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Random coincidence points, invariant approximation theorems, nonstarshaped domain and q-normed spaces

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Abstract. We present coincidence points results satisfying a generalized contractive condition in the setting of a nonstarshaped domain of a q-normed space which is not necessarily a locally convex space. Invariant approximation results are also obtained as application.

Keywords. Random best approximation, random fixed point, $\mathcal{S}$-nonexpansive random operators, coincidence point, weakly commuting mappings, property (N), q-normed space.

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1 Introduction and preliminaries

In the material to be produced here, the following definitions have been used.

Let (Ω, Σ) be a measurable space with Σ a sigma algebra of subsets of Ω and let $\mathcal{M}$ be a subset of a metric space $(\mathcal{X}, d)$. We denote by $2^{\mathcal{X}}$ the family of all subsets of $\mathcal{X}$, by $\mathrm{CB}(\mathcal{X})$ the family of all nonempty closed and bounded subsets of $\mathcal{X}$, and by $\mathcal{H}$ the Hausdorff metric on $\mathrm{CB}(\mathcal{X})$, induced by the metric d. For any $x \in \mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{X}$, by $d(x, \mathcal{A})$ we denote the distance between x and $\mathcal{A}$, that is, $d(x, \mathcal{A}) = \inf\{d(x, a) : a \in \mathcal{A}\}$.

A mapping $\mathcal{T}: \Omega \to 2^{\mathcal{X}}$ is called measurable (respectively, weakly measurable) if, for any closed (respectively, open) subset $\mathcal{B}$ of $\mathcal{X}$, $\mathcal{T}^{-1}(\mathcal{B}) = \{\omega \in \Omega : \mathcal{T}(\omega) \cap \mathcal{B} \neq \emptyset\} \in \Sigma$. Note that, if $\mathcal{T}(\omega) \in \mathrm{CB}(\mathcal{X})$ for every $\omega \in \Omega$, then $\mathcal{T}$ is weakly measurable if and only if it is measurable.

A mapping $\xi: \Omega \to \mathcal{X}$ is said to be a measurable selector of a measurable mapping $\mathcal{T}: \Omega \to 2^{\mathcal{X}}$ if ξ is measurable and, for any $\omega \in \Omega$, $\xi(\omega) \in \mathcal{T}(\omega)$. A mapping $\mathcal{S}: \Omega \times \mathcal{X} \to \mathcal{X}$ is called a random operator if, for any $x \in \mathcal{X}$, $\mathcal{T}(\cdot, x)$ is measurable. A mapping $\mathcal{T}: \Omega \times \mathcal{X} \to \mathrm{CB}(\mathcal{X})$ is called a multivalued random operator if for every $x \in X$, $\mathcal{T}(\cdot, x)$ is measurable. A measurable mapping $\xi: \Omega \to \mathcal{X}$ is called a random fixed point of a random operator $\mathcal{T}: \Omega \times \mathcal{X} \to \mathcal{X}$ if for every $\omega \in \Omega$, $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$. A measurable mapping $\xi: \Omega \to \mathcal{X}$ is called a random coincidence of $\mathcal{T}: \Omega \times \mathrm{CB}(\mathcal{X}) \to \mathcal{X}$ and $\mathcal{S}: \Omega \times \mathcal{X} \to \mathcal{X}$ if

$\mathcal{S}(\omega, \xi(\omega)) \in \mathcal{T}(\omega, \xi(\omega))$ for all $\omega \in \Omega$. We denote by $\mathcal{F}(\mathcal{T})$ the set of fixed points of $\mathcal{T}$ and by $\mathcal{C}(\mathcal{S}, \mathcal{T})$ the set of coincidence points of $\mathcal{S}$ and $\mathcal{T}$.

Let $\mathcal{X}$ be a linear space. A q-norm on $\mathcal{X}$ is a real-valued function $\|\cdot\|_q$ on $\mathcal{X}$ with $0 < q \leq 1$, satisfying the following conditions:

(a) $\|x\|_q \geq 0$ and $\|x\|_q = 0$ iff $x = 0$,

(b) $\|\lambda x\|_q = |\lambda|^q \|x\|_q$,

(c) $\|x + y\|_q \leq \|x\|_q + \|y\|_q$

for all $x, y \in \mathcal{X}$ and all scalars λ. The pair $(\mathcal{X}, \|\cdot\|_q)$ is called a q-normed space. It is a metric space with $d_q(x, y) = \|x - y\|_q$ for all $x, y \in \mathcal{X}$, defining a translation invariant metric d_q on $\mathcal{X}$. If $q = 1$, we obtain the concept of a normed linear space. It is well known that the topology of every Hausdorff locally bounded topological linear space is given by some q-norm, $0 < q \leq 1$. The spaces l_q and $\mathcal{L}_q[0,1]$, $0 < q \leq 1$, are q-normed spaces. A q-normed space is not necessarily a locally convex space. Recall that, if $\mathcal{X}$ is a topological linear space, then its continuous dual space $\mathcal{X}^*$ is said to separate the points of $\mathcal{X}$ if for each $x \neq 0$ in $\mathcal{X}$, there exists a $g \in \mathcal{X}^*$ such that $g(x) \neq 0$. In this case the weak topology on $\mathcal{X}$ is well-defined. We mention that, if $\mathcal{X}$ is not locally convex, then $\mathcal{X}^*$ need not separate the points of $\mathcal{X}$. For example, if $\mathcal{X} = \mathcal{L}_q[0,1]$, $0 < q < 1$, then $\mathcal{X}^* = \{0\}$ [28, pp. 36–37]. However, there are some non-locally convex spaces (such as the q-normed space l_q, $0 < q < 1$) whose dual separates the points [15].

Let $\mathcal{L}_q$, $0 < q \leq 1$, be the space of all measurable functions $f(t)$ on $\mathcal{J} = [a, b]$ with $\int_a^b |f(t)|^q dt < \infty$ (we identify functions which are equal almost everywhere). For all $f \in \mathcal{L}_q$, $0 < q \leq 1$, let the function $\|f\|_q$ be defined by

$$\|f\|_q = \Big(\int_a^b |f(t)|^q dt \Big)^{\frac{1}{q}}. \tag{1.1}$$

This expression is an example of a quasinorm on a topological linear space [15].

Let $\mathcal{X}$ be a q-normed space and $\mathcal{M}$ a nonempty subset of $\mathcal{X}$. The Hausdorff metric $\mathcal{H}_q$ on $\mathrm{CB}(\mathcal{X})$ induced by the q-norm of $\mathcal{X}$ is defined by

$$\mathcal{H}_q(\mathcal{A}, \mathcal{B}) = \max\Big\{ \sup_{a \in \mathcal{A}} d_q(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d_q(\mathcal{A}, b) \Big\} \tag{1.2}$$

for all $\mathcal{A}, \mathcal{B} \in \mathrm{CB}(\mathcal{X})$, where $d_q(x, \mathcal{B}) = \inf\{\|x - y\|_q : y \in \mathcal{B}\}$ for each $x \in \mathcal{X}$.

Example 1.1 ([25]). We will show that the q-th power of the quasi-norm $\|f\|_q$ in $\mathcal{L}_q$ defined by (1.1) is a q-norm on $\mathcal{L}_q$. For each $f \in \mathcal{L}_q$ the q-th power of the quasi-norm in $\mathcal{L}_q$ is defined by

$$\|f\|_q^q = \int_a^b |f(t)|^q dt. \tag{1.3}$$

The norm defined by (1.3) is a q-norm on $\mathcal{L}_q$:

(a) For each $f \in \mathcal{L}_q$, $\|f\|_q \geq 0$. If $\|f\|_q^q = 0$, then $f(t) = 0$ almost everywhere,

(b) $\|af\|_q^q = \int_a^b |af(t)|^q dt = |a|^q \int_a^b |f(t)|^q dt = |a|^q \|f\|_q^q$ for all scalars a and all $f \in \mathcal{L}_q$,

(c) $\|f+g\|_q^q = \int_a^b |f(t)+g(t)|^q dt \leq \int_a^b |f(t)|^q dt + \int_a^b |g(t)|^q dt \leq \|f\|_q^q + \|g\|_q^q$ for all $f, g \in L_q$.

Thus all the properties of a q-norm, $0 < q \leq 1$, are satisfied. Hence the q-th power of the quasi-norm q in $\mathcal{L}_q$ is a q-norm on $\mathcal{L}_q$.

A set $\mathcal{M}$ is said to have property (N) [14, 23] if

(1) $\mathcal{T} : \mathcal{M} \to \mathrm{CB}(\mathcal{M})$,

(2) $(1-k_n)p + k_n \mathcal{T} x \subseteq \mathcal{M}$ for some $p \in \mathcal{M}$ and a fixed sequence of real numbers k_n $(0 < k_n < 1)$ converging to 1 and for each $x \in \mathcal{M}$.

Each p-starshaped set has the property (N) with respect to any map $\mathcal{T} : \mathcal{M} \to \mathrm{CB}(\mathcal{M})$, but the converse does not hold in general.

Let $\mathcal{S} : \mathcal{M} \to \mathcal{X}$ be a single-valued map. A multivalued map $\mathcal{T} : \mathcal{M} \to \mathrm{CB}(\mathcal{X})$ is said to be an $\mathcal{S}$-contraction if for a fixed constant k, $0 \leq k < 1$, and for all $x, y \in \mathcal{X}$

$$\mathcal{H}_q(\mathcal{T}(x), \mathcal{T}(y)) \leq k^q \|\mathcal{S}(x) - \mathcal{S}(y)\|_q.$$

If $k = 1$ in the above inequality, then $\mathcal{T}$ is called $\mathcal{S}$-nonexpansive. Indeed, if $\mathcal{S} = \mathcal{I}$ (the identity map on $\mathcal{X}$), then each $\mathcal{S}$-contraction is a contraction. Let $\mathcal{T} : \mathcal{M} \to \mathrm{CB}(\mathcal{M})$. The mapping $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ is said to be $\mathcal{T}$-weakly commuting if for all $x \in \mathcal{M}$, $\mathcal{S}\mathcal{S}x \in \mathcal{T}\mathcal{S}x$.

A random operator $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ is continuous (respectively, nonexpansive, $\mathcal{S}$-nonexpansive) if, for each $\omega \in \Omega$, $\mathcal{T}(\omega, \cdot)$ is continuous (respectively, nonexpansive, $\mathcal{S}$-nonexpansive). Let $\mathcal{T} : \Omega \times \mathcal{X} \to \mathrm{CB}(\mathcal{X})$ be a random operator. Then a random operator $\mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ is $\mathcal{T}$-weakly commuting if $\mathcal{S}(\omega, \cdot)$ is $\mathcal{T}$-weakly commuting for each $\omega \in \Omega$.

Let $\mathcal{M}$ be a subset of a normed space $\mathcal{X}$ for each $x \in \mathcal{X}$. Define

$$\mathcal{P}_{\mathcal{M}}(x) = \{y \in \mathcal{M} : \|x - y\| = \mathrm{dist}(x, \mathcal{M})\},$$

the set of the best $\mathcal{M}$-approximants to x. The set $\mathcal{P}_{\mathcal{M}}(x)$ is always a bounded subset of $\mathcal{X}$ and it is closed or convex if $\mathcal{M}$ is closed or convex (see [8]).

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. The theory of random fixed point theorems was initiated by the Prague school of probabilistic in the 1950s. The interest in this subject enhanced after the publication of the survey paper by Bharucha Reid [7]. Random fixed point theorems for contraction mappings were first studied by Spacek [34] and Hans [11, 12]. Itoh [17–19] gave several random fixed point theorems for various single and multivalued random operators. Random fixed point theory has received much attention in recent years (see [2, 26, 27, 36]).

Random coincidence point theorems and random approximations are stochastic generalizations of classical coincidence point and approximation theorems, and have application in probability theory and nonlinear analysis. The random fixed point theory for self-maps and non-self-maps has been developed during the last decade by various author, (see e.g. [2, 13]). Recently, this theory has been further extended for 1-set contractive, nonexpansive, semi-contractive and completely continuous random maps, etc.

Random fixed point theorems have been applied in many instances in the field of random best approximation theory and several interesting and meaningful results have been studied. The theory of approximation has become so vast that it intersects with every other branch of analysis and plays an important role in the applied sciences and engineering. Approximation theory is concerned with the approximation of functions of a certain kind by other functions. In this point of view, in the year 1963, Meinardus [22] was the first to observe the general principle and to use a Schauder fixed point theorem for finding a deterministic version of a fixed point theorem as best approximation. Afterwards, a number of results were developed in this direction under different conditions following the line made by Meinardus (see e.g. [8, 30, 31]).

On the other hand, in the year 2000, Shahzad and Latif [32, Theorem 3.2] proved the random coincidence point, which is further extended by Shahzad and Nawab [33, Theorem 3.1]. Shahzad and Nawab [33, Theorem 3.8] have also given the invariant approximation result for single-valued mappings and extended and complemented the results of Beg and Shahzad [4, 6]. The result of Shahzad and Latif [32, Theorem 3.2] and Xu [35, Theorem 1] was also generalized and improved by Khan et al. [20, Theorem 3.13], in the sense that the maps $\mathcal{S}$ and $\mathcal{T}$ need not be commuting for the existence of random coincidence, $\mathcal{T}(\omega, \cdot)$ is not necessarily $\mathcal{S}(\omega, \cdot)$-nonexpansive, and $\mathcal{S}$ is not affine. As application, random invariant approximation results have also been obtained for single-valued mappings.

The purpose of this paper is to generalize the results of Khan et al. [20] for more generalized nonexpansive mappings in a q-normed space. In this way, related results of Beg and Shahzad [3–5], Nashine [24], Shahzad and Latif [32], Shahzad and Latif [33] and Xu [35] are improved and generalized for multivalued random operators in a q-normed space. Incidently, these results also give a multivalued random version as generalization of Dotson [9], Nashine [25], Sahab et al. [29] and Singh [30, 31] and many more related results in a q-normed space.

The following result is also needed in the sequel.

Theorem 1.2 ([10]). *Let $(\mathcal{X}, d)$ be a complete separable metric space, let (Ω, Σ) be a measurable space, and let $\mathcal{T}\colon \Omega \times \mathcal{X} \to \mathrm{CB}(\mathcal{X})$ and $\mathcal{S}\colon \Omega \times \mathcal{X} \to \mathcal{X}$ be mappings such that*

(i) *$\mathcal{T}(\omega, \cdot)$, $\mathcal{S}(\omega, \cdot)$ are continuous for all $\omega \in \Omega$,*

(ii) *$\mathcal{T}(\omega, \cdot)$, $\mathcal{S}(\omega, \cdot)$ are measurable for all $x \in \mathcal{X}$,*

(iii) *they satisfy, for each $\omega \in \Omega$,*

$$\begin{aligned}
&\mathcal{H}(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y))\\
&\le \alpha(\omega) \max \big\{d(\mathcal{S}(\omega, x), \mathcal{S}(\omega, y)),\\
&\qquad\qquad d(\mathcal{S}(\omega, x), \mathcal{T}(\omega, x)), d(\mathcal{S}(\omega, y), \mathcal{T}(\omega, y)),\\
&\qquad\qquad \frac{1}{2}\big[d(\mathcal{S}(\omega, x), \mathcal{T}(\omega, y)) + d(\mathcal{S}(\omega, y), \mathcal{T}(\omega, x))\big]\big\}\\
&\quad + \beta(\omega) \max \big\{d(\mathcal{S}(\omega, x), \mathcal{T}(\omega, x)), d(\mathcal{S}(\omega, y), \mathcal{T}(\omega, y))\big\}\\
&\quad + \gamma(\omega)\big[d(\mathcal{S}(\omega, x), \mathcal{T}(\omega, y)) + d(\mathcal{S}(\omega, y), \mathcal{T}(\omega, x))\big]
\end{aligned}$$

for every $x, y \in \mathcal{X}$, where $\alpha, \beta, \gamma\colon \Omega \to [0, 1)$ are measurable mappings such that for all $\omega \in \Omega$, $\beta(\omega) > 0$, $\gamma(\omega) > 0$, $\alpha(\omega) + \beta(\omega) + 2\gamma(\omega) = 1$.

If $\mathcal{S}(\Omega \times \mathcal{X}) = \mathcal{X}$ for each $\omega \in \Omega$, then there is a measurable mapping $\xi\colon \Omega \to \mathcal{X}$ such that $\mathcal{S}(\omega, \xi(\omega)) \in \mathcal{T}(\omega, \xi(\omega))$ for all $\omega \in \Omega$ (i.e., $\mathcal{T}$ and $\mathcal{S}$ have a random coincidence point).

2 Results on random coincidence points and common random fixed points

In the following, random coincidence point and common random fixed point theorems for multivalued random operator are presented.

Theorem 2.1. *Let* (Ω, Σ) *be a measurable space, let* $\mathcal{M}$ *be a nonempty subset of a* q*-normed space* $\mathcal{X}$*, and let* $\mathcal{S}: \Omega \times \mathcal{M} \to \mathcal{M}$ *be a random operator such that* $\mathcal{S}(\omega, \mathcal{M}) = \mathcal{M}$ *for each* $\omega \in \Omega$*. Assume that* $\mathcal{T}: \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ *is a continuous random operator that satisfies, for each* $\omega \in \Omega$*,* $x, y \in \mathcal{M}$ *and* $\lambda \in [0, 1]$

$$\begin{aligned}
&\mathcal{H}_q(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \\
&\leq \max \Big\{\|\mathcal{S}(\omega, x) - \mathcal{S}(\omega, y)\|_q, \\
&\qquad \operatorname{dist}(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, x)), \operatorname{dist}(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, y)), \\
&\qquad \frac{1}{2}\big[\operatorname{dist}(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, y)) + \operatorname{dist}(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, x))\big]\Big\} \\
&\quad + \Big[\frac{1 - (k(\omega))^q}{2(k(\omega))^q}\Big] \max \big\{\operatorname{dist}(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, x)), \operatorname{dist}(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, y))\big\} \\
&\quad + \Big[\frac{1 - (k(\omega))^q}{4(k(\omega))^q}\Big]\big[\operatorname{dist}(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, y)) + \operatorname{dist}(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, x))\big],
\end{aligned} \tag{2.1}$$

where $k: \Omega \to (0, 1)$ *are measurable mappings. Suppose that* $\mathcal{M}$ *has the property* (N)*. Then* $\mathcal{S}$ *and* $\mathcal{T}$ *have a random coincidence point if one of the following conditions is satisfied:*

1. $\mathcal{M}$ *is separable compact and* $\mathcal{S}$ *is continuous,*
2. $\mathcal{X}$ *is a Banach space,* $\mathcal{M}$ *is separable weakly compact,* $\mathcal{S}$ *is weakly continuous and* $(\mathcal{S} - \mathcal{T})(\omega, \cdot)$ *is demiclosed at* 0*,*
3. $\mathcal{X}$ *is a Banach space,* $\mathcal{M}$ *is separable weakly compact,* $\mathcal{T}$ *is completely continuous and* $\mathcal{S}$ *is continuous,*
4. $\mathcal{M}$ *is separable complete,* $\mathcal{T}(\mathcal{M})$ *is bounded and* $(\mathcal{S} - \mathcal{T})(\mathcal{M})$ *is closed.*

Moreover, if for each $\omega \in \Omega$ *and any* $x \in \mathcal{M}$*,* $\mathcal{S}(\omega, x) \in \mathcal{T}(\omega, x)$ *implies*

$$\mathcal{S}(\omega, \mathcal{S}(\omega, x)) = \mathcal{S}(\omega, x),$$

and if $\mathcal{S}$ *is a* $\mathcal{T}$*-weakly commuting random operator, then* $\mathcal{T}$ *and* $\mathcal{S}$ *have a common random fixed point.*

Proof. Choose a fixed sequence of measurable mappings $k_n: \Omega \to (0, 1)$ such that $k_n(\omega) \to 1$ as $n \to \infty$. For $n \geq 1$, define a sequence of random operators $\mathcal{T}_n: \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ as

$$\mathcal{T}_n(\omega, x) = k_n(\omega)\mathcal{T}(\omega, x) + (1 - k_n(\omega))p \tag{2.2}$$

for all $x \in \mathcal{M}$. Then, each $\mathcal{T}_n$ is a well-defined map from $\mathcal{M}$ into $\mathrm{CB}(\mathcal{M})$ and $\omega \in \Omega$ as $\mathcal{M}$ has property (N). It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\mathcal{H}_q(\mathcal{T}_n(\omega,x),\mathcal{T}_n(\omega,y)) = [k_n(\omega)]^q \mathcal{H}_q(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))\\
&\le [k_n(\omega)]^q \max\Big\{\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,y)\|_q,\\
&\qquad\qquad \frac{1}{2}\big[\|\mathcal{S}(\omega,x)-\mathcal{T}_n(\omega,y)\|_q + \|\mathcal{S}(\omega,y)-\mathcal{T}_n(\omega,x)\|_q\big]\Big\}\\
&\quad+\Big[\frac{1-(k_n(\omega))^q}{2}\Big]\max\big\{\|\mathcal{S}(\omega,x)-\mathcal{T}_n(\omega,x)\|_q, \|\mathcal{S}(\omega,y)-\mathcal{T}_n(\omega,y)\|_q\big\}\\
&\quad+\Big[\frac{1-(k_n(\omega))^q}{4}\Big]\big[\|\mathcal{S}(\omega,x)-\mathcal{T}_n(\omega,y)\|_q + \|\mathcal{S}(\omega,y)-\mathcal{T}_n(\omega,x)\|_q\big]
\end{aligned}$$

for all $x, y \in \mathcal{M}$ and $\omega \in \Omega$. Note that for all $\omega \in \Omega$,

$$\Big(\frac{1-k_n(\omega)}{2}\Big)^q > 0, \quad \Big(\frac{1-(k_n(\omega))^q}{4}\Big) > 0$$

and

$$(k_n(\omega))^q + \Big(\frac{1-(k_n(\omega))^q}{2}\Big) + 2\Big(\frac{1-(k_n(\omega))^q}{4}\Big) = 1$$

for each n.

1. Since $\mathcal{M}$ is compact, all conditions of Theorem 1.2 are satisfied on $\mathcal{M}$ and so, there exists a coincidence random fixed point ξ_n of $\mathcal{T}_n$ and $\mathcal{S}$ such that we have $\mathcal{S}_n(\omega,\xi_n(\omega)) \in \mathcal{T}(\omega,\xi_n(\omega))$.

For each n, define $\mathcal{G}_n\colon \Omega \to \mathcal{C}(\mathcal{M})$ by

$$\mathcal{G}_n = \mathrm{cl}\{\xi_i(\omega) : i \ge n\},$$

where $\mathcal{C}(\mathcal{M})$ is the set of all nonempty compact subsets of $\mathcal{M}$. Let $\mathcal{G}\colon \Omega \to \mathrm{CB}(\mathcal{M})$ be the mapping defined as $\mathcal{G}(\omega) = \bigcap_{n=1}^{\infty} \mathcal{G}_n(\omega)$. Himmelberg [13, Theorem 4.1] implies that $\mathcal{G}$ is measurable. The Kuratowski and Ryll-Nardzewski selection theorem [21] further implies that $\mathcal{G}$ has a measurable selector $\xi\colon \Omega \to \mathcal{M}$. We show that ξ is the random fixed point of $\mathcal{T}$ and $\mathcal{S}$. Fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$, there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges to $\xi(\omega)$, that is, $\xi_m(\omega) \to \xi(\omega)$. Also, for every $\omega \in \Omega$, since $\xi_m(\omega) \in \mathcal{T}_m(\omega,\xi_m(\omega))$, we have

$$\mathcal{T}_m(\omega,\xi_m(\omega)) = k_m(\omega)\mathcal{T}(\omega,\xi_m(\omega)) + (1-k_m(\omega))p \;\to\; \mathcal{T}(\omega,\xi(\omega))$$

as $k_m(\omega) \to 1$, and $\mathcal{H}_q(\mathcal{T}(\omega,\xi_m(\omega)),\mathcal{T}(\omega,\xi(\omega))) \to 0$ for every $\omega \in \Omega$. Now,

$$\begin{aligned}
\|\xi(\omega)-\mathcal{T}(\omega,\xi(\omega))\|_q &\le \|\xi(\omega)-\xi_m(\omega)\|_q + \|\xi_m(\omega)-\mathcal{T}(\omega,\xi(\omega))\|_q\\
&\le \|\xi(\omega)-\xi_m(\omega)\|_q + \mathcal{H}_q(\mathcal{T}_m(\omega,\xi_m(\omega)),\mathcal{T}(\omega,\xi(\omega)))\\
&\to 0
\end{aligned}$$

as $m \to \infty$, for every $\omega \in \Omega$. Since $\mathcal{T}(\omega, \xi(\omega))$ is closed for each $\omega \in \Omega$, we have $\xi(\omega) \in \mathcal{T}(\omega, \xi(\omega))$. Also, from the continuity of $\mathcal{S}$, we have

$$\begin{aligned}\mathcal{S}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \lim_{m\to\infty} \xi_m(\omega)) &= \lim_{m\to\infty} \mathcal{S}(\omega, \xi_m(\omega)) \\ &= \lim_{m\to\infty} \xi_m(\omega) = \xi(\omega).\end{aligned}$$

If $\mathcal{S}$ is $\mathcal{T}$-weakly commuting at $\upsilon(\omega) \in C(\mathcal{S}, \mathcal{T})$, then $\mathcal{S}(\omega, \mathcal{S}(\omega, \upsilon(\omega))) = \mathcal{T}(\omega, \mathcal{S}(\omega, \upsilon(\omega)))$, and hence

$$\mathcal{S}(\omega, \upsilon(\omega)) = \mathcal{S}(\omega, \mathcal{S}(\omega, \upsilon(\omega))) \in \mathcal{T}(\omega, \mathcal{S}(\omega, \upsilon(\omega))).$$

Thus $\mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$.

2. Since the weak topology is Hausdorff and $\mathcal{M}$ is weakly compact, it follows that $\mathcal{M}$ is strongly closed and is a complete metric space. Thus by weak continuity of $\mathcal{S}$ and Theorem 1.2, there exists a random fixed point ξ of $\mathcal{T}_n$ such that $\mathcal{S}(\omega, \xi_n(\omega)) \in \mathcal{T}_n(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$. By the definition of $\mathcal{T}(\omega, \xi_n(\omega))$, there is a $\zeta_n(\omega) \in \mathcal{T}(\omega, \xi_n(\omega)$.

For each n, define $\mathcal{G}_n \colon \Omega \to \mathcal{WC}(\mathcal{M})$ by

$$\mathcal{G}_n = w\text{-cl}\{\xi_i(\omega) : i \geq n\},$$

where $\mathcal{WC}(\mathcal{M})$ is the set of all nonempty weakly compact subsets of $\mathcal{M}$ and w-cl denotes the weak closure. Define a mapping $\mathcal{G} \colon \Omega \to \mathcal{W}\mathrm{CB}(\mathcal{M})$ by

$$\mathcal{G}(\omega) = \bigcap_{n=1}^{\infty} \mathcal{G}_n(\omega).$$

Because $\mathcal{M}$ is weakly compact and separable, the weak topology on $\mathcal{M}$ is a metric topology. Then Himmelberg [13, Theorem 4.1] implies that $\mathcal{G}$ is w-measurable. The Kuratowski and Ryll-Nardzewski selection theorem [21] further implies that $\mathcal{G}$ has a measurable selector $\xi \colon \Omega \to \mathcal{M}$. We show that ξ is the random fixed point of $\mathcal{T}$. Fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$, there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges weakly to $\xi(\omega)$, that is, $\xi_m(\omega) \to \xi(\omega)$.

Now, from weak continuity of $\mathcal{S}$, we have

$$\begin{aligned}\mathcal{S}(\omega, \xi_m(\omega)) - \zeta_m(\omega) &= k_n(\omega)\zeta_m(\omega) + (1 - k_n(\omega))p - \zeta_m(\omega) \\ &= (1 - k_m(\omega))(p - \zeta_m(\omega)).\end{aligned}$$

Since $\mathcal{M}$ is bounded and $k_m(\omega) \to 1$, it follows that

$$\|\mathcal{S}(\omega, \xi_m(\omega)) - \zeta_m(\omega)\|_q \to 0.$$

Now,

$$y_m = \mathcal{S}(\omega, \xi_m(\omega)) - \zeta_m(\omega)) = (\mathcal{S} - \mathcal{T})(\omega, \xi_m(\omega))$$

and $y_m \to 0$. Since $(\mathcal{S} - \mathcal{T})(\omega, \cdot)$ is demiclosed at 0, we have $0 \in (\mathcal{S} - \mathcal{T})(\omega, \xi(\omega))$. This implies that $\mathcal{S}(\omega, \xi(\omega)) \in \mathcal{T}(\omega, \xi(\omega))$. As in the proof of 1., this implies $\mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$.

3. As in 2., there exists a random fixed point ξ_n of $\mathcal{T}_n$ such that

$$\xi_n = \mathcal{S}(\omega, \xi_n(\omega)) = \mathcal{T}_n(\omega, \xi_n(\omega)) \quad \text{for each } \omega \in \Omega.$$

For each n, define $\mathcal{G}_n\colon \Omega \to \mathcal{WC}(\mathcal{M})$ by $\mathcal{G}_n = w\text{-cl}\{\xi_i(\omega) : i \geq n\}$, where $\mathcal{WC}(\mathcal{M})$ is the set of all nonempty weakly compact subsets of $\mathcal{M}$ and w-cl denotes the weak closure. Defined a mapping $\mathcal{G}\colon \Omega \to \mathcal{W}\mathrm{CB}(\mathcal{M})$ by $\mathcal{G}(\omega) = \bigcap_{n=1}^{\infty} \mathcal{G}_n(\omega)$. Because $\mathcal{M}$ is weakly compact and separable, the weak topology on $\mathcal{M}$ is a metric topology. Then Himmelberg [13, Theorem 4.1] implies that $\mathcal{G}$ is w-measurable. The Kuratowski and Ryll-Nardzewski selection theorem [21] further implies that $\mathcal{G}$ has a measurable selector $\xi\colon \Omega \to \mathcal{M}$. We show that ξ is the random fixed point of $\mathcal{T}$. Fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$, there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges weakly to $\xi(\omega)$, that is, $\xi_m(\omega) \to^w \xi(\omega)$. Since $\mathcal{T}$ is completely continuous, $\mathcal{T}(\omega, \xi_m(\omega)) \to \mathcal{T}(\omega, \xi(\omega))$ as $m \to \infty$. Since $k_m(\omega) \to 1$, we get

$$\xi_m(\omega) = (1 - k_m)q + k_m \mathcal{T}(\omega, \xi_m(\omega)) = \mathcal{T}(\omega, \xi(\omega)).$$

Thus $\mathcal{T}(\omega, \xi_m(\omega)) \to \mathcal{T}^2(\omega, \xi(\omega))$ as $m \to \infty$ and consequently $\mathcal{T}^2(\omega, \xi(\omega)) = \mathcal{T}(\omega, \xi(\omega))$ implies that $\mathcal{T}(\omega, \zeta(\omega)) = \zeta(\omega)$, where $\zeta(\omega) = \mathcal{T}(\omega, \xi(\omega))$. But, since

$$\mathcal{S}(\omega, \xi_m(\omega)) = \xi_m(\omega) \to \mathcal{T}(\omega, \xi(\omega)) = \zeta(\omega),$$

using the continuity of $\mathcal{S}$ and the uniqueness of the limit, we have

$$\mathcal{S}(\omega, \zeta(\omega)) = \zeta(\omega).$$

Hence

$$\mathcal{S}(\omega, \zeta(\omega)) = \mathcal{T}(\omega, \zeta(\omega)) = \zeta(\omega).$$

4. By Theorem 1.2, for each $n \geq 1$, there exists $\xi_n(\omega) \in \mathcal{M}$ such that $\mathcal{S}(\omega, \xi_n(\omega)) \in \mathcal{T}_n(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$. This implies that there is a $\zeta_n(\omega) \in \mathcal{T}(\omega, \xi_n(\omega)$ such that

$$\begin{aligned}
\mathcal{S}(\omega, \xi_n(\omega)) - \zeta_n(\omega) &= k_n(\omega)\zeta_n(\omega) + (1 - k_n(\omega))p - \zeta_n(\omega) \\
&= (1 - k_n(\omega))(p - \zeta_n(\omega)).
\end{aligned}$$

Since $\mathcal{T}(\mathcal{M})$ is bounded and $k_n(\omega) \to 1$, it follows that

$$\|\mathcal{S}(\omega, \xi_m(\omega)) - \zeta_m(\omega)\|_q \to 0 \quad \text{as } n \to \infty.$$

As $(\mathcal{S} - \mathcal{T})(\omega, \cdot)$ is closed, we have $0 \in (\mathcal{S} - \mathcal{T})(\omega, \xi(\omega))$. This implies that $\mathcal{S}(\omega, \xi(\omega)) \in \mathcal{T}(\omega, \xi(\omega))$. As in the proof of 1., this implies $\mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$.

□

If in Theorem 2.1, $\mathcal{S}(\omega, x) = x$ for all $(\omega, x) \in \Omega \times \mathcal{M}$, then we get the following random fixed point theorem.

Corollary 2.2. *Let (Ω, Σ) be a measurable space, let $\mathcal{M}$ be a nonempty subset of a q-normed space $\mathcal{X}$ and let $\mathcal{T}: \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ be a continuous random operator that satisfies, for each $\omega \in \Omega$, $x, y \in \mathcal{M}$ and $\lambda \in [0, 1]$*

$$\begin{aligned}
&\mathcal{H}_q(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \\
&\le \max\Big\{\|x - y\|_q, \operatorname{dist}(x, \mathcal{T}_\lambda(\omega, x)), \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, y)), \\
&\qquad\qquad \frac{1}{2}\big[\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, y)) + \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, x))\big]\Big\} \\
&\quad + \left[\frac{1 - (k(\omega))^q}{2(k(\omega))^q}\right] \max\big\{\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, x)), \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, y))\big\} \\
&\quad + \left[\frac{1 - (k(\omega))^q}{4(k(\omega))^q}\right]\big[\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, y)) + \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, x))\big],
\end{aligned} \tag{2.3}$$

where $k: \Omega \to (0, 1)$ are measurable mappings. Suppose that $\mathcal{M}$ has the property (N), then there exists a measurable map $\xi: \Omega \to \mathcal{M}$ such that $\xi(\omega) \in \mathcal{T}(\omega, \xi(\omega))$ if one of the following conditions is satisfied:

1. *$\mathcal{M}$ is separable compact,*
2. *$\mathcal{X}$ is Banach space, $\mathcal{M}$ is separable weakly compact, and $(\mathcal{I} - \mathcal{T})(\omega, \cdot)$ is demiclosed at 0, where $\mathcal{I}$ is the identity operator,*
3. *$\mathcal{X}$ is Banach space, $\mathcal{M}$ is separable weakly compact, $\mathcal{T}$ is completely continuous,*
4. *$\mathcal{M}$ is separable complete, $\mathcal{T}(\mathcal{M})$ is bounded and $(\mathcal{I} - \mathcal{T})(\mathcal{M})$ is closed, where $\mathcal{I}$ is the identity operator.*

Corollary 2.3. *Let (Ω, Σ) be a measurable space, let $\mathcal{M}$ be a nonempty subset of a q-normed space $\mathcal{X}$, and let $\mathcal{S}: \Omega \times \mathcal{M} \to \mathcal{M}$ be a random operator such that $\mathcal{S}(\omega, \mathcal{M}) = \mathcal{M}$ for each $\omega \in \Omega$. Assume that $\mathcal{T}: \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ is* *a continuous random operators that satisfies, for each* $\omega \in \Omega$, $x, y \in \mathcal{M}$ *and* $\lambda \in [0,1]$

$$\begin{aligned}&\mathcal{H}_q(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))\\&\le \max\big\{\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,y)\|_q,\\&\qquad\qquad \operatorname{dist}(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)),\operatorname{dist}(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y)),\\&\qquad\qquad \frac{1}{2}\big[\operatorname{dist}(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y))+\operatorname{dist}(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))\big]\big\}.\end{aligned} \tag{2.4}$$

Suppose that $\mathcal{M}$ *has the property* (N), *then* $\mathcal{S}$ *and* $\mathcal{T}$ *have a random coincidence point under each of the conditions of Theorem* 2.1.

Moreover, if for each $\omega \in \Omega$ *and any* $x \in \mathcal{M}$, $\mathcal{S}(\omega,x) \in \mathcal{T}(\omega,x)$ *implies* $\mathcal{S}(\omega,\mathcal{S}(\omega,x)) = \mathcal{S}(\omega,x)$, *and if* $\mathcal{S}$ *is a* $\mathcal{T}$*-weakly commuting random operator, then* $\mathcal{T}$ *and* $\mathcal{S}$ *have a common random fixed point.*

If in Corollary 2.3, $\mathcal{S}(\omega,x) = x$ for all $(\omega,x) \in \Omega \times \mathcal{M}$, then we obtain the following random fixed point theorem.

Corollary 2.4. *Let* (Ω,Σ) *be a measurable space, let* $\mathcal{M}$ *be a nonempty subset of a* q*-normed space* $\mathcal{X}$ *and let* $\mathcal{T}\colon \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ *be a continuous random operator that satisfies, for each* $\omega \in \Omega$, $x, y \in \mathcal{M}$ *and* $\lambda \in [0,1]$

$$\begin{aligned}&\mathcal{H}_q(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))\\&\le \max\big\{\|x-y\|_q,\operatorname{dist}(x,\mathcal{T}_\lambda(\omega,x)),\operatorname{dist}(y,\mathcal{T}_\lambda(\omega,y)),\\&\qquad\qquad \frac{1}{2}\big[\operatorname{dist}(x,\mathcal{T}_\lambda(\omega,y))+\operatorname{dist}(y,\mathcal{T}_\lambda(\omega,x))\big]\big\}.\end{aligned} \tag{2.5}$$

Then there exists a measurable map $\xi\colon \Omega \to \mathcal{M}$ *such that* $\xi(\omega) \in \mathcal{T}(\omega,\xi(\omega))$ *under each of the conditions of Corollary* 2.2.

3 Results on random invariant approximation

As application of Theorem 2.1, we have the following results on random invariant approximation.

Theorem 3.1. *Let* (Ω,Σ) *be a measurable space, let* $\mathcal{M}$ *be a subset of a* q*-normed space of* $\mathcal{X}$, *let* $\mathcal{S}\colon \Omega \times \mathcal{M} \to \mathcal{M}$ *and let* $\mathcal{T}\colon \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ *be continuous. Suppose that*

(a) *$\mathcal{T}$ and $\mathcal{S}$ satisfy for all $x, y \in \mathcal{P}_{\mathcal{M}}(x_0)$, for all $\omega \in \Omega$ and $\lambda \in [0,1]$*

$$\begin{aligned}
&\mathcal{H}_q(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))\\
&\le \max\Big\{\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,x_0)\|_q,\\
&\qquad\qquad \operatorname{dist}(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), \operatorname{dist}(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y)),\\
&\qquad\qquad \frac{1}{2}\big[\operatorname{dist}(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y)) + \operatorname{dist}(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))\big]\Big\}\\
&\quad + \Big[\frac{1-(k(\omega))^q}{2(k(\omega))^q}\Big]\max\big\{\operatorname{dist}(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), \operatorname{dist}(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y))\big\}\\
&\quad + \Big[\frac{1-(k(\omega))^q}{4(k(\omega))^q}\Big]\big[\operatorname{dist}(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y)) + \operatorname{dist}(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))\big],
\end{aligned} \tag{3.1}$$

where $k\colon \Omega \to (0,1)$ are measurable mappings,

(b) *$\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty and has the property (N),*

(c) *$\mathcal{P}_{\mathcal{M}}(x_0)$ is both $\mathcal{T}$-invariant and $\mathcal{S}$-invariant.*

Then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{C}(\mathcal{S},\mathcal{T}) \neq \emptyset$ if one of the following conditions is satisfied:

1. *$\mathcal{P}_{\mathcal{M}}(x_0)$ is separable compact and $\mathcal{S}$ is continuous,*
2. *$\mathcal{X}$ is a Banach space, $\mathcal{P}_{\mathcal{M}}(x_0)$ is separable weakly compact, $\mathcal{S}$ is weakly continuous and $(\mathcal{S}-\mathcal{T})(\omega,\cdot)$ is demiclosed at 0,*
3. *$\mathcal{X}$ is a Banach space, $\mathcal{P}_{\mathcal{M}}(x_0)$ is separable weakly compact, $\mathcal{T}$ is completely continuous and $\mathcal{S}$ is continuous,*
4. *$\mathcal{P}_{\mathcal{M}}(x_0)$ is separable complete, $\mathcal{T}(\mathcal{M})$ is bounded and $(\mathcal{S}-\mathcal{T})(\mathcal{M})$ is closed.*

Moreover, if for each $\omega \in \Omega$ and any $x \in \mathcal{M}$, $\mathcal{S}(\omega,x) \in \mathcal{T}(\omega,x)$ implies $\mathcal{S}(\omega,\mathcal{S}(\omega,x)) = \mathcal{S}(\omega,x)$, and if $\mathcal{S}$ is a $\mathcal{T}$-weakly commuting random operator, then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$.

Proof. Since $\mathcal{P}_{\mathcal{M}}(x_0)$ is both $\mathcal{T}$-invariant and $\mathcal{S}$-invariant, it follows that $\mathcal{S}\colon \Omega \times \mathcal{P}_{\mathcal{M}}(x_0) \to \mathcal{P}_{\mathcal{M}}(x_0)$, $\mathcal{T}\colon \Omega \times \mathcal{P}_{\mathcal{M}}(x_0) \to \mathrm{CB}(\mathcal{P}_{\mathcal{M}}(x_0))$. The results now follow from Theorem 2.1. □

Theorem 3.2. *Let (Ω,Σ) be a measurable space, let $\mathcal{M}$ be a subset of a q-normed space of $\mathcal{X}$, let $\mathcal{S}\colon \Omega \times \mathcal{M} \to \mathcal{M}$ and $\mathcal{T}\colon \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ be such that $\mathcal{T}(\omega,\cdot)\colon \partial\mathcal{M} \cap \mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $\mathcal{S}(\omega,x_0) \in \mathcal{T}(\omega,x_0) = \{x_0\}$, $\omega \in \Omega$. Suppose that*

(a) *$\mathcal{T}$ and $\mathcal{S}$ satisfy for all $x \in \mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$, for all $\omega \in \Omega$ and $\lambda \in [0,1]$*

$$\mathcal{H}_q(\mathcal{T}(\omega,x),\mathcal{T}(\omega,y))$$
$$\leq \begin{cases} \|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,x_0)\|_q \qquad \text{if } y = x_0, \\ \max\{\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,y)\|_q, \\ \qquad \operatorname{dist}(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), \operatorname{dist}(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y)), \\ \qquad \frac{1}{2}\big[\operatorname{dist}(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y)) + \operatorname{dist}(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))\big]\big\} \\ + \big[\frac{1-(k(\omega))^q}{2(k(\omega))^q}\big] \max\big\{\operatorname{dist}(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,x)), \operatorname{dist}(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,y))\big\} \\ + \big[\frac{1-(k(\omega))^q}{4(k(\omega))^q}\big]\big[\operatorname{dist}(\mathcal{S}(\omega,x),\mathcal{T}_\lambda(\omega,y)) + \operatorname{dist}(\mathcal{S}(\omega,y),\mathcal{T}_\lambda(\omega,x))\big], \\ \qquad \qquad \text{if } y \in \mathcal{P}_{\mathcal{M}}(x_0), \end{cases} \tag{3.2}$$

where $k\colon \Omega \to (0,1)$ are measurable mappings,

(b) *$\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty and has the property (N),*

(c) *$\mathcal{S}(\omega,\mathcal{P}_{\mathcal{M}}(x_0)) = \mathcal{P}_{\mathcal{M}}(x_0)$, i.e., $\mathcal{P}_{\mathcal{M}}(x_0)$ is $\mathcal{S}$-invariant.*

Then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{C}(\mathcal{S},\mathcal{T}) \neq \emptyset$, under each of the conditions of Theorem 3.1. *Moreover, if for each $\omega \in \Omega$ and any $x \in \mathcal{M}$, $\mathcal{S}(\omega,x) \in \mathcal{T}(\omega,x)$ implies*

$$\mathcal{S}(\omega,\mathcal{S}(\omega,x)) = \mathcal{S}(\omega,x),$$

and if $\mathcal{S}$ is a $\mathcal{T}$-weakly commuting random operator, then

$$\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset.$$

Proof. Let $y \in \mathcal{P}_{\mathcal{M}}(x_0)$. Then

$$\|y - x_0\|_q = \operatorname{dist}(x,\mathcal{M}).$$

Note that for any $t(\omega) \in (0,1)$,

$$\|t(\omega)x_0 + (1-t(\omega))y - x_0\|_q = [1-t(\omega)]^q \|y-x_0\|_q < \operatorname{dist}(x_0,\mathcal{M}).$$

It follows that the line segment $\{t(\omega)x_0 + (1-t(\omega))y : 0 < t(\omega) < 1\}$ and the set $\mathcal{M}$ are disjoint. Thus y is not in the interior of $\mathcal{M}$ and so $y \in \partial\mathcal{M} \cap \mathcal{M}$. Since $\mathcal{T}(\partial\mathcal{M} \cap \mathcal{M}) \subset \mathcal{M}$, $\mathcal{T}x$ must be in $\mathcal{M}$. Let $z \in \mathcal{T}(\omega,y)$.

$$\begin{aligned} \|z - x_0\|_q &\leq \mathcal{H}_q(\mathcal{T}(\omega,y),\mathcal{T}(\omega,x_0)) \\ &\leq \|\mathcal{S}(\omega,y) - \mathcal{S}(\omega,x_0)\|_q \\ &= \|\mathcal{S}(\omega,y) - x_0\|_q = \operatorname{dist}(x_0,\mathcal{M}). \end{aligned}$$

Now $z \in \mathcal{M}$ and $\mathcal{S}(\omega, y) \in \mathcal{P}_{\mathcal{M}}(x_0)$ imply $z \in \mathcal{P}_{\mathcal{M}}(x_0)$. Thus $\mathcal{T}(\omega, \mathcal{P}_{\mathcal{M}}(x_0)) \subseteq \mathcal{P}_{\mathcal{M}}(x_0)$. Hence $\mathcal{T}$ maps $\mathcal{P}_{\mathcal{M}}(x_0)$ into $\mathrm{CB}(\mathcal{P}_{\mathcal{M}}(x_0))$. Thus, the result follows from Theorem 2.1. □

Define

$$\mathcal{C}_{\mathcal{M}}^{\mathcal{S}}(x_0) = \{x \in \mathcal{M} : \mathcal{S}x \in \mathcal{P}_{\mathcal{M}}(x_0)\}$$

and (see [1])

$$\mathcal{D}_{\mathcal{M}}^{\mathcal{S}}(x_0) = \mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{C}_{\mathcal{M}}^{\mathcal{S}}(x_0).$$

Theorem 3.3. *Let* (Ω, Σ) *be a measurable space, let* $\mathcal{M}$ *be a subset of a* q*-normed space of* $\mathcal{X}$, *let* $\mathcal{S}: \Omega \times \mathcal{M} \to \mathcal{M}$ *and* $\mathcal{T}: \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ *be such that* $\mathcal{T}(\omega, \cdot): \partial\mathcal{M} \cap \mathcal{M} \to \mathcal{M}$, *where* $\partial\mathcal{M}$ *stands for the boundary of* $\mathcal{M}$. *Let* $x_0 \in \mathcal{X}$ *and* $\mathcal{S}(\omega, x_0) \in \mathcal{T}(\omega, x_0) = \{x_0\}$, $\omega \in \Omega$. *Suppose that*

(a) $\mathcal{T}$ *and* $\mathcal{S}$ *satisfy for all* $x \in \mathcal{D}_{\mathcal{M}}^{\mathcal{S}}(x_0) (= \mathcal{D}) \cup \{x_0\}$, *for all* $\omega \in \Omega$ *and* $\lambda \in [0, 1]$

$$\begin{aligned}
&\mathcal{H}_q(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \\
&\leq \begin{cases}
\|\mathcal{S}(\omega, x) - \mathcal{S}(\omega, x_0)\|_q \qquad \text{if } y = x_0, \\
\max\{\|\mathcal{S}(\omega, x) - \mathcal{S}(\omega, y)\|_q, \\
\qquad \mathrm{dist}(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, x)), \mathrm{dist}(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, y)), \\
\qquad \frac{1}{2}\big[\mathrm{dist}(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, y)) + \mathrm{dist}(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, x))\big]\big\} \\
+ \Big[\frac{1-(k(\omega))^q}{2(k(\omega))^q}\Big] \max\big\{\mathrm{dist}(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, x)), \mathrm{dist}(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, y))\big\} \\
+ \Big[\frac{1-(k(\omega))^q}{4(k(\omega))^q}\Big]\big[\mathrm{dist}(\mathcal{S}(\omega, x), \mathcal{T}_\lambda(\omega, y)) + \mathrm{dist}(\mathcal{S}(\omega, y), \mathcal{T}_\lambda(\omega, x))\big], \\
\qquad\qquad \text{if } y \in \mathcal{D},
\end{cases}
\end{aligned} \tag{3.3}$$

where $k: \Omega \to (0, 1)$ *are measurable mappings,*

(b) $\mathcal{D}$ *is nonempty and has the property* (N),

(c) $\mathcal{S}(\omega, \mathcal{D}) = \mathcal{D}$, *i.e.,* $\mathcal{D}$ *is* $\mathcal{S}$*-invariant,*

(d) $\mathcal{S}$ *is nonexpansive on* $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$.

Then $\mathcal{P}_{\mathcal{M}}(x_0) \cap C(\mathcal{S}, \mathcal{T}) \neq \emptyset$ *under each of the conditions of Theorem* 3.1.

Moreover, if for each $\omega \in \Omega$ *and any* $x \in \mathcal{M}$, $\mathcal{S}(\omega, x) \in \mathcal{T}(\omega, x)$ *implies* $\mathcal{S}(\omega, \mathcal{S}(\omega, x)) = \mathcal{S}(\omega, x)$, *and if* $\mathcal{S}$ *is a* $\mathcal{T}$*-weakly commuting random operator, then* $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$.

Proof. Let $y \in \mathcal{D}$, then $\mathcal{S}(\omega, y) \in \mathcal{D}$, since $\mathcal{S}(\omega, \mathcal{D}) = \mathcal{D}$ for each $\omega \in \Omega$. Also, if $y \in \partial\mathcal{M}$ then $\mathcal{T}(\omega, y) \in \mathcal{M}$, since $\mathcal{T}(\omega, \partial\mathcal{M}) \subseteq \mathcal{M}$ for each $\omega \in \Omega$. Let $z \in \mathcal{T}(\omega, y)$, then

$$\begin{aligned}\|z - x_0\|_q &\le \mathcal{H}_q(\mathcal{T}(\omega, y), \mathcal{T}(\omega, x_0))\\ &\le \|\mathcal{S}(\omega, y) - \mathcal{S}(\omega, x_0)\|_q\\ &= \|\mathcal{S}(\omega, y) - x_0\|_q = \operatorname{dist}(x_0, \mathcal{M}).\end{aligned}$$

Now $z \in \mathcal{M}$ and $\mathcal{S}(\omega, y) \in \mathcal{P}_{\mathcal{M}}(x_0)$ imply $z \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}(\omega, y)$ is also closest to x_0, hence $\mathcal{T}(\omega, y) \in \mathcal{P}_{\mathcal{M}}(x_0)$. Consequently, $\mathcal{P}_{\mathcal{M}}(x_0)$ is $\mathcal{T}(\omega, \cdot)$-invariant, that is, $\mathcal{T}(\omega, \cdot) \subseteq \mathcal{P}_{\mathcal{M}}(x_0)$. As $\mathcal{S}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$, we have for each $\omega \in \Omega$

$$\begin{aligned}\|\mathcal{S}\mathcal{T}(\omega, y) - x_0\|_q &= \|\mathcal{S}\mathcal{T}(\omega, y) - \mathcal{S}(\omega, x_0)\|_q \le \|\mathcal{T}(\omega, y) - x_0\|_q\\ &= \|\mathcal{T}(\omega, y) - \mathcal{T}(\omega, x_0)\|_q \le \|\mathcal{S}(\omega, y) - \mathcal{S}(\omega, x_0)\|_q\\ &= \|\mathcal{S}(\omega, y) - x_0\|_q.\end{aligned}$$

Thus, $\mathcal{S}\mathcal{T}(\omega, y) \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}(\omega, y) \in \mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0)$ and hence $\mathcal{T}(\omega, y) \in \mathcal{D}$. So, $\mathcal{T}$ maps $\mathcal{P}_{\mathcal{M}}(x_0)$ into $\mathrm{CB}(\mathcal{P}_{\mathcal{M}}(x_0))$ and $\mathcal{S}(\omega, \cdot)$ is a self-map on $\mathcal{D}$. Hence, all the conditions of the Theorem 3.1 are satisfied. Thus, there exists a measurable map $\xi\colon \Omega \to \mathcal{D}$ such that

$$\xi(\omega) = \mathcal{T}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \xi(\omega)) \quad \text{for each } \omega \in \Omega. \qquad \square$$

If in Theorem 3.1, Theorem 3.2 and Theorem 3.3 $\mathcal{S}(\omega, x) = x$ for all $(\omega, x) \in \Omega \times \mathcal{P}_{\mathcal{M}}(x_0)$, then we get the following random best approximation theorem.

Corollary 3.4. *Let* (Ω, Σ) *be a measurable space, let* $\mathcal{M}$ *be a subset of a* q*-normed space of* $\mathcal{X}$*, let* $\mathcal{T}\colon \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ *satisfy for all* $x, y \in \mathcal{P}_{\mathcal{M}}(x_0)$*, for all* $\omega \in \Omega$ *and* $\lambda \in [0, 1]$

$$\begin{aligned}&\mathcal{H}_q(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y))\\ &\le \max\Big\{\|x - y\|_q, \operatorname{dist}(x, \mathcal{T}_\lambda(\omega, x)), \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, y)),\\ &\qquad \frac{1}{2}\big[\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, y)) + \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, x))\big]\Big\}\\ &\quad + \Big[\frac{1 - (k(\omega))^q}{2(k(\omega))^q}\Big] \max\big\{\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, x)), \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, y))\big\}\\ &\quad + \Big[\frac{1 - (k(\omega))^q}{4(k(\omega))^q}\Big]\big[\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, y)) + \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, x))\big], \end{aligned} \tag{3.4}$$

where $k\colon \Omega \to (0, 1)$ *are measurable mappings. If* $\mathcal{P}_{\mathcal{M}}(x_0)$ *is nonempty and has the property* (N) *and* $\mathcal{P}_{\mathcal{M}}(x_0)$ *is* $\mathcal{T}$*-invariant, then* $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$ *under each of the conditions of Theorem* 3.1.

Corollary 3.5. *Let* (Ω, Σ) *be a measurable space, let* $\mathcal{M}$ *be a subset of a* q*-normed space of* $\mathcal{X}$ *and let* $\mathcal{T}\colon \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ *be such that* $\mathcal{T}(\omega, \cdot)\colon \partial\mathcal{M} \cap \mathcal{M} \to \mathcal{M}$, *where* $\partial\mathcal{M}$ *stands for the boundary of* $\mathcal{M}$. *Let* $x_0 \in \mathcal{X}$ *and* $\mathcal{T}(\omega, x_0) = \{x_0\}$ *for all* $\omega \in \Omega$. *Suppose that* $\mathcal{T}$ *satisfies for all* $x \in \mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$, *for all* $\omega \in \Omega$ *and* $\lambda \in [0, 1]$

$$\mathcal{H}_q(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \leq \begin{cases} \|x - x_0\|_q \qquad \text{if } y = x_0, \\ \max\{\|x - y\|_q, \\ \qquad \operatorname{dist}(x, \mathcal{T}_\lambda(\omega, x)), \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, y)), \\ \qquad \frac{1}{2}\big[\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, y)) + \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, x))\big]\big\} \\ + \big[\frac{1-(k(\omega))^q}{2(k(\omega))^q}\big] \max\big\{\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, x)), \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, y))\big\} \\ + \big[\frac{1-(k(\omega))^q}{4(k(\omega))^q}\big]\big[\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, y)) + \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, x))\big], \\ \qquad \text{if } y \in \mathcal{P}_{\mathcal{M}}(x_0), \end{cases} \tag{3.5}$$

where $k\colon \Omega \to (0, 1)$ *are measurable mappings. If* $\mathcal{P}_{\mathcal{M}}(x_0)$ *is nonempty and has the property* (N). *Then* $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$ *under each of the conditions of Corollary* 3.4.

Corollary 3.6. *Let* (Ω, Σ) *be a measurable space, let* $\mathcal{M}$ *be a subset of a* q*-normed space of* $\mathcal{X}$ *and let* $\mathcal{T}\colon \Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ *be such that* $\mathcal{T}(\omega, \cdot)\colon \partial\mathcal{M} \cap \mathcal{M} \to \mathcal{M}$, *where* $\partial\mathcal{M}$ *stands for the boundary of* $\mathcal{M}$. *Let* $x_0 \in \mathcal{X}$ *and* $\mathcal{T}(\omega, x_0) = \{x_0\}$ *for all* $\omega \in \Omega$. *Suppose that* $\mathcal{T}$ *satisfies for all* $x \in \mathcal{D} \cup \{x_0\}$, *for all* $\omega \in \Omega$ *and* $\lambda \in [0, 1]$

$$\mathcal{H}_q(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \leq \begin{cases} \|x - x_0\|_q \qquad \text{if } y = x_0, \\ \max\{\|x - y\|_q, \\ \qquad \operatorname{dist}(x, \mathcal{T}_\lambda(\omega, x)), \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, y)), \\ \qquad \frac{1}{2}\big[\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, y)) + \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, x))\big]\big\} \\ + \big[\frac{1-(k(\omega))^q}{2(k(\omega))^q}\big] \max\big\{\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, x)), \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, y))\big\} \\ + \big[\frac{1-(k(\omega))^q}{4(k(\omega))^q}\big]\big[\operatorname{dist}(x, \mathcal{T}_\lambda(\omega, y)) + \operatorname{dist}(y, \mathcal{T}_\lambda(\omega, x))\big], \\ \qquad \text{if } y \in \mathcal{D}, \end{cases} \tag{3.6}$$

where $k:\Omega \to (0,1)$ *are measurable mappings. If* $\mathcal{D}$ *is nonempty and has the property* (N)*, then* $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$ *under each of the conditions of Theorem* 3.1.

Corollary 3.7. *Let* (Ω, Σ) *be a measurable space, let* $\mathcal{M}$ *be a subset of a* q*-normed space of* $\mathcal{X}$*, let* $\mathcal{S}:\Omega \times \mathcal{M} \to \mathcal{M}$ *and* $\mathcal{T}:\Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$*. Suppose that* $\mathcal{T}$ *and* $\mathcal{S}$ *satisfy for all* $x, y \in \mathcal{P}_{\mathcal{M}}(x_0)$*, for all* $\omega \in \Omega$ *and* $\lambda \in [0,1]$

$$\begin{aligned}
&\mathcal{H}_q(\mathcal{T}(\omega,x), \mathcal{T}(\omega,y)) \\
&\leq \max\Big\{\|\mathcal{S}(\omega,x) - \mathcal{S}(\omega,x_0)\|_q, \\
&\qquad\qquad \operatorname{dist}(\mathcal{S}(\omega,x), \mathcal{T}_\lambda(\omega,x)), \operatorname{dist}(\mathcal{S}(\omega,y), \mathcal{T}_\lambda(\omega,y)), \\
&\qquad\qquad \frac{1}{2}\big[\operatorname{dist}(\mathcal{S}(\omega,x), \mathcal{T}_\lambda(\omega,y)) + \operatorname{dist}(\mathcal{S}(\omega,y), \mathcal{T}_\lambda(\omega,x))\big]\Big\}. \qquad (3.7)
\end{aligned}$$

If $\mathcal{P}_{\mathcal{M}}(x_0)$ *is nonempty and has the property* (N) *and if* $\mathcal{P}_{\mathcal{M}}(x_0)$ *is both* $\mathcal{T}$*-invariant and* $\mathcal{S}$*-invariant, then* $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{C}(\mathcal{S}, \mathcal{T}) \neq \emptyset$ *under each of the conditions of Theorem* 3.1.

Moreover, if for each $\omega \in \Omega$ *and any* $x \in \mathcal{M}$*,* $\mathcal{S}(\omega,x) \in \mathcal{T}(\omega,x)$ *implies* $\mathcal{S}(\omega, \mathcal{S}(\omega,x)) = \mathcal{S}(\omega,x)$*, and if* $\mathcal{S}$ *is a* $\mathcal{T}$*-weakly commuting random operator, then* $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$.

Corollary 3.8. *Let* (Ω, Σ) *be a measurable space, let* $\mathcal{M}$ *be a subset of a* q*-normed space of* $\mathcal{X}$ *and let* $\mathcal{T}:\Omega \times \mathcal{M} \to \mathrm{CB}(\mathcal{M})$ *satisfy for all* $x, y \in \mathcal{P}_{\mathcal{M}}(x_0)$*, for all* $\omega \in \Omega$ *and* $\lambda \in [0,1]$

$$\begin{aligned}
&\mathcal{H}_q(\mathcal{T}(\omega,x), \mathcal{T}(\omega,y)) \\
&\leq \max\Big\{\|x - x_0\|_q, \operatorname{dist}(x, \mathcal{T}_\lambda(\omega,x)), \operatorname{dist}(y, \mathcal{T}_\lambda(\omega,y)), \\
&\qquad\qquad \frac{1}{2}\big[\operatorname{dist}(x, \mathcal{T}_\lambda(\omega,y)) + \operatorname{dist}(y, \mathcal{T}_\lambda(\omega,x))\big]\Big\}. \qquad (3.8)
\end{aligned}$$

If $\mathcal{P}_{\mathcal{M}}(x_0)$ *is nonempty and has the property* (N) *and if* $\mathcal{P}_{\mathcal{M}}(x_0)$ *is* $\mathcal{T}$*-invariant, then* $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$ *under each of the conditions of Corollary* 3.4.

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