PAPER • **OPEN ACCESS**

Oscillatory Behavior of Nonlinear Delay Differential Equation

To cite this article: D Seethalakshmi and S Balamuralitharan 2018 *J. Phys.: Conf. Ser.* **1000** 012037

IOP Conf. Series: Journal of Physics: Conf. Series **1000** (2018) 012037 doi:10.1088/1742-6596/1000/1/012037

Oscillatory Behavior of Nonlinear Delay Differential Equation

D Seethalakshmi[1], S Balamuralitharan[2]

[1]Department of Mathematics, M.Kumarasamy College of Engineering (Autonomous), Thalavapalayam, Karur -639 113, Tamil Nadu, India.
[2] Department of Mathematics, SRM Institute of Science and Technology, Faculty of Engineering and Technology, Kattankulathur – 603 203,Tamil Nadu, India.

seetha_subash@rediffmail.com, balamurali.maths@gmail.com

Abstract. This paper is imposed an oscillatory behavior of nonlinear delay differential equation (NDDE). In particular, the main results got improve those studied in the literature survey. A variety of oscillatory behaviorsare given.

1. Introduction

In this research, there has been an involving importance on the oscillatory behavior of NDDE. For example, see the recent papers [1] – [15]. In this paper, we discuss our attention of NDDE of the form

$$(\xi)' + p(t)\chi = 0, \quad \text{(A)}$$

where $\xi = |x'(t-\tau)|^{\alpha-1} x'(t-\tau), \quad \chi = f\left[x(t-\tau)\right].$

Here we will study the properties of hypotheses (H1) - (H5):

(H1) $\alpha > 1$ is a real constant;

(H2) $p \in C[t_0, \infty), p(t) > 0;$

(H3) $\xi \in C^1[t_0, \infty), \lim_{t\to\infty} x(t-\tau) = \infty$;

(H4) $f \in C(-\infty,\infty)$, f is non-decreasing o$(-\infty,\infty)$, $f \in C^1(M)$, $M = (-\infty, 0) \cup (0, \infty)$, $\chi > 0 \ \ for \ x(t-\tau) \neq 0.$

(H5) $\xi \in C^1[T_x, \infty)$ The equation of (A) we assume $x(t-\tau) \in C^1[T_x, \infty)$, $T_x \geq t_0$. which has the hypotheses (H5) and satisfies (A) on $[T_x, \infty)$. Hence we obtain the result of (A) that satisfy

$\sup\{|x(t-\tau): \ t \geq T|\} > 0 \ \ for\ all\ T \geq T_x$. The condition $\int_0^\infty p(s)\, ds = \infty$ is oscillatory for the equation of (A). This paper is to discuss oscillatory results that are consider for $\alpha = 2$, equation (A) for the NDDE

$$(\Omega)' + p(t)\chi = 0, \quad (1)$$

where $\Omega = |x'(t-\tau)| \ x'(t-\tau).$

2. Main Results of NDDE

In this section, we will prove the main results of this research work.

Theorem 2.1 Let $\alpha = 2$. Let $f^{'}\left(x\left(t-\tau\right)\right)$ be increasing open interval $\left(-\infty,-t\right)$ and decreasing open interval $\left(t,\infty\right)$, $t \geq 0$. Let us assume that

$$\int_0^\infty p(s)\left| f\left[c\left(s-\tau\right)\right]\right| ds = \infty \quad \text{for all } c \neq 0 \tag{2}$$

And moreover

$$\int_0^\infty \left(x(t-\tau)^2 p(s) - \frac{x(t-\tau)^{'}}{f^{'}\left[\pm\lambda(s)\right]} \right) ds = \infty \ \ for \ \ some \ \ \lambda > 0 \tag{3}$$

Then equation (1) is oscillatory.

Proof : Assume that Equ .(1) has a positive solution of $\tau = x(t-\tau)$. Then

$$(\Omega)^{'} = -p(t)\,\chi < \mathbf{0}. \tag{4}$$

Hence Ω is decreasing. Therefore, either $\tau' = x^{'}(t-\tau) > 0, or\, \tau' = x^{'}(t-\tau) < 0.$ since

$$0 > (\Omega)^{'} = 2\Omega,$$

We assume that $\tau'' < 0$. Then yields $\tau \to -\infty \ \ as \ \ t \to \infty$. This is a contradiction. So we conclude that $\tau > 0,\ \tau' > 0,\ \ \tau'' < 0$ and

$$\left[(\tau')^2\right]^{'} = -p(t)\,\chi.$$

We define
$$y(t-\tau) = \frac{[\tau\tau']^2}{\chi} \tag{5}$$

Then $y(t-\tau) > 0$ and

$$y^{'}(t-\tau) = 2\tau\,\frac{[\tau']^3}{\chi} + \frac{\left[(\tau\tau')^2\right]^{'}}{\chi} - \frac{[\tau\tau']^2 f^{'}[\tau](\tau')^2}{\chi^2}$$

$$= 2\,\frac{\tau'}{\tau} y(t-\tau) - (\tau)^2 p(t) - y(t-\tau)\frac{f^{'}[\tau](\tau')^2}{\chi} \tag{6}$$

We declare that $\tau' \to 0 \ as \ t \to \infty$. To prove it is contradiction, that is $\tau' \to 2c \, as \ t \to \infty, c > 0.$ Then $\tau' \geq 2c$ that on integration implies t_1 to t

$$\tau \geq \tau_1 + 2c\left(t - t_1\right) \geq ct \tag{7}$$

We get (4) from t_1 to t and substitute (7)

$$-[\tau']^2 + [\tau']^2 = \int_{t_1}^{t} p(s)\, f\left[x(s-\tau)\right] ds > \int_{t_1}^{t} p(s)\, f\left[c(s-\tau)\right] ds$$

Putting $t \to \infty$ we have

$$\int_{t_1}^{\infty} p(s)\, f[c(s-\tau)] ds < \infty.$$

It contains that $\tau' \to 0 \ as \ t \to \infty$. Hence , for any $\lambda > 0$ there exists a t_1 such that $\lambda/2 > \tau'.\ t \geq t_1$. Integrate from $t_1 \ to \ t$ we have

$\tau \le \tau_1 + \dfrac{\lambda}{2}(t-t_1) \le \lambda t,\ t \ge t_2 \ge t_1$ (8)

$\chi' \ge f'\left[\lambda(t-\tau)\right]$

Conversely since τ' is decreasing and $\tau' \to \boldsymbol{0}$ *as* $t \to \infty$.

It consider that $\tau' \ge \tau'' \ge (\tau')^2$ (9)

Equating (8) and (9) substitute with (6) we have

$$y'(t-\tau) \le \tau^2 p(t) + 2\frac{\tau'}{\tau} y(t-\tau) - \frac{\tau' f'\left[\lambda(t-\tau)\right]}{\tau^2} y^2(t-\tau)$$

$$= -\tau^2 p(t) - \frac{\tau' f'\left[\lambda(t-\tau)\right]}{\tau^2}\left[\left(y(t-\tau) - \frac{\tau}{f'\left[\lambda(t-\tau)\right]}\right)^2 - \frac{\tau^2}{\left(f'\left[\lambda(t-\tau)\right]\right)^2}\right] \quad (10)$$

Integrating the above inequality from t_2 to t we conclude in the point of (3) that $y(t-\tau) \to -\infty$ as $t \to \infty$. This is a contradiction. Hence the proof is complete.

For $\alpha = 2$ theorem 2.1 gives.

Corollary 2.1 Let $f'\left(x(t-\tau)\right)$ be increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, $t \ge 0$. Let us consider that (2) holds for any $c \ne 0$ and

$$\int_0^\infty \left(x(s-\tau)\, p(s) - \frac{x'(s-\tau)}{x(s-\tau)\, f'\left[\pm\lambda(s)\right]} \right) ds = \infty \text{ for some } \lambda > 0. \quad (11)$$

Then we have

$(\xi)' + p(t)\,\chi = 0$ (12)

is oscillatory

Theorem 2.2 Let $\alpha = 2$. Let $f'(\tau)$ be decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, $t \ge 0$. Let us consider that (2) holds for any $c \ne 0$. If

$$\int_0^\infty \left(\left(x(s-\tau)\right)^2 p(s) - M\, x(s-\tau)\,\tau' \right) ds = \infty \quad \textit{for some} \quad M > 0. \quad (13)$$

Then Eqn. (1) is oscillatory.

Proof: Let M > 0. In theorem 2.1, we prove that $\tau' > 0$, $\chi < 0$ *and* $\xi \to 0$ *as* $t \to \infty$. Then there exists c > 0 such that $\tau > c$. If $y(t-\tau)$ be defined by (5), then $y(t-\tau) > 0$ and (6) is determined. It is uncomplicated to verify that

$\Omega' \ge f'(c)(\tau')^{-1}(\tau')^2$ (14)

Since $\tau' \to 0$ then for any $\lambda > \boldsymbol{0}$ we have $\tau' < \lambda$. see from (14) that

$$\Omega' \ge f'(c)\,\lambda^{-1}(\tau')^2 = K\,(\tau')^2$$

Where λ is taken such that $f'(c)\,\lambda^{-1} = 1/(M)$ *and* $K = f^{\frac{1}{2}}\tau$. We have

$$y'(t-\tau) \le -(\tau^2)\, p(t) + 2\frac{\tau}{\tau'} y(t-\tau) - K\frac{\tau'}{\tau} y^2(t-\tau)$$

$$= -\tau^2\ p(t)\ -\ K\ \frac{\tau'}{\tau^2}\left[\left(y(t-\tau)-\ \frac{\tau}{K}\right)^2\ -\ \frac{\tau^2}{K^2}\right]$$

$$\leq\ -\ \tau^2\ p(t)\ +\frac{1}{K}\tau\tau'.(15)$$

Integrate from t_1 *to* t, and then putting $t\to\infty$. This is contradiction. The proof is over.

For $\alpha\ =\ \boldsymbol{2}$ theorem 2.2 gives

Corollary 2.2 Let $\alpha\ =\boldsymbol{2}$. Let us assume that

$$\int_0^\infty x(s-\tau)^2 p(s)\,ds=\ \infty\ (16)$$

and for some M >0

$$\int_0^\infty\left(x(s-\tau)^2\ p(s)-Mx(s-\tau)^{-1}\ \tau'\right)ds\ =\ \infty\ (17)$$

Then we have

$$(\Omega)'\ +\ p(t)\ \chi\ =\ 0\ (18)$$

is oscillatory.

Theorem 2.3

Let us assume that

$$h(\tau)\ \mathrm{sgn}(\tau)\geq\ f(\tau)\mathrm{sgn}(\tau)\ ,\ \tau\neq 0 \qquad (19)$$

and (H1 –H5) holds. Then Eqn. (1) is oscillatory.

Proof: Let us consider that τ is a positive solution of Eq. (1) . Then $\tau'>0$, $\tau''<0$ and

$$\left([\tau']^2\right)'=\ -\ p(t)\ \ h[\tau]\ \ \leq -\ p(\tau)\ \chi.$$

Let $y(t-\tau)$ be defined by (5) . Then $y(t-\tau)\ >0$ and

$$y'(t-\tau)\leq\ 2\ \frac{\tau'}{\tau}y(t-\tau)-\tau^2p(t)-y(t-\tau)\frac{f'[\tau]\tau\tau'}{f[\tau]}$$

The proof is same to the previous proof and so it can be absent.

Example 1 Consider the second order NDDE

$$\tau''+\ p(t)\ln\left(\ 1+|\tau|\right)\mathrm{sgn}\ \tau\ =\ 0.(20)$$

From Eq.(20),

$$\int_0^\infty p(s)\ln\left(1+c(s-\tau)\right)ds\ \ =\infty \qquad \textit{for any } c>0 \text{ and}$$

$$\int_0^\infty\left(x(s-\tau)\,p(s)-\ \frac{1+\lambda(s)}{x'(s-\tau)}x(s-\tau)\right)ds\ \ =\ \infty \qquad \textit{for some } \lambda>0.$$

We take $\tau=t-\pi,\ p(t)=t\ +\ \pi/4.$ *therefore we get*

$$\left(e^t\left(\tau+\ \frac{1}{2}\ \tau\left(t-\frac{\pi}{2}\right)\ +\frac{1}{2}\ \tau\ \left(t-\frac{\pi}{2}\right)\right)'\ \right)'+12\sqrt{33}\,e^{\,t}\ \ \tau\ \left(t-\ \frac{1}{2}\ \arctan\frac{\sqrt{33}}{4}\right)=0.$$

Hence, equation (1) is oscillatory.

Remark 1 Let (H4 and H5) and (19) holds. We consider that all results of Theorem 2.1 is verified. Then Eqn. (1) is oscillatory. It gives the same oscillatory criteria proof to Theorem 2.2 and 2.3 for (1) with $\alpha = \boldsymbol{2}$.

It is an open problem how to conclude oscillatory behavior similar to Theorem 2.1 to2.3 for Eqn. (1) with $\alpha = \boldsymbol{2}$. The following theorem gives a different result.

Theorem 2.4
We assume that

$$\int_{t_0}^{\infty} \frac{dx}{|f(\pm\tau)|^{1/\alpha}} < \infty \text{ and}$$

$$\int_{t_0}^{\infty} x'(s-\tau)\left(\int_{s}^{\infty} p(x)\,dx\right)^{1/\alpha} ds = \infty.$$ Then Eq.(1) is oscillatory.

Proof:
Let us assume that τ is a positive solution of (1). Here $\tau' > 0$ *and* $\tau'' < 0$. Integrate (1) from t to s we have

$$-[\tau']^{\alpha} + [\tau']^{\alpha} = \int_{t}^{s} p(x)\, f\,[\tau]\; dx \ \ge\ f[\tau] \int_{t}^{s} p(s)\, ds.$$

Using conditions of $u'(t-\tau)$ *and* putting $s \to \infty$ we have

$$(\tau')^{\alpha} \ge (\tau'')^{\alpha} \ge f[\tau] \int_{t}^{\infty} p(s)\, ds. \quad (21)$$

It contains from (21) that

$$\frac{\tau^2}{f^{1/\alpha}[\tau]} \ \ge\ \tau'\left(\int_{t}^{\infty} p(x)\,dx\right)^{\frac{1}{\alpha}}$$

integrate from t_1 *to* t gives

$$\int_{\tau_1-\tau}^{\tau} \frac{ds}{f^{1/\alpha}(s)} \ \ge\ \int_{t_1-\tau}^{t-\tau} x'(s-\tau)\left(\int_{s}^{\infty} p(x)\,dx\right)^{1/\alpha} ds. \quad (22)$$

The left side of the inequality (22) is bounded; on the other hand the right side of the inequality (22) as $t \to \infty$. Hence the proof is complete.

Remark 2: Theorem 2.4 cannot be developed to equation (1) with $f(x-\tau) = x(t-\tau)$ and $\alpha > \boldsymbol{1}$.

For $\alpha = \frac{\boldsymbol{1}}{\boldsymbol{2}}$ theorems 2.4 gives:

Let us assume that

$$\int_{t_0}^{\infty} \frac{du}{|f(\pm\tau)|^2} < \infty$$

and

$$\int_{t_0}^{\infty} \tau'\left(\int_{s}^{\infty} p(x)\,dx\right)^2 ds = \infty.$$

Then Eqn. (1) is oscillatory.

References

[1] H.J. Li, C.C. Yeh, An integral criterion for oscillation of nonlinear differential equations, *Japonica Math.* 41 (1) , 185-188.

[2] D.D. Mirzov, On the oscillation of a system of nonlinear differential equations, Diff. Uravnenija 9 , 581-583.

[3] Hong, H.-L.lian, W.-Cyeh, C.-C Oscillation criteria for half-linear differential equations with functional argument, Nonlinear World 3, 849-855.

[4] Jozefdzurina Oscillation Criteria for second order Nonlinear Retarded Differnetial Equations, Math. Slovaca, 54 , No. 3, 245-253

[5] O. Lopes, Forced oscillations in nonlinear neutral differential equations, SIAM J. Appl. Math. 29-1 196-207.

[6] J. K. Hale, S. M. VerduynLunel, Introduction to Functional Differential Equations, Applied Mathematical Sciences 99, Springer-Verlag, 2003.

[7] I. Gyori, G. Ladas, Oscillation Theory of Delay Differential Equations, Oxford Mathematical Monographs, Oxford University Press, 1991.

[8] P. S. Gromova, Stability of solutions of nonlinear equations in the asymptotically critical case, Mathematical Notes 1:6 472-479.

[9] T. Erneux, Applied Delay Differential Equations, Springer, 2009.

[10] P. Das, N. Misra, A necessary and sufficient condition for the solution to a functional differential equation to be oscillatory, J. Math. Anal. Appl. 205 78-87.

[11] R.P. Agarwal, S.L. Shien, C.C. Yeh, Oscillation criteria for second-order retarded differential equations, Math. Comput. Modelling 26 , 1- 11.

[12] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, Oscillation Theory of Differential equations with Deviating Arguments, Marcel Dekker Inc., New York .

[13] P.J.Y. Wong, R.P. Agarwal, Oscillatory behavior of solutions of certain second order nonlinear differential equations, J. Math. Anal. Appl. 198, 337-354.

[14] M. Peng, W. Ge, L. Huang, Q. Xu, A correction on the oscillatory behavior of solutions of certain second order nonlinear differential equations, Appl. Mat. Comp. 104 , 207-215.