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Planar graph characterization of γ - Uniquely colorable graphs

M Yamuna and A Elakkiya
Department of Mathematics, School of Advanced Sciences, VIT University, Vellore-632014, India

E-mail: myamuna@vit.ac.in

Abstract. A graph G = (V, E) is uniquely colorable if the chromatic number χ (G) = n and every n – coloring of G induces the same partition of V. Uniquely colorable graphs whose chromatic partition contains at least one γ - set is γ - uniquely colorable graphs In this paper, we characterize the planarity of γ - uniquely colorable graphs.

1. Introduction

Dominating sets has been used in graph theory for characterizing graph based on various properties. In [1] Bing Zhou investigated the dominating – χ – color number, $d_\chi(G)$, of a graph G. That is the maximum number of color classes that are also dominating when G is colored using χ (G) colors and also proved that $d_\chi(G \vee H) = d_\chi(G) + d_\chi(H)$ where G ∨ H is the join of G and H. In [2] Arumugam, et al introduced three fundamental parameters which involve independent domination and chromatic number and presented several interesting results and unsolved problems on these parameters. In [3], Ramachandran et al examined the relation between the domination number, chromatic number and dominating χ-color number of Harary graph $H_{k,\ n}$. Characterizing planar graphs based on graph properties is a common problem discussed by various authors. In [4], Yamuna and Karthika haveobtained a characterization of planar graphs when G and $\overline{G}$ are γ - stable. In [5], Enciso and Dutton have classified planar graph based on the complement of G. They have proved that, if G is a planar graph, then $\gamma(\overline{G}) \le 4$.

2. Terminology

We consider only simple connected undirected graphs G = (V, E) with n vertices and m edges. H is a subgraph of G, if vertex set of H is contained in vertex set of G and (u v) ∈ E (H) implies (u v) ∈ E (G). A subgraph H is said to be an induced subgraph of G if for every pair u, v of vertices, (u v) ∈ E (H) implies (u v) ∈ E (G) and is denoted by 〈 H 〉. Two graphs are said to be homeomorphic if one graph can obtained from the other by the creation of edges in series or by the merging of edges in series. In graph theory, K_5 and $K_{3,3}$ are called Kuratowski′s graph. A path is a trail in which all vertices (except perhaps the first and last ones) are distinct, P_n denotes the path with n vertices. A cycle is a circuit in which no vertex except the first (which is also the last) appears more than once. C_n is a cycle with n vertices. K_n is a complete graph with n vertices. The complement of a graph G is a graph $\overline{G}$ on the same vertices ∍ two distinct vertices of $\overline{G}$are adjacent if and only if they are not adjacent in G. A graph G = (V, E) is uniquely colorable if the chromatic number and every n –

coloring of G induces the same partition of V. For properties related to graph theory we refer toHarary [6].

A set of vertices D in G is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set – abbreviated MDS. The cardinality of any MDS for G is called the domination number of G and it is denoted by $\gamma(G)$. For properties related to domination we refer to Haynes, Hedetniemi and Slater [7].

3. Result and discussion

3.1. Planarity of γ- uniquely colorable graphs

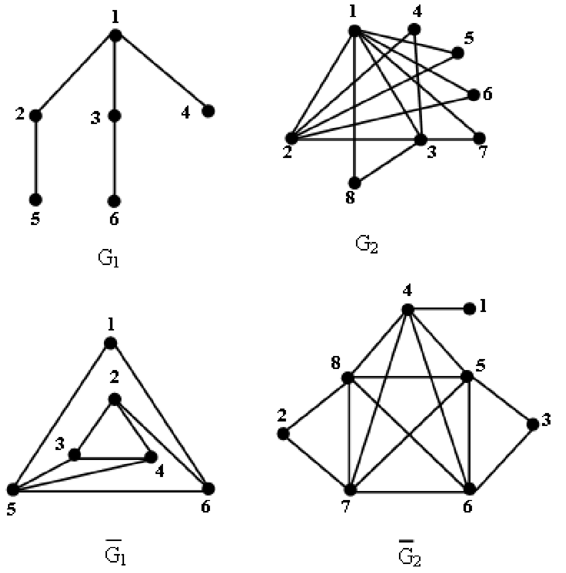


Figure 1. Uniquely colorable graphs

In Figure 1. G_1 and G_2 are uniquely colorable. $\overline{G_1}$is planar and $\overline{G_2}$ is non – planar. So when G is γ - uniquely colorable the complement graph may or may not be planar. We proceed to characterize planarity of $\overline{G}$ when G is γ - uniquely colorable.

We recollect Kuratowski′s theorem perhaps the most interesting and important results on graph planarity.

We shall use this theorem to classify the cases under which the complement of γ - uniquely colorable graph is planar. Whenever $\gamma(G) = 1$, $\overline{G}$ is disconnected and vice versa. In any γ - chromatic partition, the sets considered in the partition are independent implies $\gamma(G) \leq 4$. So in the remaining part of the paper it is sufficient to restrict our discussion to cases where $2 \leq \gamma(G) \leq 4$. In the remaining part of the paper, we stick onto the following notation.

1. P is a γ - chromatic partition for G
2. $P = \{ V_1, V_2,\dots,V_k \}$, $V_1 = \{ a_1,a_2,\dots, a_{k1} \}$, $V_2 = \{ b_1, b_2, \dots, b_{k2} \},\dots, V_k = \{q_1, q_2,\dots, q_{k17} \}$.
3. V_1 is always a γ - set.

Since each partition is independent $| V_i | \leq 4$ (else$\overline{G}$ is always non – planar).

To classify the planarity of $\overline{G}$, we follow the following pattern

a. $| P | = 2, \gamma(G) = 2, 3.$
b. $| P | = 3, \gamma(G) = 2.$
c. $| P | = 4, \gamma(G) = 2.$

a. $| P | = 2, \gamma (G) = 2, | V_1| = | V_2 | = 2$

Since γ(G) = 2 , every partition of G must contain at least two vertices. G has only four vertices, implies $\overline{G}$ is planar.

b. | P | = 2 , γ (G) = 2, | V_1| = 2, | V_2 | = 3

| V (G) | = 5. If $\overline{G}$ is non planar, then $\overline{G}$ is K_5 is the only possibility implies, G is a disconnected graph. Since we have fixed G, $\overline{G}$ has connected graph this is not possible implies $\overline{G}$ is planar.

c. | P | = 2 , γ (G) = 2, | V_1| = 2, | V_2 | = 4

If possible assume that $\overline{G}$ is non planar.

1. If K_5 is a subgraph of $\overline{G}$ say $\langle$ a_1, b_1, b_2, b_3, $b_4\rangle$ is K_5, thenin G $\langle$ a_1, b_1, b_2, b_3, $b_4\rangle$ is independent. In $\overline{G}$ a_2 is adjacent to at least one of a_1, b_1, b_2, b_3, b_4 (else $\overline{G}$ is a disconnected graph) implies in G a_2 is not adjacent to at least one of a_1, b_1, b_2, b_3, b_4 (say not adjacent to a1). Since in G $\langle$ a_1, b_1, b_2, b_3, $b_4\rangle$ is disconnected graph implies K_5 is not a subgraph of $\overline{G}$.
2. If $\overline{G}$ is $K_{3,3}$, then $\langle$ a_1, a_2, b_1, b_2, b_3, $b_4\rangle$ is disconnected in G implies $\overline{G}$ is not $K_{3,3}$.
3. If $\overline{G}$ is a graph that is homeomorphic to K_5, then$\overline{G}$ is K_5 with one edge subdivided.

If the subdivided edge is as seen in Fig. 2

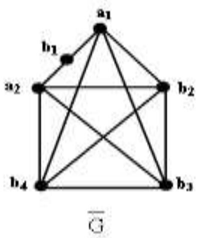


Figure 2. Subdivided Edge in $\overline{G}$.

In G { a_1, b_1 } and { a_2, b_1 } are independent. { a_1, b_1} , { a_2, b_2, b_3, b_4 } and { a_2, b_1 }, { a_1, b_2, b_3, b_4 } are chromatic partitions for G, a contradiction to the assumption that G is uniquely colorable implies $\overline{G}$ is not a graph homeomorphic to K_5. If the subdivided edge is as seen in Fig. 3

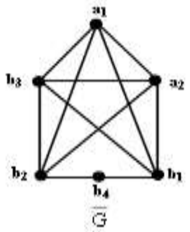


Figure 3. Subdivided Edge in $\overline{G}$.

In G, { b_2, b_4 } , { b_1, b_4 } are independent sets. {{ b_2, b_4 } , { a_1, a_2, b_1, b_3 }} and { { b_1, b_4 }, { a_1, a_2, b_2, b_3 } } are chromatic partitions for G, a contradiction to the assumption that G is uniquely colorable.

d. | P | = 2, γ(G) = 3, | V_1 | = 3, | V_2 | = 3.

If $\overline{G}$ is non planar, then we get a contradiction as in case c.

e. | P | = 3, γ(G) = 2, | V_1 | = 2, | V_2 | = 2, | V_3 | = 2.

If $\overline{G}$ is non planar, then we get a contradiction as in case c.

f. | P | = 3, γ(G) = 2, | V_1 | = 2, | V_2 | = 2, | V_3 | = 3.

If possible assume that $\overline{G}$ is non planar. If K_5 is a subgraph of $\overline{G}$ say $\langle$ a_1, a_2, b_1, b_2, $c_1\rangle$, then in G {{ a_1, a_2, b_1, b_2, c_1 } { c_2 }, { c_3 }} is a γ - chromatic partition for G other than P, a contradiction to our assumption that G is uniquely colorable.

If a subgraphhomeomorphic to K_5 is contained in $\overline{G}$, then the following are possible.

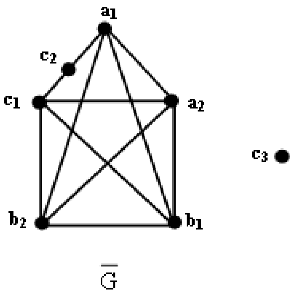


Figure 4. Subdivided Edge in $\overline{G}$.

Here P_1 = {{ c_1, c_2 }, { c_3 }, { a_1, a_2, b_1, b_2 }} is a chromatic partition for G other than P, a contradiction to our assumption that G is uniquely colorable.

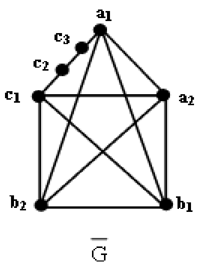


Figure 5. Subdivided Edge in $\overline{G}$.

Here P_1 = {{ c_1, c_2 }, { c_3 }, { a_1, a_2, b_1, b_2 }} is a chromatic partition for G other than P, a contradiction to our assumption that G is uniquely colorable.

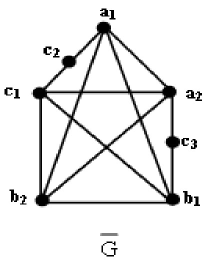


Figure 6.

Here P_1 = {{a_1, c_2 }, { a_2, c_3 }, { b_1, b_2, c_1 }} and P_2 = {{ a_1, a_2, b_2 }, { b_1, c_3 }, {c_1, c_3 }} are the two chromatic partition for G other than P, a contradiction to our assumption that G is uniquely colorable implies $\overline{G}$ is not a graph homeomorphic to K_5.

If $K_{3,3}$ is a subgraph of $\overline{G}$ say ⟨ a_1, a_2, b_1, b_2, c_1, c_2 ⟩ is $K_{3,3}$ as seen in Fig. 7.

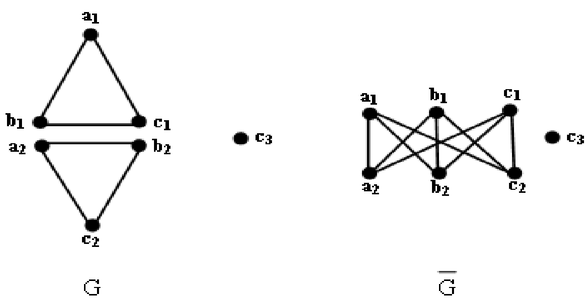


Figure 7.

In graph G, we have edges as in Fig. 7 (more edges will be present , edges required for discussion is drawn in the Fig.). We know that P = {{ a_1, a_2 }, { b_1, b_2 }, {c_1, c_2, c_3 } } is a γ - chromatic partition for G. Let X_1 = { a_1, b_1 , c_1 }, X_2 = { a_2, b_2 , c_2 }, X_3 = { c_3 }. ⟨X_i⟩, i = 1, 2 is complete , implies any independent set with three vertices includes at least one vertex from every X_i, i = 1, 2, 3. Let { a_1, a_2, c_3 } be a possible independent set. { b_1, b_2 }, { c_1, c_2 }, { b_1, c_2 }, { b_2 c_1, } are independent sets.

P_1 = { { b_1, c_2 }, { b_2, c_1 }, { a_1, a_2, a_3 }}

P_2 = {{ b_1, c_1 }, { b_2, c_2 }, { a_1, a_2, a_3 }} are the chromatic partition for G such that $| P_i | = 3$, i = 1, 2, a contradiction to our assumption that G is uniquely colorable.

If $\overline{G}$ is the graph homeomorphic to $K_{3,3}$ as seen in Fig.8 b then G is as seen in Fig.8 a

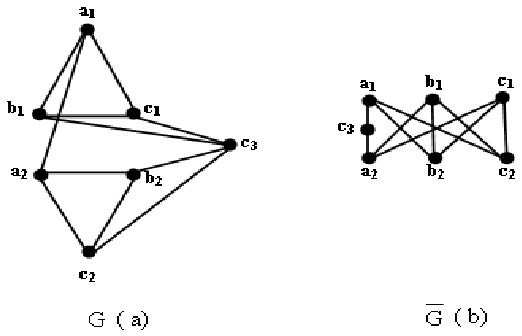


Figure8.

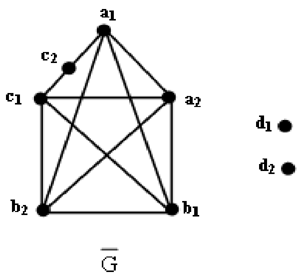
From the graph G, we observe that an independent set with three vertices is not possible. Hence a γ - chromatic partition P = { V_1, V_2, V_3 } such that $| V_3 | = 3$ is not possible , a contradiction to our assumption that P is a partition for G , implies $\overline{G}$ is not a graph homeomorphic to $K_{3,3}$

g. $| P | = 4$, $\gamma(G) = 2$, $| V_i | = 2$, i = 1 to 4.

If possible assume that $\overline{G}$ is non planar.

If K_5 is a subgraph of $\overline{G}$ say $\langle a_1, a_2, b_1, b_2, c_1 \rangle$ is K_5, then P_1 = {{ a_1, a_2, b_1, b_2, c_1 }, { c_2 }, { d_1 }, { d_2 } } is a γ - chromatic partition for G other than P, a contradiction to our assumption that G is uniquely colorable graph.

If a subgraphhomeomorphic to K_5 is contained in $\overline{G}$, then the following cases are possible.



e

Figure9.

Chromatic partition for the graph in Fig. 9 is P_1= { a_1, a_2, b_1, b_2}, { c_1, c_2 }, { d_1 }, { d_2}.

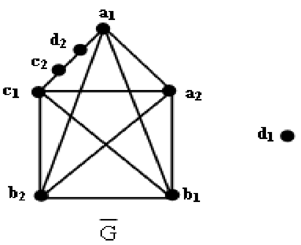


Figure10.

Chromatic partition for the graph in Fig. 10 is P_1= { a_1, a_2, b_1, b_2}, { c_1, c_2 }, { d_1 }, { d_2}.

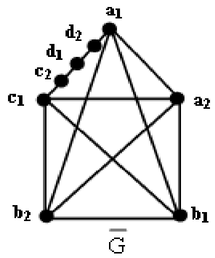


Figure11.

Chromatic partition for the graph in Fig.11 is $P_1 = \{ a_1, a_2, b_1, b_2\}, \{ c_1, c_2 \}, \{ d_1 , d_2\}$.

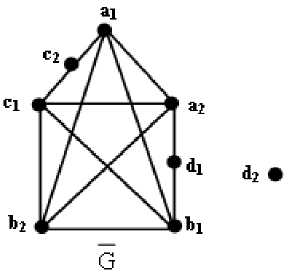


Figure12.

Chromatic partition for the graph in Fig. 12 is $P_1 =\{ a_1, c_2 \}, \{ a_2, d_1 \}, \{ b_1, b_2, c_1 \}, \{ d_2 \}$.

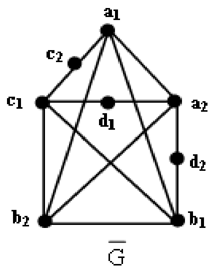


Figure13.

Chromatic partition for the graph in Fig. 13 is $P_1 = \{ a_1, c_2 \}, \{ a_2, d_1 \}, \{ b_1, b_2, c_1 \}, \{ d_2 \}$.

In all cases, P_1 is a chromatic partition for G other than P, a contradiction to our assumption that G is uniquely colorable graph.

If $K_{3,3}$is a subgraph of $\overline{G}$ say $\langle a_1, a_2, b_1, b_2, c_1, c_2\rangle$ is $K_{3,3}$ as seen in Fig. 14

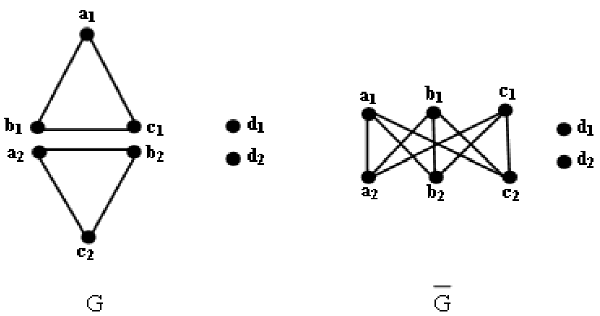


Figure14.

In graph G, we have edges as seen in Fig.14 (more edges will be present, edges required for discussion is drawn in the Fig.14). We know that $P = \{ \{ a_1, a_2 \}, \{ b_1, b_2 \}, \{ c_1, c_2 \}, \{ d_1, d_2 \} \}$ is a γ - chromatic partition for G. Let $X_1 = \{ a_1, b_1, c_1 \}$ $X_2 = \{ a_2, b_2 , c_2 \}$, $X_3 = \{ d_1, d_2 \}$. If there is no edge between d_1 and d_2 in G, then $P_1 = \{ \{ a_1, a_2 \}, \{ b_1, b_2 \}, \{ c_1, c_2 \}, \{ d_1, d_2 \} \}$ and $P_2 = \{ \{ a_1, b_2 \}, \{ b_1, a_2 \}, \{ c_1, c_2 \}, \{ d_1, d_2 \} \}$ are two possible chromatic partition for G such that $| V_i | = 2$, i = 1 to 4, a contradiction to our assumption that G is uniquely colorable. If there is an edge between d_1 and d_2, then any possible independent set including d_i has a vertex from X_1 or X_2. If d_1 with a vertex in X_1 is independent and d_2 with a vertex in X_2 is independent say { a_1, d_1 }, { a_2, d_2 } are independent then P_1 = { { a_1, d_1 }, { a_2, d_2 },{ b_1, b_2 }, { c_1, c_2 }} and P_2 = { { a_1, d_1 }, { a_2, d_2 },{ b_1, c_1 }, { b_2, c_2 }} are two possible chromatic partitions for G , a contradiction to our assumption that G is uniquely colorable.

If both d_1 and d_2 forms an independent set with same vertex in some X_i, i = 1, 2 say { a_1, d_1 }, { a_1, d_2 } are independent sets, then d_1 and d_2 cannot be collectively adjacent to all the remaining vertices. If d_1 and d_2 are collectively adjacent to all the remaining vertices, then the structure of G is as seen in Fig. 15

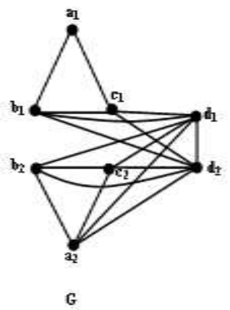


Figure15.

From the graph G in above Fig. 15, we observe that γ - chromatic partition for G such that| V_i | = 2, i = 1 to 4 is not possible, a contradiction to our assumption. So d_1 and d_2 are not adjacent to at least one more vertex in X_i, i = 1, 2. If d_1 and d_2 are not adjacent to two vertices in X_1 say a_1, b_1 then the graph G has edges as seen in Fig. 16

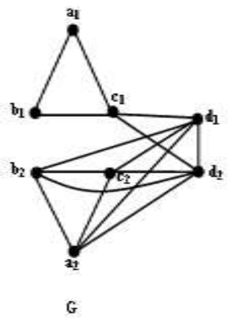


Figure16.

From the graph G in Fig. 16, we observe that γ - chromatic partition for G such that | V_i | = 2, i = 1 to 4 is not possible , a contradiction to our assumption . So d1 and d2 are collectively not adjacent to at least three vertices in X_i, i = 1, 2. If d_1 and d_2 are not collectively adjacent to every vertex in X_1 the G is a disconnected graph which is not possible. Hence , we conclude that there are distinct pair of independent sets between the set { X_1, X_3} and { X_2, X_3 } say { a_1, d_1}and { a_2, d_2 }form an independent pair. P_1 = { { a_1, d_1 }, { a_2, d_2 },{ b_1, b_2 }, { c_1, c_2 } } and P_2 = { { a_1, d_1 }, { a_2, d_2 },{ b_1, c_1 }, { b_2, c_2 }} are two possible chromatic partitions for G such that | V_i | = 2, i = 1 to 4, a contradiction to our assumption that G is uniquely colorable.

In every other case, where the number of edges is more than the cases considered here, a partition of the form | P | = 4 such that | V_i | = 2, i = 1 to 4 is not possible. For any additional non – independency between vertices the number of independent set increases and hence the above discussion is true in this case also.

If a subgraphhomeomorphic to $K_{3,3}$ is constructed in $\overline{G}$, then the following cases are possible.

Case 1

If $\overline{G}$ is the graph homeomorphic to $K_{3,3}$as seen in Fig. 17.

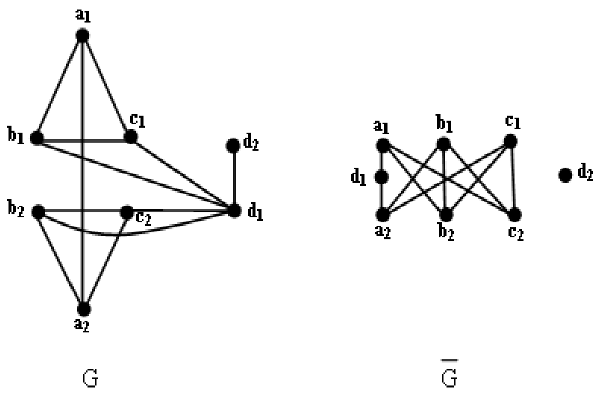


Figure17.

Since $\overline{G}$ is a connected graph, d_2 is adjacent to at least one vertex other than d_1, implies d_2 not adjacent to at least one vertex in $X_1 \cup X_2$ say vertex a1 then $P_1 = \{ \{ a_1, d_2 \}, \{ a_2, d_1 \},\{ b_1, b_2 \}, \{ c_1, c_2 \}\}$ and $P_2 = \{ \{ a_1, d_2 \}, \{ a_2, d_1 \},\{ b_1, c_1 \}, \{ b_2, c_2 \}\}$ (note that any pair of four independent sets in G includes at least two pairs from X_1, X_2) are two possible chromatic partition for G , a contradiction to our assumption that G is uniquely colorable.

Case 2

If $\overline{G}$ is the graph homeomorphic to $K_{3,3}$as seen in Fig. 18.

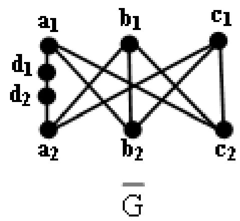


Figure18.

In this case $P_1 = \{ \{ a_1, d_1 \}, \{ a_2, d_2 \},\{ b_1, c_2 \}, \{ b_2, c_1 \}\}$ and $P_2 = \{ \{ a_1, d_1 \}, \{ a_2, d_2 \},\{ b_1, b_2 \}, \{ c_1, c_2 \}\}$ are two possible chromatic partition for G, a contradiction to our assumption that G is uniquely colorable.

Case 3

If $\overline{G}$ is the graph homeomorphic to $K_{3,3}$as seen in Fig. 19.

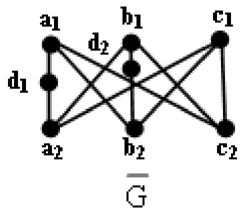


Figure 19.

In this case $P_1 = \{ \{ a_1, d_1 \}, \{ a_2, b_1 \},\{ d_2, b_2 \},\{ c_1, c_2 \},\}$and $P_2 = \{ \{ a_1, b_2 \}, \{ a_2, d_1 \},\{ b_1, d_2 \}, \{ c_1, c_2 \}\}$ are two possible chromatic partition for G , a contradiction to our assumption that G is uniquely colorable implies, $\overline{G}$ is not a graph homeomorphic to K_5.

In section 3.1, we have classified the possible cases under which $\overline{G}$ is planar. In all the remaining cases $\overline{G}$ may be planar or non – planar. We continue further to construct the following as counter example for the remaining cases.

1. Constructing a uniquely colorable graph G so that $\overline{G}$ is planar.
2. Constructing a uniquely colorable graph G so that $\overline{G}$ is non – planar.

Choose a random γ - chromatic partition a random γ - set not discussed in section. From the introduction, we know that it is sufficient to restrict to the cases when γ(G) ≤ 4 and the cardinality of each partition is also less than or equal to 4. Assume that | P | = k, γ(G) = h, h ≤ 4. Let P = { V_1, V_2, … , V_k }. Let V_1 = { a_1,a_2,…, a_{k1} }, V_2 = { b_1, b_2, …, b_{k2} },…, V_k = {q_1, q_2,…, q_{k17} }, 2 ≤k_i≤ 4, i = 1 to k. Note that | V_1 | ≤ | V_i |, i = 2, 3,…, k.

3.2. $\bar{G}$ is planar

3.2.1. Construction of G. Choose a random pair of vertices from the sets { V_1, V_2 }, { V_2, V_3},…,.{ V_{k-1}, V_k }. For our construction, let us choose the pairs as (a_1, b_1) (b_1, c_1),….,(p_1, q_1). We construct G as follows.

1. V (G) = $V_1 \cup V_2 \cup$,…., $\cup V_k$
2. We draw edges in the following fashion
   $\langle V_i \rangle$, i = 1 to k is a null graph. ⟨ a_1, b_1⟩, ⟨ b_1, c_1⟩, …., ⟨ p_1, q_1⟩ is also a null graph. Draw edges between the remaining pair of vertices.
   E (G) = E (K_n) – { E (K_{k_1}) + E (K_{k_2}) +,…., E ($K_{k_{17}}$) } – { (a_1, b_1) (b_1, c_1),….,(p_1, q_1) } where n = | V (G) |.
   From the way the graph is constructed, we note the following
   1. P = { V_1, V_2, ,…, V_k }forms a γ - chromatic partition for G.
   2. Any independent set between V_i, V_j, i ≠ j is always of cardinality 2.
   3. | V_i | ≥ 2, i = 1 to k.

When we try for γ - chromatic partition for G the available distinct independent sets of size at least 2 { V_1, V_2,…., V_k }, (a_1, b_1) (b_1, c_1),….,(o_1, p_1) if | P | is odd.
Let us assume that P is a partition such that | V_i | = 2, i = 1 to k. If we try to create a new partition other than P, then the available independent pairs is either $\frac{k}{2}$or $\frac{k-1}{2}$.
| $\bigcup_{i=1}^{k} V_i$| = 2k. The remaining $2k - \frac{k}{2}$or $2k - \frac{k-1}{2}$ vertices are adjacent to each other except for the pairs in every V_i. This simply means that we cannot form another $k - \frac{k}{2}$ or $k - \frac{k-1}{2}$ distinct independent pairs implies another partition of same cardinality is not possible which implies G is uniquely colorable. From this discussion and the graph construction, we can conclude that if at least one pair of independent set from(V_i, V_j) is considered in any partition, then since the remaining vertices in V_i and $V_j$are adjacent to every other vertices. It is not possible to find a γ - chromatic partition for G of same cardinality as P that is, any other γ - chromatic partition P_1 for G has cardinality at least two greater than P implies, G is uniquely colorable in all cases.

3.2.2. V_1 is a γ - set for G. a_1 dominates V (G) – { V_1 } – { b_1 }, a_2 dominates b_1 implies V (G) is a dominating set for G. Also | V_1 | ≤ | V_i |, i = 2 to k implies V_1 is a γ- set for G.
3.2.3.$\overline{G}$ is planar. In G, $\langle V_i \rangle$, i = 1 to k and (a_1, b_1) (b_1, c_1), …., (p_1, q_1) are the only independent sets implies, these independent sets are complete in $\overline{G}$. Since | V_i | ≤4 , i = 1 to k, each complete block is planar . Also every block is connected to the other by an edge, implies $\overline{G}$ is planar. The general structure of $\overline{G}$ is as seen in Fig. 20.

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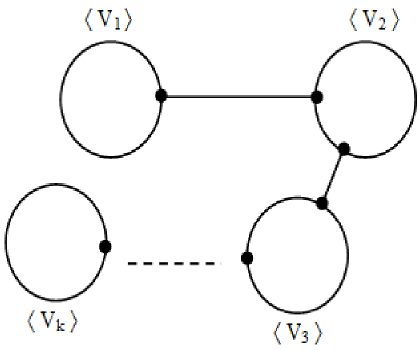

$\overline{G}$

Figure20.

Note that $\langle V_i \rangle$ may be K_2, K_3 or K_4.
Example

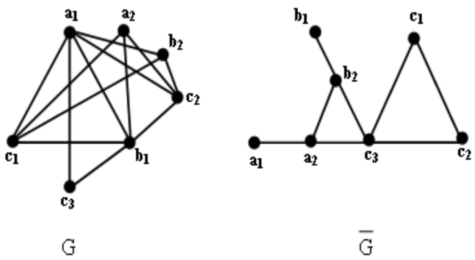

Figure 21.

3.3. Non – planar graph construction
We construct a graph G as follows

1. $V (G) = V_1 \cup V_2 \cup ,\ldots, \cup V_k$
2. We draw edges in the following fashion
   Draw edges from a_1 to every vertex in $V (G) – V_1$
   Draw edges from b_1 to every vertex in $V (G) – V_2$
   .
   .
   .
   Draw edges from q_1 to every vertex in $V (G) – V_k$

$E (G) = E (K_n) – \{ E (K_{k_1}) + E (K_{k_2}) +,\ldots, E (K_{k_{17}}) \}$
$P = \{ V_1, V_2, \ldots, V_k \}$ is a γ- chromatic partition for G. a_1 forms an independent set only with vertices in V_1, b_1 forms an independent set only with vertices in $V_2,\ldots$, q_1 forms an independent set only with vertices in V_k, implies a minimum possible partition for G should be of size k. Since $\{ a_1, b_1,\ldots,q_1\}$ are not adjacent only with vertices in $V_1, V_2,\ldots, V_k$ respectively. Either a_1 can be in a single partition or can be combined with vertices in V_1. This is true for the vertices in $\{ b_1, c_1,\ldots, q_1 \}$. So even if the remaining vertices are combined in different sets to form a γ - chromatic partition for G, these vertices will remain back in distinct sets. So P is the only possible γ- chromatic partition of size k, implies G is uniquely colorable.

3.3.1. V_1 is a γ - set for G. Vertex a_1 is adjacent to $V(G) – \{ V_1 \}$ implies V (G) is a dominating set for G. Also $| V_1 | \leq | V_i |$, i = 2 to k implies V_1 is a γ - set for G.

3.3.2. $\overline{G}$ *is non – planar.* V (G) – { a_1, b_1, …, q_1 } is an independent set in G. Also the cases considered for this construction are such that | V (G) – { a_1, b_1,…, q_1 } | ≥ 5, implies we have at least five independent vertices in G. This vertices form K_5 in $\bar{G}$ implies $\bar{G}$ is non – planar.

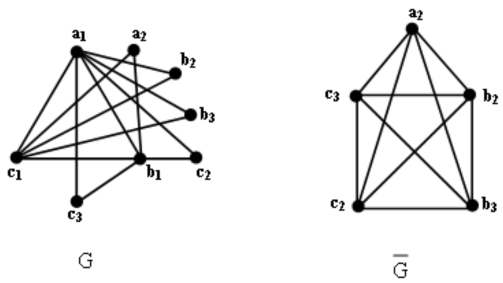

Figure 22.

4. Conclusion

From the above discussion, we conclude that when G is γ - uniquely colorable, $\overline{G}$ is planar only when

i. γ (G) = 2, 2 ≤ | P | ≤ 4
ii. γ (G) = 3, | P | = 2

In all the remaining cases, $\overline{G}$ can be planar and non – planar.

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