Arab Journal of Mathematical Sciences

Arab J Math Sci 23 (2017) 186–195

On the genus of nil-graph of ideals of commutative rings

T. Tamizh Chelvam*, K. Selvakumar, P. Subbulakshmi

Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627012, Tamil Nadu, India

Received 28 December 2015; received in revised form 9 September 2016; accepted 14 September 2016
Available online 2 October 2016

Abstract. Let R be a commutative ring with identity and let $\text{Nil}(R)$ be the ideal of all nilpotent elements of R. Let $\mathbb{I}(R) = \{I : I \text{ is a non-trivial ideal of } R \text{ and there exists a non-trivial ideal } J \text{ such that } IJ \subseteq \text{Nil}(R)\}$. The *nil-graph* of ideals of R is defined as the simple undirected graph $\mathbb{AG}_N(R)$ whose vertex set is $\mathbb{I}(R)$ and two distinct vertices I and J are adjacent if and only if $IJ \subseteq \text{Nil}(R)$. In this paper, we study the planarity and genus of $\mathbb{AG}_N(R)$. In particular, we have characterized all commutative Artin rings R for which the genus of $\mathbb{AG}_N(R)$ is either zero or one.

Keywords: Nil-graph of ideals; Commutative ring; Annihilating-ideal; Planar; Genus

2010 Mathematics Subject Classification: primary 05C75; 05C25; secondary 13A15; 13M05

1. Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. In the literature, there are many papers assigning graphs to rings, groups and semigroups, see [2,9,12]. The first graph construction from a commutative ring is the zero-divisor graph by Beck [9]. The zero-divisor graph was later studied by D.D. Anderson et al. [3] and Anderson et al. [2]. There are several other graphs associated with commutative rings such as the total graph [1], the annihilator graph [7] and the dot-product graph [8]. These consider the elements in the commutative ring as vertices. In ring theory, the structure of a ring R is more closely

* Corresponding author.
E-mail addresses: tamche59@gmail.com (T. Tamizh Chelvam), selva_158@yahoo.co.in (K. Selvakumar), shunlaxmi@gmail.com (P. Subbulakshmi).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2016.09.004

tied to its ideals behavior than to its elements, and so it is more appropriate to define a graph with ideals instead of elements as vertices. Some of the graph constructions with ideals of a commutative ring as vertices are the annihilating ideal graph [10] and the nil-ideal graph [14]. Several authors [4,5,17–21] studied various properties of these graphs including diameter, girth, domination and genus. In this paper, we are interested in certain topological properties of the nil-graph of ideals of commutative rings.

Throughout this paper, R is a commutative ring with identity which is not an integral domain. An ideal I of R is said to be an annihilating-ideal if there exists a non-zero ideal J of R such that $IJ = (0)$. We denote the set of non-zero annihilating ideals of R by $\mathbb{A}^*(R)$. Behboodi et al. [10,11] introduced and investigated the annihilating-ideal graph of R. The *annihilating-ideal graph* of R is defined as the simple undirected graph $\mathbb{AG}(R)$ whose vertex set is $\mathbb{A}^*(R)$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. Shaveisi et al. [14] generalized the annihilating-ideal graph of R and introduced the nil-graph of ideals of R. Let $\mathrm{Nil}(R)$ be the ideal of all nilpotent elements of R and $\mathbb{I}(R) = \{I : I \text{ is a non-trivial ideal of } R \text{ and there exists a non-trivial ideal } J \text{ such that } IJ \subseteq \mathrm{Nil}(R)\}$. The *nil-graph of ideals* of R is defined as the undirected simple graph $\mathbb{AG}_N(R)$ whose vertex set is $\mathbb{I}(R)$ and two distinct vertices I and J are adjacent if and only if $IJ \subseteq \mathrm{Nil}(R)$. It is easy to see that $\mathbb{AG}(R)$ is a subgraph of $\mathbb{AG}_N(R)$.

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. The girth of G is the length of a shortest cycle in G and is denoted by $gr(G)$. If G has no cycles, we define the girth of G to be infinite. The *corona* of two graphs G_1 and G_2 is the graph $G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

Let S_k denote the sphere with k handles, where k is a non-negative integer, that is, S_k is an oriented surface with k handles. The genus of a graph G, denoted $g(G)$, is the smallest integer n such that the graph can be embedded in S_n. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. We say that a graph G is planar if $g(G) = 0$, and toroidal if $g(G) = 1$. Note that a planar graph G has an embedding in the plane. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. Kuratowski's theorem says that a graph G is *planar* if and only if it contains no subdivision of K_5 or $K_{3,3}$. Also, if H is a subgraph of a graph G, then $g(H) \leq g(G)$. For details about the notion of embedding of a graph in a surface one can refer to A.T. White [22]. Several authors [6,13,15,16,21] studied the genus of graphs from commutative rings. In particular several characterizations are obtained for planar and toroidal nature of graphs from commutative rings. The purpose of this paper is to study the embeddings of the nil-graph of ideals $\mathbb{AG}_N(R)$. This paper is organized as follows.

In Section 2, we characterize all commutative Artin rings R for which the nil-graph of ideals $\mathbb{AG}_N(R)$ is planar. In Section 3, we characterize all commutative Artin rings R for which the nil-graph of ideals $\mathbb{AG}_N(R)$ is of genus one. Now we state a result which provides a characterization for $\mathbb{AG}_N(R)$ to be complete.

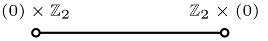

Fig. 1. $\mathbb{AG}_N(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

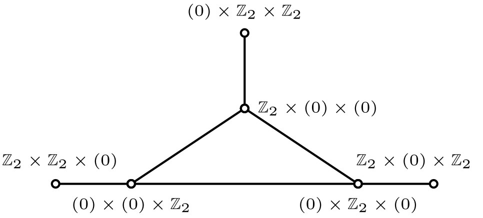

Fig. 2. $\mathbb{AG}_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

Theorem 1.1 ([14]). *Let R be a commutative ring. Then $\mathbb{AG}_N(R)$ is complete if and only if one of the following conditions holds:*

(i) *$(R, Nil(R))$ is a local ring;*
(ii) *$R \cong F_1 \times F_2$, where F_1 and F_2 are two fields.*

2. Planarity of nil-graph of ideals

In this section, we characterize all commutative Artin rings R for which $\mathbb{AG}_N(R)$ is planar. Let us see some examples of nil-graph of ideals.

Example 2.1. Two nil-graph of ideals are given in Figs. 1 and 2.

Now we obtain a characterization for $\mathbb{AG}_N(R)$ to be planar for some classes of rings R.

Theorem 2.2. *Let $R = F_1 \times F_2 \times \cdots \times F_n$ be a commutative ring, where each F_i is a field and $n \geq 2$. Then $\mathbb{AG}_N(R)$ is planar if and only if $n = 2$ or 3.*

Proof. If $n = 2$, then $\mathbb{AG}_N(R) \cong K_2$(refer to Fig. 1). If $n = 3$, then $\mathbb{AG}_N(R) \cong K_3 \circ K_1$(refer to Fig. 2). Hence $\mathbb{AG}_N(R)$ is planar in both cases.

Conversely, assume that $\mathbb{AG}_N(R)$ is planar. Suppose that $n > 3$. Consider the non-zero proper ideals $I_1 = F_1 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $I_2 = (0) \times F_2 \times (0) \times (0) \times \cdots \times (0)$, $I_3 = F_1 \times F_2 \times (0) \times (0) \times \cdots \times (0)$, $J_1 = (0) \times (0) \times F_3 \times (0) \times \cdots \times (0)$, $J_2 = (0) \times (0) \times (0) \times F_4 \times \cdots \times (0)$ and $J_3 = (0) \times (0) \times F_3 \times F_4 \times \cdots \times (0)$ of R. Note that $I_i J_k = (0) = \text{Nil}(R)$ for all i, k and so $K_{3,3}$ is a subgraph of $\mathbb{AG}_N(R)$, a contradiction to $\mathbb{AG}_N(R)$ being planar. Hence $n = 2$ or $n = 3$. □

Theorem 2.3. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$. Then $\mathbb{AG}_N(R)$ is planar if and only if R is a local ring with at most four non-trivial ideals.*

Proof. Suppose that R is a local ring with at most four non-trivial ideals. By Theorem 1.1, $\mathbb{AG}_N(R)$ is complete and so $\mathbb{AG}_N(R) \cong K_t$, where $t \leq 4$. Hence $\mathbb{AG}_N(R)$ is planar.

Conversely, assume that $\mathbb{AG}_N(R)$ is planar. Assume that $n > 1$. Let $I_1 = R_1 \times (0) \times \mathfrak{m}_3 \times \cdots \times \mathfrak{m}_n$, $I_2 = (0) \times R_2 \times \mathfrak{m}_3 \times \cdots \times \mathfrak{m}_n$, $I_3 = \mathfrak{m}_1 \times (0) \times \mathfrak{m}_3 \times \cdots \times \mathfrak{m}_n$,

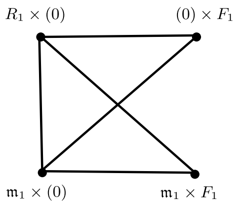

Fig. 3. $\mathbb{AG}_N(R_1 \times F_1)$.

$I_4 = (0) \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times \cdots \times \mathfrak{m}_n$ and $I_5 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times \cdots \times \mathfrak{m}_n$. Then I_i $(1 \leq i \leq 5)$ are non-zero proper ideals in R and $I_iI_j \subseteq \mathrm{Nil}(R)$ for all $i \neq j$ and so K_5 is a subgraph of $\mathbb{AG}_N(R)$, a contradiction. Hence $n = 1$ and so R is a local ring. Since $\mathbb{AG}_N(R)$ is planar and by Theorem 1.1, R contains at most four non-trivial ideals. $\square$

Theorem 2.4. *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and each F_j is a field, $n \geq 1$ and $m \geq 1$. Then $\mathbb{AG}_N(R)$ is planar if and only if $R = R_1 \times F_1$ and $\mathfrak{m}_1$ is the only non-trivial ideal in R_1.*

Proof. If $R = R_1 \times F_1$ and $\mathfrak{m}_1$ is the only non-trivial ideal in R_1, then $\mathbb{AG}_N(R)$ is isomorphic to the graph given in Fig. 3. Hence $\mathbb{AG}_N(R)$ is planar.

Conversely, assume that $\mathbb{AG}_N(R)$ is planar. Suppose that $n \geq 2$. Then $I_1 = R_1 \times (0) \times (0) \times \cdots \times (0)$, $I_2 = (0) \times R_2 \times (0) \times \cdots \times (0)$, $I_3 = \mathfrak{m}_1 \times (0) \times (0) \times \cdots \times (0)$, $I_4 = (0) \times \mathfrak{m}_2 \times (0) \times \cdots \times (0)$ and $I_5 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0) \times \cdots \times (0)$ are non-trivial ideals in R and $I_iI_j \subseteq \mathrm{Nil}(R)$ for all $i \neq j$. From this we get that K_5 is a subgraph of $\mathbb{AG}_N(R)$, a contradiction. Hence $n = 1$.

Suppose that $m \geq 2$. Now $I_1 = R_1 \times (0) \times (0) \times \cdots \times (0)$, $I_2 = (0) \times F_1 \times (0) \times \cdots \times (0)$, $I_3 = \mathfrak{m}_1 \times F_1 \times (0) \times \cdots \times (0)$, $J_1 = (0) \times (0) \times F_2 \times \cdots \times (0)$, $J_2 = \mathfrak{m}_1 \times (0) \times F_2 \times \cdots \times (0)$ and $J_3 = \mathfrak{m}_1 \times (0) \times (0) \times \cdots \times (0)$ are non-trivial ideals in R with $I_iJ_j \subseteq \mathrm{Nil}(R)$ for all i, j and so $K_{3,3}$ is a subgraph of $\mathbb{AG}_N(R)$, a contradiction. Hence $m = 1$.

Suppose I is any non-trivial ideal in R_1. Trivially $I \subset \mathfrak{m}_1$. Consider the non-zero proper ideals $I_1 = R_1 \times (0)$, $I_2 = \mathfrak{m}_1 \times (0)$, $I_3 = I \times (0)$, $J_1 = (0) \times F_1$, $J_2 = \mathfrak{m}_1 \times F_1$ and $J_3 = I \times F_1$ of R. Then $I_iJ_j \subseteq \mathrm{Nil}(R)$ for all i, j and so $K_{3,3}$ is a subgraph of $\mathbb{AG}_N(R)$, a contradiction. Hence $\mathfrak{m}_1$ is the only non-trivial ideal in R_1. $\square$

It is well known that every commutative Artin ring R is isomorphic to the direct product of finitely many local rings. Using this, we have the following corollary which gives a characterization for $\mathbb{AG}(R)$ to be planar for a commutative Artinian ring R.

Corollary 2.5. *Let R be a commutative Artin ring with identity. Then $\mathbb{AG}_N(R)$ is planar if and only if one of the following conditions holds:*

(i) *R is a local ring with at most four non-trivial ideals;*
(ii) *$R \cong F_1 \times F_2$ or $R \cong F_1 \times F_2 \times F_3$, where each F_i is a field;*
(iii) *$R \cong R_1 \times F_1$ and $\mathfrak{m}_1$ is the only non-trivial ideal in R_1, where $(R_1, \mathfrak{m}_1)$ is a local ring and F_1 is a field.*

An undirected graph is said to be an *outerplanar* graph if it can be drawn in the plane without crossings in such a way that all the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says that a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$. Note that every outerplanar graph is planar. Now, let us obtain a characterization for $\mathbb{AG}_N(R)$ to be outerplanar.

Theorem 2.6. *Let R be a commutative Artin ring with identity. Then $\mathbb{AG}_N(R)$ is outerplanar if and only if one of the following conditions holds:*

(i) $R = F_1 \times F_2$ *or* $R = F_1 \times F_2 \times F_3$ *where F_i are fields;*
(ii) $(R, \mathfrak{m})$ *is a local ring which contains at most 3 non-trivial ideals;*
(iii) $R = R_1 \times F_1$ *where $(R_1, \mathfrak{m}_1)$ is a local ring, $\mathfrak{m}_1$ is the only non-trivial ideal in R_1 and F_1 is a field.*

Proof. Suppose $\mathbb{AG}_N(R)$ is outerplanar. Since every outerplanar graph is planar and by Corollary 2.5(ii) and Figs. 1 and 2, we get $R \cong F_1 \times F_2$ or $R \cong F_1 \times F_2 \times F_3$. By Corollary 2.5(iii) and Fig. 3, $R = R_1 \times F_1$ where $(R_1, \mathfrak{m}_1)$ is a local ring, $\mathfrak{m}_1$ is the only non-trivial ideal in R_1 and F_1 is a field.

Suppose that $(R, \mathfrak{m})$ is a local ring which contains at least 4 non-trivial ideals. Then by Theorem 1.1, $\mathbb{AG}_N(R) \cong K_t$ for $t \geq 4$ and so $\mathbb{AG}_N(R)$ is not outerplanar, a contradiction. Hence R contains at most three non-trivial ideals. □

3. Genus of nil-graph of ideals

In this section, we discuss the genus of the nil-graph of ideals of a commutative ring. In particular, we characterize all commutative Artin rings R for which $\mathbb{AG}_N(R)$ has genus one. The following two results about the genus of a complete graph and a complete bipartite graph are very useful in the subsequent sections.

Lemma 3.1. *Let $m, n \geq 3$ be integers and for a real number x, $\lceil x \rceil$ is the least integer that is greater than or equal to x. Then $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$. In particular, $g(K_n) = 1$ if $n = 5, 6, 7$.*

Lemma 3.2. *Let $m, n \geq 3$ be integers and for a real number x, $\lceil x \rceil$ is the least integer that is greater than or equal to x. Then $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$. In particular, $g(K_{4,4}) = g(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$.*

Theorem 3.3. *Let $R = F_1 \times F_2 \times \cdots \times F_n$ be a commutative ring, where each F_i is a field and $n \geq 2$. Then $g(\mathbb{AG}_N(R)) = 1$ if and only if $n = 4$.*

Proof. Assume that $g(\mathbb{AG}_N(R)) = 1$. Suppose $n \geq 5$. Then $I_1 = F_1 \times (0) \times (0) \times (0) \times (0) \times \cdots \times (0)$, $I_2 = (0) \times F_2 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $I_3 = F_1 \times F_2 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $J_1 = (0) \times (0) \times F_3 \times (0) \times (0) \times \cdots \times (0)$, $J_2 = (0) \times (0) \times (0) \times F_4 \times (0) \times \cdots \times (0)$, $J_3 = (0) \times (0) \times (0) \times (0) \times F_5 \times \cdots \times (0)$, $J_4 = (0) \times (0) \times F_3 \times F_4 \times (0) \times \cdots \times (0)$, $J_5 = (0) \times (0) \times F_3 \times (0) \times F_5 \times \cdots \times (0)$, $J_6 = (0) \times (0) \times (0) \times F_4 \times F_5 \times \cdots \times (0)$ and $J_7 = (0) \times (0) \times F_3 \times F_4 \times F_5 \times \cdots \times (0)$ are non-zero proper ideals in R and $I_i J_j \subseteq \text{Nil}(R)$ for all i, j. From this we have that $K_{3,7}$ is a subgraph of $\mathbb{AG}_N(R)$. By Lemma 3.2, $g(\mathbb{AG}_N(R)) \geq 2$, a contradiction. Hence $n \leq 4$ and by Theorem 2.2, $n = 4$.

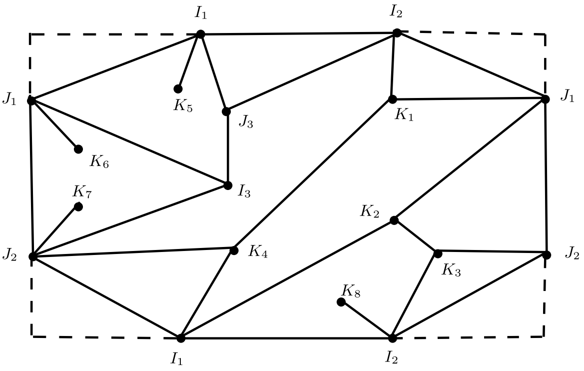

Fig. 4. Torus embedding of $\mathbb{AG}_N(F_1 \times F_2 \times F_3 \times F_4)$.

Conversely, suppose that $n = 4$. Consider the non-zero proper ideals $I_1 = F_1 \times (0) \times (0) \times (0)$, $I_2 = (0) \times F_2 \times (0) \times (0)$, $I_3 = F_1 \times F_2 \times (0) \times (0)$, $J_1 = (0) \times (0) \times F_3 \times (0)$, $J_2 = (0) \times (0) \times (0) \times F_4$, $J_3 = (0) \times (0) \times F_3 \times F_4$, $K_1 = F_1 \times (0) \times (0) \times F_4$, $K_2 = (0) \times F_2 \times (0) \times F_4$, $K_3 = F_1 \times (0) \times F_3 \times (0)$, $K_4 = (0) \times F_2 \times F_3 \times (0)$, $K_5 = (0) \times F_2 \times F_3 \times F_4$, $K_6 = F_1 \times F_2 \times (0) \times F_4$, $K_7 = F_1 \times F_2 \times F_3 \times (0)$ and $K_8 = F_1 \times (0) \times F_3 \times F_4$ of R. Then $I_iJ_j \subseteq \text{Nil}(R)$ for all i, j and so $K_{3,3}$ is a subgraph of $\mathbb{AG}_N(R)$. Therefore by Lemma 3.2, $g(\mathbb{AG}_N(R)) \geq 1$, whereas an embedding given in Fig. 4 explicitly shows that $g(\mathbb{AG}_N(R)) = 1$. □

Theorem 3.4. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$, $1 \leq i \leq n$. Then $g(\mathbb{AG}_N(R)) = 1$ if and only if one of the following conditions holds:*

(i) *R is a local ring with p non-zero proper ideals where $5 \leq p \leq 7$;*
(ii) *$R = R_1 \times R_2$ and $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only non-trivial ideals in R_1 and R_2 respectively.*

Proof. Assume that $g(\mathbb{AG}_N(R)) = 1$. Suppose that $n \geq 3$. Consider the non-zero proper ideals $I_1 = \mathfrak{m}_1 \times (0) \times (0) \times \cdots \times (0)$, $I_2 = (0) \times \mathfrak{m}_2 \times (0) \times \cdots \times (0)$, $I_3 = (0) \times (0) \times \mathfrak{m}_3 \times \cdots \times (0)$, $I_4 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0) \times \cdots \times (0)$, $I_5 = \mathfrak{m}_1 \times (0) \times \mathfrak{m}_3 \times \cdots \times (0)$, $I_6 = (0) \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times \cdots \times (0)$, $I_7 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times \cdots \times (0)$ and $I_8 = R_1 \times (0) \times (0) \times \cdots \times (0)$ of R. Then $I_iI_j \subseteq \text{Nil}(R)$ for all $i \neq j$ and so K_8 is a subgraph of $\mathbb{AG}_N(R)$. By Lemma 3.1, $g(\mathbb{AG}_N(R)) \geq 2$, a contradiction. Hence $n \leq 2$.

Assume that $n = 1$. Then R is a local ring and so by Theorem 1.1, $\mathbb{AG}_N(R)$ is complete and so $\mathbb{AG}_N(R) \cong K_p$, where p is the number of non-trivial ideals in R. If $p \geq 8$, then by Lemma 3.1, $g(\mathbb{AG}_N(R)) \geq 2$, a contradiction. If $p \leq 4$, then by Theorem 2.3, $g(\mathbb{AG}_N(R)) = 0$, a contradiction. Hence $5 \leq p \leq 7$.

Assume that $n = 2$. Let n_i be the number of non-trivial ideals in R_i, for $i = 1, 2$. Suppose that $n_i \geq 2$ for some i. Without loss of generality, we assume that $n_2 \geq 2$ and $K_2 \subset \mathfrak{m}_2$ is a non-zero proper ideal of R_2. Consider the set Ω of non-zero proper ideals $I_1 = (0) \times \mathfrak{m}_2$, $I_2 = \mathfrak{m}_1 \times (0)$, $I_3 = R_1 \times (0)$, $I_4 = (0) \times R_2$, $I_5 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $I_6 = (0) \times K_2$, $I_7 = \mathfrak{m}_1 \times K_2$, $I_8 = \mathfrak{m}_1 \times R_2$ and $J = R_1 \times \mathfrak{m}_2$ of R. Then $I_4I_8 \not\subseteq \text{Nil}(R)$ and $I_iI_j, I_4J, I_8J \subseteq \text{Nil}(R)$ for all $i \neq j$ and $i, j \neq 4, 8$ and so the subgraph induced by Ω in $\mathbb{AG}_N(R)$ contains a subgraph which is isomorphic to a subdivision of K_8. By Lemma 3.1, $g(\mathbb{AG}_N(R)) \geq 2$, a contradiction. Hence $n_i = 1$ for $i = 1, 2$.

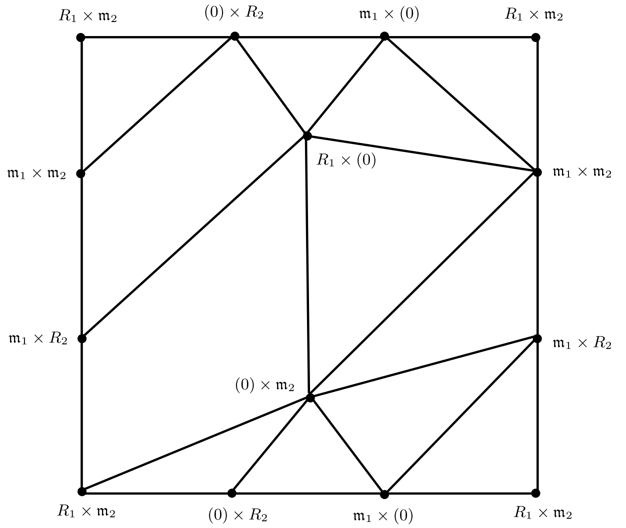

Fig. 5. Torus embedding of $\mathbb{AG}_N(R_1 \times R_2)$.

Conversely, suppose that R is a local ring with p non-zero proper ideals where $5 \leq p \leq 7$. By Theorem 1.1, $\mathbb{AG}_N(R) \cong K_p$. By Lemma 3.1, $g(\mathbb{AG}_N(R)) = 1$.

Suppose that $R = R_1 \times R_2$ and $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only non-trivial ideals in R_1 and R_2 respectively. Then $I_1 = \mathfrak{m}_1 \times (0)$, $I_2 = (0) \times \mathfrak{m}_2$, $I_3 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $I_4 = R_1 \times (0)$ and $I_5 = (0) \times R_2$ are non-trivial ideals in R with $I_iI_j \subseteq \text{Nil}(R)$ for all $i \neq j$ and so K_5 is a subgraph of $\mathbb{AG}_N(R)$. By Lemma 3.1, $g(\mathbb{AG}_N(R)) \geq 1$. An embedding of $g(\mathbb{AG}_N(R))$ in a torus is given in Fig. 5 and hence $g(\mathbb{AG}_N(R)) = 1$. $\square$

Theorem 3.5. *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and each F_j is a field, $n \geq 1$ and $m \geq 1$. Then $g(\mathbb{AG}_N(R)) = 1$ if and only if one of the following conditions holds:*

(i) *$R = R_1 \times F_1 \times F_2$ and $\mathfrak{m}_1$ is the only non-trivial ideal in R_1;*
(ii) *$R = R_1 \times F_1$ and $2 \leq n_1 \leq 3$ where n_1 is the number of non-trivial ideals in R_1.*

Proof. Assume that $g(\mathbb{AG}_N(R)) = 1$. Suppose that $n \geq 2$. Consider the non-zero proper ideals $I_1 = \mathfrak{m}_1 \times (0) \times (0) \times \cdots \times (0)$, $I_2 = R_1 \times (0) \times \cdots \times (0)$, $I_3 = (0) \times \mathfrak{m}_2 \times (0) \times \cdots \times (0)$, $I_4 = (0) \times R_2 \times (0) \times \cdots \times (0)$, $J_1 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0) \times \cdots \times (0)$, $J_2 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0) \times \cdots \times (0) \times F_1 \times (0) \times \cdots \times (0)$, $J_3 = \mathfrak{m}_1 \times (0) \times \cdots \times (0) \times F_1 \times (0) \times \cdots \times (0)$, $J_4 = (0) \times \mathfrak{m}_2 \times (0) \times \cdots \times (0) \times F_1 \times \cdots \times (0)$ and $J_5 = (0) \times \cdots \times (0) \times F_1 \times (0) \times \cdots \times (0)$ of R. Note that $I_iJ_j \subseteq \text{Nil}(R)$ for all i, j and so $K_{4,5}$ is a subgraph of $\mathbb{AG}_N(R)$. By Lemma 3.2, $g(\mathbb{AG}_N(R)) \geq 2$, a contradiction. Hence $n = 1$.

Suppose that $m \geq 3$. Consider the non-zero proper ideals $I_1 = (0) \times (0) \times (0) \times F_3 \times \cdots \times (0)$, $I_2 = \mathfrak{m}_1 \times (0) \times (0) \times F_3 \times \cdots \times (0)$, $I_3 = \mathfrak{m}_1 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $I_4 = R_1 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $J_1 = (0) \times F_1 \times (0) \times (0) \times \cdots \times (0)$, $J_2 = (0) \times (0) \times F_2 \times (0) \times \cdots \times (0)$, $J_3 = (0) \times F_1 \times F_2 \times (0) \times \cdots \times (0)$, $J_4 = \mathfrak{m}_1 \times F_1 \times (0) \times (0) \times \cdots \times (0)$ and $J_5 = \mathfrak{m}_1 \times (0) \times F_2 \times (0) \times \cdots \times (0)$ of R. Then $I_iJ_j \subseteq \text{Nil}(R)$ for all i, j and so $K_{4,5}$ is a subgraph of $\mathbb{AG}_N(R)$. By Lemma 3.2, $g(\mathbb{AG}_N(R)) \geq 2$, a contradiction. Hence $m \leq 2$.

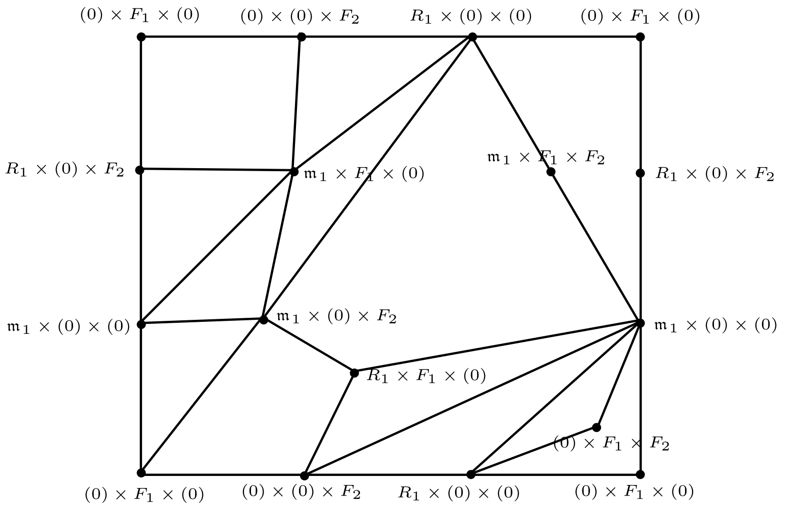

Fig. 6. Torus embedding of $\mathbb{AG}_N(R_1 \times F_1 \times F_2)$.

Assume that $m = 2$. Suppose that $\{\mathfrak{m}_1, I\}$ are the non-trivial ideals in R_1. Then $I \subset \mathfrak{m}_1$. Consider the non-zero proper ideals $I_1 = (0)\times F_1\times(0)$, $I_2 = \mathfrak{m}_1\times F_1\times(0)$, $I_3 = I\times F_1\times(0)$, $I_4 = R_1 \times F_1 \times (0)$, $J_1 = (0) \times (0) \times F_2$, $J_2 = \mathfrak{m}_1 \times (0) \times F_2$, $J_3 = I \times (0) \times F_2$, $J_4 = I \times (0) \times (0)$ and $J_5 = \mathfrak{m}_1 \times (0) \times (0)$ of R. Here we have $I_iJ_j \subseteq \text{Nil}(R)$ for all i, j and so $K_{4,5}$ is a subgraph of $\mathbb{AG}_N(R)$. By Lemma 3.2, $g(\mathbb{AG}_N(R)) \geq 2$, a contradiction. Hence $\mathfrak{m}_1$ is the only non-trivial ideal in R_1.

Assume that $m = 1$. Suppose that $n_1 \geq 4$, the number of non-zero proper ideals of R_1. Let $\{\mathfrak{m}_1, K_1, K_2, K_3\}$ be the distinct non-zero proper ideals in R_1. Then $K_i \subset \mathfrak{m}_1$ for $i = 1, 2, 3$. Consider the ideals $I_1 = (0)\times F_1$, $I_2 = \mathfrak{m}_1\times F_1$, $I_3 = K_1\times F_1$, $I_4 = K_2\times F_1$, $I_5 = K_3\times F_1$, $J_1 = R_1 \times (0)$, $J_2 = \mathfrak{m}_1 \times (0)$ $J_3 = K_1 \times (0)$, $J_4 = K_2 \times (0)$ and $J_5 = K_3 \times (0)$ of R. Then $I_iJ_j \subseteq \text{Nil}(R)$ for all i, j and so $K_{5,5}$ is a subgraph of $\mathbb{AG}_N(R)$. By Lemma 3.2, $g(\mathbb{AG}_N(R)) \geq 2$, a contradiction. By Theorem 2.4, $n_1 \neq 1$ and hence $2 \leq n_1 \leq 3$.

Conversely, assume that $R = R_1 \times F_1 \times F_2$ and $\mathfrak{m}_1$ is the only non-trivial ideal in R_1. Let $I_1 = (0) \times F_1 \times (0)$, $I_2 = \mathfrak{m}_1 \times F_1 \times (0)$, $I_3 = R_1 \times F_1 \times (0)$, $J_1 = (0) \times (0) \times F_2$, $J_2 = \mathfrak{m}_1 \times (0) \times F_2$ and $J_3 = \mathfrak{m}_1 \times (0) \times (0)$. Then $I_1, I_2, I_3, J_1, J_2, J_3$ are non-trivial ideals in R and $I_iJ_j \subseteq \text{Nil}(R)$ for all i, j and so $K_{3,3}$ is a subgraph of $\mathbb{AG}_N(R)$. By Lemma 3.2, $g(\mathbb{AG}_N(R)) \geq 1$. A torus embedding of $\mathbb{AG}_N(R_1 \times F_1 \times F_2)$ is given in Fig. 6 and hence $g(\mathbb{AG}_N(R)) = 1$.

Suppose that $R = R_1 \times F_1$ and $n_1 = 3$. Assume that $\mathfrak{m}_1, K_1$ and K_2 are the distinct non-trivial ideals in R_1. Consider the non-zero proper ideals $I_1 = (0) \times F_1$, $I_2 = \mathfrak{m}_1 \times F_1$, $I_3 = K_1 \times F_1$, $I_4 = K_2 \times F_1$, $J_1 = R_1 \times (0)$, $J_2 = \mathfrak{m}_1 \times (0)$, $J_3 = K_1 \times (0)$ and $J_4 = K_2 \times (0)$ of R. Then $I_iJ_j \subseteq \text{Nil}(R)$ for all i, j and so $K_{4,4}$ is a subgraph of $\mathbb{AG}_N(R)$. By Lemma 3.2, $g(\mathbb{AG}_N(R)) \geq 1$. A torus embedding of $\mathbb{AG}_N(R_1 \times F_1)$ is given in Fig. 7 and hence $g(\mathbb{AG}_N(R)) = 1$.

Suppose that $R = R_1 \times F_1$ and $n_1 = 2$. Let $I_1 = (0) \times F_1$, $I_2 = \mathfrak{m}_1 \times F_1$, $I_3 = K_1 \times F_1$, $J_1 = R_1 \times (0)$, $J_2 = \mathfrak{m}_1 \times (0)$ and $J_3 = K_1 \times (0)$. Then $I_iJ_j \subseteq \text{Nil}(R)$ for all i, j and so $K_{3,3}$ is a subgraph of $\mathbb{AG}_N(R)$. By Lemma 3.2, $g(\mathbb{AG}_N(R)) \geq 1$. Fig. 8 explicitly gives an embedding for $g(\mathbb{AG}_N(R))$ in a torus and hence $g(\mathbb{AG}_N(R)) = 1$.

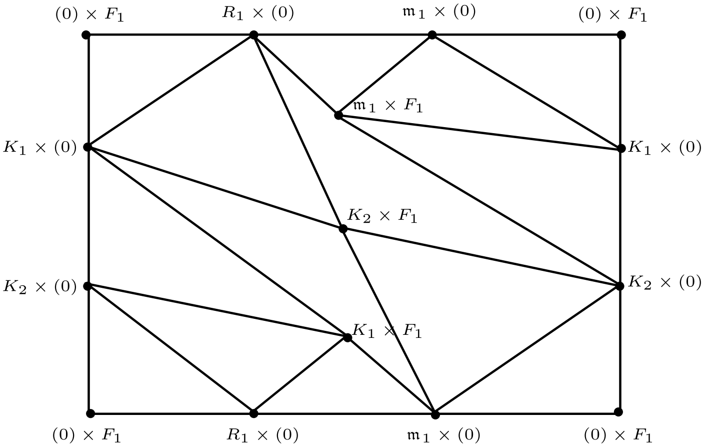

Fig. 7. Torus embedding of $\mathbb{AG}_N(R_1 \times F_1)$.

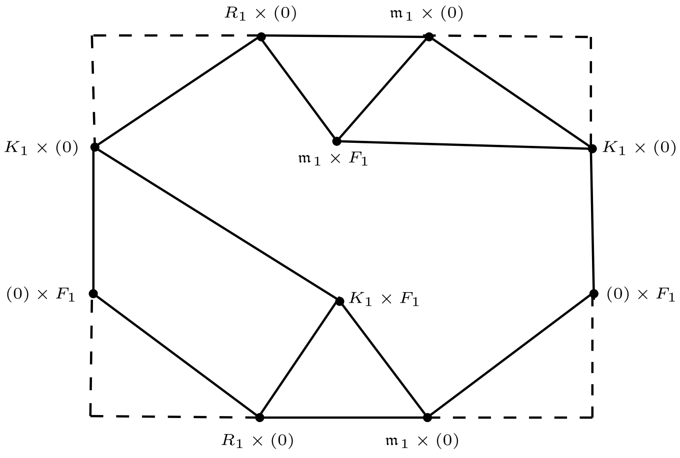

Fig. 8. Torus embedding of $\mathbb{AG}_N(R_1 \times F_1)$.

Acknowledgments

Authors thank the referee for comments to improve the presentation of the contents in the acceptable form. This work is supported by the UGC-BSR One-time grant (F.19-109/2013(BSR)) of University Grants Commission, Government of India for the first author. Also the work reported here is supported by the UGC Major Research Project (F.42-8/2013(SR)) awarded to the second author by the University Grants Commission, Government of India.

References

[1] D.F. Anderson, A. Badawi, The total graph of a commutative ring, J. Algebra 320 (2008) 2706–2719.
[2] D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra 217 (1999) 434–447.
[3] D.D. Anderson, M. Naseer, Becks coloring of a commutative ring, J. Algebra 159 (1993) 500–514.

[4] T. Asir, T. Tamizh Chelvam, On the genus two characterizations of unit, unitary cayley and co-maximal graphs, ARS Combin. in press.

[5] T. Asir, T. Tamizh Chelvam, On the genus of generalized unit and unitary cayley graphs of a commutative ring, Acta Math. Hungar. 142 (2) (2014) 444–458. http://dx.doi.org/10.1007/s10474-013-0365-1.

[6] T. Asir, T. Tamizh Chelvam, On the intersection graph of gamma sets in the total graph I, J. Algebra Appl. 12 (4) (2013) #1250198. http://dx.doi.org/10.1142/S0219498812501988.

[7] A. Badawi, On the annihilator graph of a commutative ring, Comm. Algebra 42 (2014) 108–121. http://dx.doi.org/10.1080/00927872.2012.707262.

[8] A. Badawi, On the dot product graph of a commutative ring, Comm. Algebra 43 (2015) 43–50.

[9] I. Beck, Coloring of a commutative ring, J. Algebra 116 (1988) 208–226.

[10] M. Behboodi, Z. Rakeei, The annihilating-ideal graph of commutative rings I, J. Algebra Appl. 10 (4) (2011) 727–739.

[11] M. Behboodi, Z. Rakeei, The annihilating-ideal graph of commutative rings II, J. Algebra Appl. 10 (4) (2011) 741–753.

[12] H.R. Maimani, M. Salimi, A. Sattari, S. Yassemi, Comaximal graph of commutative rings, J. Algebra 319 (2008) 1801–1808.

[13] H.R. Maimani, C. Wickham, S. Yassemi, Rings whose total graphs have genus at most one, Rocky Mountain J. Math. 42 (2012) 1551–1560.

[14] F. Shaveisi, R. Nikandish, The nil-graph of ideals of a commutative ring, Bull. Malays. Math. Sci. Soc. 39 (2016) 3–11.

[15] T. Tamizh Chelvam, T. Asir, On the genus of the total graph of a commutative ring, Comm. Algebra 41 (2013) 142–153. http://dx.doi.org/10.1080/00927872.2011.624147.

[16] T. Tamizh Chelvam, S. Nithya, Crosscap of the ideal based zero-divisor graph, Arab J. Math. Sci. 22 (1) (2016) 29–37. http://dx.doi.org/10.1016/j.ajmsc.2015.01.003.

[17] T. Tamizh Chelvam, K. Selvakumar, Central sets in annihilating-ideal graph of a commutative ring, J. Combin. Math. Combin. Comput. 88 (2014) 277–288.

[18] T. Tamizh Chelvam, K. Selvakumar, Domination in the directed zero-divisor graph of ring of matrices, J. Combin. Math. Combin. Comput. 91 (2014) 155–163.

[19] T. Tamizh Chelvam, K. Selvakumar, On the intersection graph of gamma sets in the zero-divisor, Discrete Math. Algebra Appl. 7 (1) (2015) # 1450067. http://dx.doi.org/10.1142/S1793830914500670.

[20] T. Tamizh Chelvam, K. Selvakumar, On the connectivity of the annihilating-ideal graphs, Discuss. Math. Gen. Algebra Appl. 35 (2015) 195–204. http://dx.doi.org/10.7151/dmgaa.1241.

[21] T. Tamizh Chelvam, K. Selvakumar, V. Ramanathan, On the planarity of the k-zero-divisor hypergraphs, AKCE Int. J. Graphs Comb. 12 (2015) 169–176. http://dx.doi.org/10.1016/j/akcej201511.011.

[22] A.T. White, Graphs, Groups and Surfaces, North-Holland, Amsterdam, 1973.