\tau d\xi$$
$$+\int_0^\alpha K^{-1}M(\alpha-\xi)h(\xi)d\xi+\int_0^\alpha K^{-1}M(\alpha-\xi)\mathscr{B}x(\xi)d\xi,\quad \alpha\in V,$$

is fulfilled.

**Theorem 4.6.** *If* $\mathbf{H_0}$-$\mathbf{H_7}$ *are fulfilled, then* (23)-(24) *is approximately controllable on* $V$ *if*

$$\widetilde{P}_K P\Big(1+\frac{1}{\alpha}\widetilde{P}_K P^2 P_{\mathscr{B}}^2 c\Big)\Big[j\big(\widetilde{l_g}+P\widetilde{L_g}(1+P_F)\big)\Big]<1.$$

## 5. An example.

Considering an integro-differential system along control

$$\frac{\partial}{\partial\alpha}\big[u(\alpha,z)-u_{zz}(\alpha,z)\big]\in\frac{\partial^2}{\partial z^2}u(\alpha,z)+\int_0^\alpha m(\alpha-\xi)\frac{\partial^2}{\partial z^2}u(\xi,z)d\xi$$
$$+\widehat{\mu}(\alpha,x)+\widehat{F}(\alpha,u(\alpha-p,z)),\ \alpha\in[0,c],\ p>0,\ z\in[0,\pi],\tag{25}$$
$$u(\alpha,0)=u(\alpha,\pi)=0,\quad \alpha\in[0,c],\tag{26}$$
$$u(\alpha,z)=\psi(\alpha,z),\quad z\in[0,\pi],\quad \alpha\in(-\infty,0],\tag{27}$$

To change this framework into abstract structure (1)-(2), assume $\mathscr{Z}=L^2([0,\pi])$ and $\mathscr{A}:D(\mathscr{A})\subset\mathscr{Z}\to\mathscr{Z}$, $K:D(K)\subset\mathscr{Z}\to\mathscr{Z}$ defined by $\mathscr{A}y=y''$, and $Ky=y-y''$ where each domain $D(\mathscr{A})$ and $D(K)$ is presented by $\{y\in\mathscr{Z}: y,y'$ are absolutely continuous, $y(0)=y(\pi)=0\}$. Then $M(\alpha)$ which is compact, analytic and self-adjoint. Additionally $\mathscr{A}$ and $K$ can be given by $Ay=\sum_{k=1}^\infty k^2\langle y,u_k\rangle u_k$, $y\in D(\mathscr{A})$, $Ky=\sum_{k=1}^\infty(1+k^2)\langle y,u_k\rangle u_k$, $y\in D(K)$ where $u_k(z)=\sqrt{\frac{2}{\pi}}\sin(kz)$, $k=1,2,3,\cdots$ is the orthonormal of vectors of $\mathscr{A}$. Additionally for $u\in\mathscr{Z}$, one can get

$$K^{-1}u=\sum_{k=1}^\infty\frac{1}{(1+k^2)}\langle u,u_k\rangle u_k,$$
$$\mathscr{A}K^{-1}u=\sum_{k=1}^\infty\frac{k^2}{(1+k^2)}\langle u,u_k\rangle u_k$$

and the kernal $m(\alpha-\xi)$ is continuous, then there exists $m_1>0$ such that $|m(\alpha-\xi)|\le m_1$.

Phase space $\mathcal{P}_g$ along the norm is presented as

$$\|\phi\|_{\mathcal{P}_g}=\int_{-\infty}^0 g(\xi)\sup_{\xi\le\theta\le0}(\|\psi(\theta)\|)_{L^2}d\xi.$$

In the above equation, $g(\xi)=e^{2\xi}$, $\xi<0$ and $j=\int_{-\infty}^0 g(\xi)d\xi=\frac{1}{2}$.

Since the analytic resolvent $M(\alpha)$ is compact, there exist constants $k_2$, $k_3>0$ such that $\|M(\alpha)\|\le k_2$ and $\|(-\mathscr{A})^\gamma M(\alpha-\xi)\|\le k_3(\alpha-\xi)^{-\gamma}$ for each $\alpha\in V$ and $0<\gamma<1$. Assume $u(\alpha)(z)=u(\alpha,z)$. Define

$$F(\alpha,u)(\cdot)=\widehat{F}(\alpha,u(\cdot)),$$
$$M(\alpha-\xi)z(\xi)=m(\alpha-\xi)\frac{\partial^2}{\partial z^2}u(\xi,z),$$

and $\mathscr{B}:\mathscr{V}\to\mathscr{Z}$ is interpreted by $\mathscr{B}z(\alpha)(z)=\widehat{\mu}(\alpha,z)$. Therefore, $\mathscr{A}K^{-1}$ is compact and bounded with $\|K^{-1}\|\le 1$.

In this way, by applying the ideas introduced above (25)-(27) may be composed as (1)-(2). Additionally, we introduce few appropriate requirements on functions introduced above to prove assumptions on Theorem 3.4 and has come to the conclusion that (25)-(27) is approximately controllable on $V$.

**Acknowledgments.** The authors would like to express their warmest thanks to the editor and anonymous referee for the careful reading of the paper and for the useful suggestions and comments.

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Received November 2019; 1st revision February 2020; 2nd revision March 2020.

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