

Ricci flow of unwarped and warped product manifolds

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Abstract

We analyse Ricci flow (normalised/un-normalised) of product manifolds –unwarped as well as warped, through a study of generic examples. First, we investigate such flows for the unwarped scenario with manifolds of the type $\mathbb{S}^n \times \mathbb{S}^m$, $\mathbb{S}^n \times \mathbb{H}^m$, $\mathbb{H}^m \times \mathbb{H}^n$ and also, similar multiple products. We are able to single out generic features such as singularity formation, isotropisation at particular values of the flow parameter and evolution characteristics. Subsequently, motivated by warped braneworlds and extra dimensions, we look at Ricci flows of warped spacetimes. Here, we are able to find analytic solutions for a special case by variable separation. For others we numerically solve the equations and draw certain useful inferences about the evolution of the warp factor, the scalar curvature as well the occurrence of singularities at finite values of the flow parameter. We also investigate the dependence of the singularities of the flow on the initial conditions. We expect our results to be useful in any physical/mathematical context where such product manifolds may arise.

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I. INTRODUCTION AND OVERVIEW

Ricci and other geometric flows [1] have been a topic of active research interest in both mathematics and physics, over the last few years. Though introduced many years ago by Hamilton [2] in mathematics and in the work of Friedan [3] in the context of σ -models and string theory, it was Perelman's proof of the Poincare conjecture using Ricci flows [4] which generated most of the current interest. Much of the subsequent work has thus focused on using ideas associated with Ricci flows, in the context of general relativity and gravitation [5]. In this article, we follow a similar line of thought with reference to metrics on warped manifolds of various types.

The definition of a warped manifold in mathematics is somewhat different from that used in physics. In mathematics, product manifolds with different overall conformal factors dependent only on a certain parameter, are termed as warped. However, in physics, warping refers to an extra coordinate dependence of one or more of these conformal factors—in other words, the conformal factor associated with one manifold in the product when assumed to be dependent on one or more of the coordinates of the other manifold, is said to be warped. Otherwise, it is termed as unwarped. We shall follow this latter definition in our work here.

We restrict our attention to Ricci flow of specific types of product manifolds. Earlier work on such manifolds can be found in [6]. We begin, following the work on non-Bianchi types in the penultimate section in [7], by looking at products of the form $\mathbb{S}^n \times \mathbb{S}^m$, $\mathbb{S}^n \times \mathbb{H}^m$, $\mathbb{H}^m \times \mathbb{H}^n$ for arbitrary m and n , and with different conformal factors (dependent on a parameter λ) associated with the canonical metric on each manifold in the product. This study essentially involves unwarped products in the sense described in the previous paragraph. We also look at multiple products (more than two) as a possible generalisation. What do we look for in our investigations? Apart from exact solutions of the flow equations (which may not be possible always), we focus attention on the following aspects: (a) blow-up/dropping to zero value of any of the conformal factors at a finite λ , (b) isotropisation (i.e. equal value of the conformal factor at a finite λ) (c) fixed points/curves of the flows (d) dependence of flow on initial conditions.

In the second half of our paper, we look at specific warped product manifolds largely inspired by currently fashionable ideas in the physics of extra dimensions. In particular, our product manifolds here are essentially similar to those used extensively in warped braneworld

models introduced in the work of Randall and Sundrum, about a decade ago [8]. A Ricci flow of such manifolds seems to provide interesting pointers in the physics of braneworlds—an aspect we discuss towards the end of our article.

We have studied normalised Ricci flows in the unwarped cases while for the warped ones we primarily investigate the unnormalised Ricci flow. In the former, we are mainly concerned with Riemannian metrics, whereas in the latter we discuss the Lorentzian case in detail because it is the one which is physically relevant.

A word here about the essential mathematical motivation of Ricci flows may not be too inappropriate. In any method to find the ‘best’ metric on a manifold the first thing to do it to define a natural evolution of metrics. One such evolution of metrics w.r.t. a ‘time’ parameter (not the physical time, in any sense) is the Ricci flow. Once we have defined the flow it is important to prove that the flow exists for all time and converges to a geometric limit. In a case the flow does not converge, the corresponding metrics degenerate and one then needs to relate the degeneration with the topology of the manifold. Among many good candidates, a specific choice is an evolution equation of vector fields (rather, a family of vector fields) on the space of metrics known as the Ricci flow equation and defined as:

$$\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij} \tag{1}$$

For the normalised Ricci flow, the corresponding equation turns out to be:

$$\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij} + \frac{2}{n}\langle R \rangle g_{ij} \tag{2}$$

where $\langle R \rangle = \frac{\int R dV}{\int dV}$. Thus the normalised flow ceases to be different from the unnormalised one if we consider non-compact, infinite volume manifolds (with a finite value for $\int R dV$). Moreover, for constant curvature manifolds the second term in the normalised flow equation reduces to $\frac{2}{n}R$. In this article, we shall first look at normalised flows in the next section. Subsequently, we shall deal with un-normalised flows.

II. RICCI FLOW OF UNWARPED PRODUCT MANIFOLDS

As mentioned in the Introduction, in this Section, we focus attention on unwarped product manifolds. We discuss each case separately by studying the evolution equations and solving

each dynamical system analytically. We present our results through figures which illustrate the nature of evolution of each conformal factor in the corresponding product manifold. In this Section, our study is based on normalised Ricci flows, since all our manifolds are of constant curvature.

A. On $\mathbb{S}^n(\sqrt{D}) \times \mathbb{S}^m(\sqrt{E})$

Here we focus on product manifolds of the aforementioned type with D and E representing the conformal factors for the \mathbb{S}^n and \mathbb{S}^m in the product. D and E are functions of the flow parameter λ .

Let us compute the various curvature quantities (Ricci tensor, Ricci scalar) of this product manifold. We denote the metric on $\mathbb{S}^n(\sqrt{a})$ by $ag_{\mathbb{S}^n}$ where $g_{\mathbb{S}^n}$ is the canonical metric on n -sphere. In the same way, the metric on the product manifold $\mathbb{S}^n(\sqrt{D}) \times \mathbb{S}^m(\sqrt{E})$ looks like: $g = Dg_{\mathbb{S}^n} + Eg_{\mathbb{S}^m}$. Let Y and V be the unit vector fields on \mathbb{S}^n and \mathbb{S}^m respectively and X be a unit vector field either on \mathbb{S}^n or \mathbb{S}^m which is perpendicular to both Y and V . It can be shown that (by Koszul's formula) $\langle \nabla_Y X, V \rangle = 0$ as $[Y, X]$ is either zero or tangent to \mathbb{S}^n and likewise with $[X, V]$. Thus $\nabla_Y X = 0$ if X is tangent to \mathbb{S}^m . Also $\nabla_Y X$ is tangent to \mathbb{S}^n if X is tangent to \mathbb{S}^n , showing that $\nabla_Y X$ can be computed on $\mathbb{S}^n(\sqrt{D})$. This shows that if X, Y are tangent to \mathbb{S}^n and U, V are tangent to $\mathbb{S}^m(\sqrt{E})$, then $\mathfrak{R}(X \wedge V) = 0$ where \mathfrak{R} is the curvature operator. Also $\mathfrak{R}(X \wedge Y) = \frac{1}{D}X \wedge Y$ and $\mathfrak{R}(U \wedge V) = \frac{1}{E}U \wedge V$. In particular, all sectional curvatures lie in the interval $[0, \max\{\frac{1}{D}, \frac{1}{E}\}]$. From this we can say:

$$Rc(X) = \frac{(n-1)}{D}X$$

$$Rc(V) = \frac{(m-1)}{E}V \text{ or we can write } Rc_g = (n-1)g_{\mathbb{S}^n} + (m-1)g_{\mathbb{S}^m} \text{ and } R = \frac{n(n-1)}{D} + \frac{m(m-1)}{E}.$$

Therefore we can make inference that $\mathbb{S}^n(\sqrt{D}) \times \mathbb{S}^m(\sqrt{E})$ always has constant scalar curvature and it is an Einstein manifold exactly when $\frac{n-1}{D} = \frac{m-1}{E}$ (this is possible either $n, m \geq 2$ or $n = m = 1$). It is also worthwhile to mention that this product manifold has constant sectional curvature only when $n = m = 1$. Now we have all the ingredients in our hand to write normalised Ricci flow equations. These are given as:

$$\frac{dD}{d\lambda} = -2(n-1) + \frac{2}{(n+m)} \left\{ \frac{n(n-1)}{D} + \frac{m(m-1)}{E} \right\} D \quad (3)$$

$$\frac{dE}{d\lambda} = -2(m-1) + \frac{2}{(n+m)} \left\{ \frac{n(n-1)}{D} + \frac{m(m-1)}{E} \right\} E \quad (4)$$

The above two equations represent a first order dynamical system. For different n and m we get different dynamical systems. For $n, m = 1$ we have D and E as constant and the metric is a flat metric on $\mathbb{S}^1 \times \mathbb{S}^1$.

1. General Solution

After some simplification, we have:

$$\frac{dD}{d\lambda} = -\frac{2m(n-1)}{m+n} + \frac{2m(m-1)}{(n+m)} \frac{D}{E} \quad (5)$$

$$\frac{dE}{d\lambda} = -\frac{2n(m-1)}{m+n} + \frac{2n(n-1)}{(n+m)} \frac{E}{D} \quad (6)$$

We may analyse the following general form for the above pair of equations. Similar equations will arise for the other product manifolds to be discussed later on. Solutions can be found for arbitrary a, b, c and d . Specific values can be inserted for each case separately. Thus, we now analyse the generic system:

$$\frac{dx}{d\lambda} = -a + b \frac{x}{y} \quad ; \quad \frac{dy}{d\lambda} = -c + d \frac{y}{x} \quad (7)$$

Dividing one equation with the other, we get:

$$\frac{dx}{dy} = \frac{y(yd - cx)}{x(bx - ay)} \quad (8)$$

which can be solved by putting $y = vx$:

$$x \frac{dv}{dx} = \frac{v^2(a+d) - v(c+b)}{(b-av)} \equiv \frac{v^2 c_1 + v c_2}{(b-av)} \quad (9)$$

Defining $c_1 = (d+a)$ and $c_2 = -(c+b)$, the general solution of x and v will look like:

$$\frac{b}{c_2} \ln\left(\frac{v}{c_1 v + c_2}\right) - \frac{a}{c_1} \ln(c_1 v + c_2) = \ln x - \ln k \quad (10)$$

For $\mathbb{S}^n(\sqrt{D}) \times \mathbb{S}^m(\sqrt{E})$, with $c_1 = 2(n-1)$ and $c_2 = -2(m-1)$ we have:

$$\ln x + \frac{m}{m+n} \ln v = \ln k, \implies x^n y^m = k \quad (11)$$

If we put the above back in the original equations, we get the profile of x or y .

$$\frac{dx}{d\lambda} = -a + \frac{b}{k} x^{\frac{m+n}{n}} \quad (12)$$

This can be integrated to get $\lambda(x)$ in terms of hypergeometric functions. For $n = m$ or $n = 1, m$, as we shall see one can obtain x and y as functions of λ . Otherwise, in general, the functions $\lambda(x)$ or $\lambda(y)$ are not invertible. We now discuss a few illustrative, special cases.

2. Special cases

a. $\mathbb{S}^1(\sqrt{D}) \times \mathbb{S}^2(\sqrt{E})$

Here $n = 1$ and $m = 2$. The equations become:

$$\frac{dD}{d\lambda} = \frac{4D}{3E} ; \quad \frac{dE}{d\lambda} = -\frac{2}{3} \quad (13)$$

The solutions are straightforward:

$$D(\lambda) = D_0 \left(\frac{E_0}{E_0 - \frac{2}{3}\lambda} \right)^2 ; \quad E(\lambda) = E_0 - \frac{2}{3}\lambda \quad (14)$$

where at $\lambda = 0$, $D = D_0, E = E_0$. One notices that D increases with increasing λ and diverges to infinity at a finite $\lambda = \frac{3}{2}E_0$. At the same value of λ , E drops to zero. The flow therefore ends at this value of λ —a curvature singularity appearing because of the shrinking of the S^1 part to zero radius. Fig.1(a) shows the aforementioned behaviour.

In general, for products of the type $\mathbb{S}^1(\sqrt{D}) \times \mathbb{S}^m(\sqrt{E})$ we end up with the following expressions for $D(\lambda)$ and $E(\lambda)$.

$$D(\lambda) = D_0 \left(\frac{E_0}{E_0 - \frac{2(m-1)}{(m+1)}\lambda} \right)^m ; \quad E(\lambda) = E_0 - \frac{2(m-1)}{(m+1)}\lambda \quad (15)$$

The conclusions for arbitrary m , are similar to those for $m = 2$.

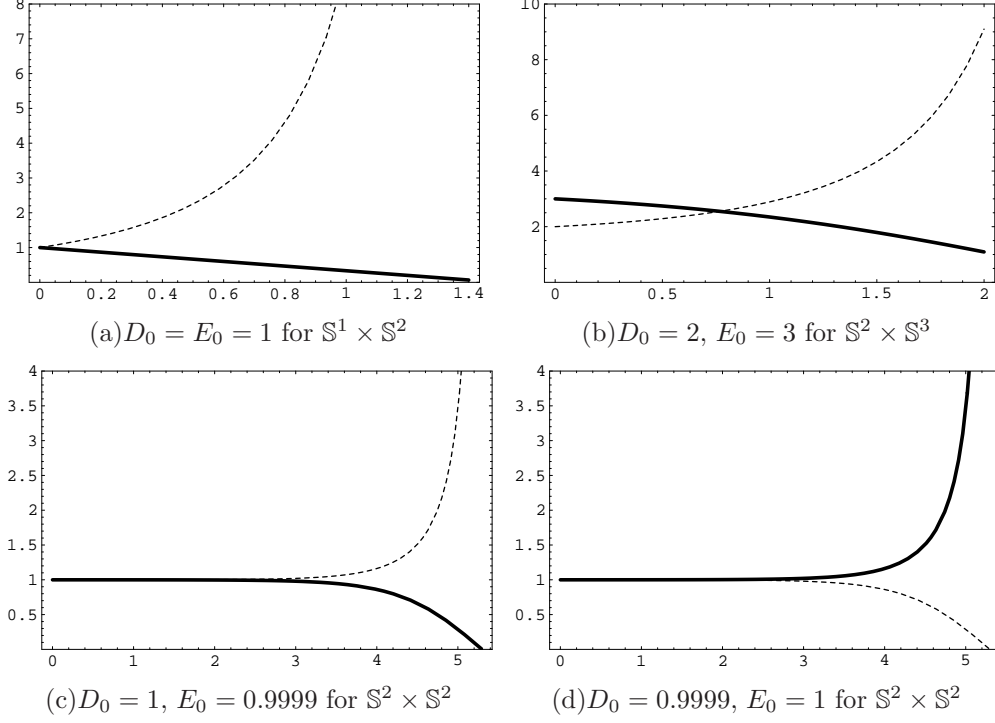


FIG. 1: $D(\lambda)$ (dashed) and $E(\lambda)$ (continuous). The horizontal axes is λ .

b. $\mathbb{S}^2(\sqrt{D}) \times \mathbb{S}^2(\sqrt{E})$

Here, we turn to the more interesting case of $\mathbb{S}^2(\sqrt{D}) \times \mathbb{S}^2(\sqrt{E})$. The equations are:

$$\frac{dD}{d\lambda} = -1 + \frac{D}{E} ; \quad \frac{dE}{d\lambda} = -1 + \frac{E}{D} \quad (16)$$

We easily note that $DE = \text{constant} = c(\text{say})$. Putting $\frac{1}{E} = \frac{D}{c}$ the first equation looks like:

$$\frac{dD}{d\lambda} = \frac{D^2 - c}{c} \quad (17)$$

and the solution with, $D(\lambda = 0) = D_0$ is:

$$D(\lambda) = \sqrt{c} \left[\frac{D_0(1 + \exp \frac{2\lambda}{\sqrt{c}}) + \sqrt{c}(1 - \exp \frac{2\lambda}{\sqrt{c}})}{D_0(1 - \exp \frac{2\lambda}{\sqrt{c}}) + \sqrt{c}(1 + \exp \frac{2\lambda}{\sqrt{c}})} \right] \quad (18)$$

From $DE = c$ one can find $E(\lambda)$ which is given as:

$$E(\lambda) = \sqrt{c} \left[\frac{D_0(1 - \exp^{\frac{2\lambda}{\sqrt{c}}}) + \sqrt{c}(1 + \exp^{\frac{2\lambda}{\sqrt{c}}})}{D_0(1 + \exp^{\frac{2\lambda}{\sqrt{c}}}) + \sqrt{c}(1 - \exp^{\frac{2\lambda}{\sqrt{c}}})} \right] \quad (19)$$

Thus, it is clear that $E_0 = \frac{c}{D_0}$. For $D_0 = E_0 = \sqrt{c}$, we get $D(\lambda)$ and $E(\lambda)$ as constant for all λ . For $D_0 > \sqrt{c}$ we have a divergence in $D(\lambda)$ while $E(\lambda)$ drops to zero at the same $\lambda = \lambda_d$. On the other hand, if $D_0 < \sqrt{c}$, the behaviour of $D(\lambda)$ and $E(\lambda)$ are reversed. In either case the value of λ at which the divergence occurs is given as:

$$\lambda_d = \frac{\sqrt{c}}{2} \ln \left(\frac{D_0 + \sqrt{c}}{D_0 - \sqrt{c}} \right) \quad (20)$$

Fig.1(c) and 1(d) demonstrate these features.

It may be noted that the above results for $\mathbb{S}^2(\sqrt{D}) \times \mathbb{S}^2(\sqrt{E})$ holds in the more general setting, for $\mathbb{S}^n(\sqrt{D}) \times \mathbb{S}^n(\sqrt{E})$ with an appropriate scaling of λ . The general equations are:

$$\frac{dD}{d\lambda} = (n-1)\left(-1 + \frac{D}{E}\right) ; \quad \frac{dE}{d\lambda} = (n-1)\left(-1 + \frac{E}{D}\right) \quad (21)$$

The solution for $D(\lambda)$ turns out to be:

$$D(\lambda) = \sqrt{c} \left[\frac{D_0(1 + \exp^{\frac{2(n-1)\lambda}{\sqrt{c}}}) + \sqrt{c}(1 - \exp^{\frac{2(n-1)\lambda}{\sqrt{c}}})}{D_0(1 - \exp^{\frac{2(n-1)\lambda}{\sqrt{c}}}) + \sqrt{c}(1 + \exp^{\frac{2(n-1)\lambda}{\sqrt{c}}})} \right] \quad (22)$$

Further λ_d is given as:

$$\lambda_d = \frac{\sqrt{c}}{2(n-1)} \ln \left(\frac{D_0 + \sqrt{c}}{D_0 - \sqrt{c}} \right) \quad (23)$$

It is worthwhile to note that in this case, $D = E = \text{constant}$ represents a fixed point (soliton) of the flow.

3. Numerical Evaluations

Though we have exact solutions formally, it is easier to analyse the nature of the flow if we obtain numerical solutions of the dynamical systems involved. This is because the analytic

solutions is the cases other than the ones quoted above are necessarily non-invertible.

We shall focus here on cases where $n \neq m$. Let us see if it is possible to have $D(\lambda_0) = E(\lambda_0)$ for $n \neq m$. Looking at the general equations one can evaluate the derivatives assuming this ($D = E$) to be true. It turns out that,

$$\left[\frac{dD}{d\lambda} \right]_{\lambda_0} = \frac{2m(m-n)}{n+m} \quad ; \quad \left[\frac{dE}{d\lambda} \right]_{\lambda_0} = \frac{2n(n-m)}{n+m} \quad (24)$$

Thus for $n = m$ there is no chance of D to be equal to E at some λ . On the other hand, this is always possible for $n \neq m$. We illustrate this fact by solving the equations numerically, as a dynamical system.

a. $\mathbb{S}^2(\sqrt{D}) \times \mathbb{S}^3(\sqrt{E})$

The equations in this case are:

$$\frac{dD}{d\lambda} = -\frac{6}{5} + \frac{12}{5} \frac{D}{E} \quad ; \quad \frac{dE}{d\lambda} = -\frac{8}{5} + \frac{4}{5} \frac{E}{D} \quad (25)$$

Fig.1(b) below demonstrates the facts mentioned above.

Similar results may be obtained for any other case with $n \neq m$. In summary, we observe the following:

- There are fixed curves defined by $\frac{D}{E} = \frac{n-1}{m-1}$.
- As one of the radii (D or E) increases (or becomes singular), the other drops to zero. This will also be apparent from a $\ln D$ vs $\ln E$ plot, which will follow the equation:
 $\ln D = -\frac{m}{n} \ln E + \frac{1}{n} \ln k$.
- Reversing the initial conditions can lead to a reversal of the behaviour of D or E. This is easily seen in the case $n = m$ where the equations go over to each other if we swap D and E . However, this is also true, as we have checked (not shown here) for $n \neq m$.
- Generically similar behaviour for all m , is also seen when one of the two manifolds is \mathbb{S}^1 and the other is \mathbb{S}^m (with $m \neq 1$).

B. On $\mathbb{S}^n(\sqrt{D}) \times \mathbb{H}^m(\sqrt{E})$

Let us now turn towards analysing another type of product manifold (as mentioned above). For this case, the metric looks like: $g = Dg_{\mathbb{S}^n} + Eg_{\mathbb{H}^m}$ where $g_{\mathbb{S}^n}$ is the canonical metric on n dimensional sphere and $g_{\mathbb{H}^m}$ is the metric on a m dimensional hyperbolic space. The normalised Ricci flow equations turn out to be:

$$\frac{dD}{d\lambda} = -2(n-1) + \frac{2}{(n+m)} \left\{ \frac{n(n-1)}{D} - \frac{m(m-1)}{E} \right\} D \quad (26)$$

$$\frac{dE}{d\lambda} = 2(m-1) + \frac{2}{(n+m)} \left\{ \frac{n(n-1)}{D} - \frac{m(m-1)}{E} \right\} E \quad (27)$$

where, as before, D and E are functions of the flow parameter λ .

1. General Solution of this system

As in the previous subsection, after some simplification we have:

$$\dot{x} = -\frac{2m(n-1)}{m+n} - \frac{2m(m-1)}{(n+m)} \frac{x}{y} \quad ; \quad \dot{y} = \frac{2n(m-1)}{m+n} + \frac{2n(n-1)}{(n+m)} \frac{y}{x} \quad (28)$$

Following the general analysis in terms of x, y and the a, b, c, d given earlier, we note that the values of a, b, c, d , for this case are:

$$a = 2m \frac{(n-1)}{(m+n)} \quad ; \quad b = -2m \frac{(m-1)}{(m+n)} \quad ; \quad c = -2n \frac{(m-1)}{(m+n)} \quad ; \quad d = 2n \frac{(n-1)}{(m+n)} \quad (29)$$

Also, $c_1 = (d+a) = 2(n-1), c_2 = -(c+b) = 2(m-1)$. Notice the difference in signs as compared to the $\mathbb{S}^n \times \mathbb{S}^m$ case. Further, the relation between x and y will be:

$$\ln x + \frac{m}{m+n} \ln y = \ln k \implies x^n y^m = k \quad (30)$$

Though this appears to be the same as for $\mathbb{S}^n \times \mathbb{S}^m$, a difference will appear in the explicit solution of $x(\lambda)$ and $y(\lambda)$. For instance,

$$\frac{dx}{d\lambda} = -a - \frac{b}{k} x^{\frac{m+n}{n}} \quad (31)$$

and, consequently, one can find $y(\lambda)$. Let us look at some special cases now.

2. $\mathbb{S}^1(\sqrt{D}) \times \mathbb{H}^2(\sqrt{E})$

In this rather simple case, the difference quoted above is easily seen. The equations are:

$$\frac{dD}{d\lambda} = -\frac{4}{3} \frac{D}{E} \quad ; \quad \frac{dE}{d\lambda} = \frac{2}{3} \quad (32)$$

The solution is:

$$D(\lambda) = D_0 \left(\frac{E_0}{E_0 + \frac{2}{3}\lambda} \right)^2 \quad ; \quad E(\lambda) = E_0 + \frac{2}{3}\lambda \quad (33)$$

where at $\lambda = 0$, $D = D_0, E = E_0$. Note the change in sign, which suggests that there is no singularity in the evolution of D , starting from $\lambda = 0$, unlike the previous case of $\mathbb{S}^1 \times \mathbb{S}^2$. The result here can easily be generalised for $\mathbb{S}^1 \times \mathbb{H}^m$.

3. $\mathbb{S}^n(\sqrt{D}) \times \mathbb{H}^n(\sqrt{E})$

Here, the equations turn out to be:

$$\frac{dD}{d\lambda} = -(n-1) - (n-1) \frac{D}{E} \quad ; \quad \frac{dE}{d\lambda} = (n-1) + (n-1) \frac{E}{D} \quad (34)$$

The features to note here are:

- There are no fixed curves, unlike the previous case for $\mathbb{S}^m \times \mathbb{S}^m$.
- There is always a possibility of having $D = E$ at some λ .
- As before DE is a constant, but evolution of $D(\lambda)$ and $E(\lambda)$ differs as compared to the previous cases.

C. $\mathbb{H}^n \times \mathbb{H}^m$

Finally we turn to products of the form $\mathbb{H}^n \times \mathbb{H}^m$. Here, the metric takes the form $g = Dg_{\mathbb{H}^n} + Eg_{\mathbb{H}^m}$ where $g_{\mathbb{H}^n(m)}$ is the canonical metric on $n(m)$ dimensional hyperbolic

space. The normalised Ricci flow equations become:

$$\frac{dD}{d\lambda} = 2(n-1) - \frac{2}{n+m} \left\{ \frac{n(n-1)}{D} + \frac{m(m-1)}{E} \right\} D \quad (35)$$

$$\frac{dE}{d\lambda} = 2(m-1) - \frac{2}{n+m} \left\{ \frac{n(n-1)}{D} + \frac{m(m-1)}{E} \right\} E \quad (36)$$

The analysis for these equations follows in the same way as for the earlier cases. We list below are few characteristic observations.

- The fixed curves are given as $\frac{D}{E} = \frac{n-1}{m-1}$ —this is similar to the $\mathbb{S}^n \times \mathbb{S}^m$ case.
- The possibility of $D = E$ at some λ exists as long as $n \neq m$.
- The flow ends at singular points, depending on initial conditions.

D. Multiple products and general trends

What could happen if we take multiple products of the \mathbb{S}^n , or \mathbb{H}^n or mixed products of various kinds? In general, it is difficult question to address. However, we can surely try out some specific examples. We illustrate things using one such example, below.

1. $\mathbb{S}^p(\sqrt{F}) \times \mathbb{S}^n(\sqrt{D}) \times \mathbb{S}^m(\sqrt{E})$

For this triple product, the metric looks like: $g = Fg_{\mathbb{S}^p} + Dg_{\mathbb{S}^n} + Eg_{\mathbb{S}^m}$ where $g_{\mathbb{S}^n}$ is the canonical metric on n dimensional sphere. The normalised Ricci flow equations are as follows:

$$\frac{dF}{d\lambda} = -2(p-1) + \frac{2}{(p+n+m)} \left\{ \frac{p(p-1)}{F} + \frac{n(n-1)}{D} + \frac{m(m-1)}{E} \right\} F \quad (37)$$

$$\frac{dD}{d\lambda} = -2(n-1) + \frac{2}{(p+n+m)} \left\{ \frac{p(p-1)}{F} + \frac{n(n-1)}{D} + \frac{m(m-1)}{E} \right\} D \quad (38)$$

$$\frac{dE}{d\lambda} = -2(m-1) + \frac{2}{(p+n+m)} \left\{ \frac{p(p-1)}{F} + \frac{n(n-1)}{D} + \frac{m(m-1)}{E} \right\} E \quad (39)$$

The above flow equations can be analysed in a way similar to the cases discussed earlier. The dynamical system is far more complicated. A few general comments can be made without solving the equations.

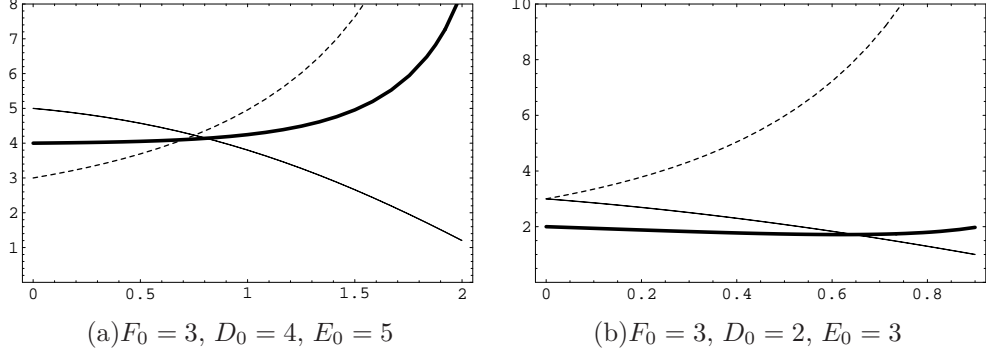


FIG. 2: $F(\lambda)$ (dashed), $D(\lambda)$ (thick) and $E(\lambda)$ (thin) for $\mathbb{S}^2 \times \mathbb{S}^3 \times \mathbb{S}^4$. The horizontal axes is λ

- If $n = m = p = 1$ then F, D and E do not evolve.
- If any one of n,m or p is equal to one (say $p = 1$), then the system of equations for D, E can be solved independently as was done for the simple products discussed in the previous subsections. These solutions can be used to find the evolution of F.
- If $n = m = p$ then the equations are simpler, but unlike the case of a product of two manifolds, we do not have any fixed curves here. If $n \neq m \neq p$ or $n = m \neq p$ there are fixed curves which can be found by just putting the R. H. S. of the above equations to zero and solving the algebraic equations.
- An intersection of all three functions D, E and F at some λ is possible as long as $p < n < m$, $n^2 + m^2 > p(n + m)$ and $p^2 + m^2 < n(p + m)$. This can be checked by using $D = E = F$ on the R. H. S. of the general equations and assuming that $\frac{dF}{d\lambda} > 0$, $\frac{dD}{d\lambda} < 0$ at the same λ . One can easily show that the above restrictions are impossible to achieve. This is also visible in the numerical example given below.

2. $\mathbb{S}^2(\sqrt{F}) \times \mathbb{S}^3(\sqrt{D}) \times \mathbb{S}^4(\sqrt{E})$

In this case the equations are:

$$\frac{dF}{d\lambda} = -\frac{14}{9} + \frac{4F}{3D} + \frac{8F}{3E} \quad ; \quad \frac{dD}{d\lambda} = -\frac{8}{3} + \frac{4D}{9F} + \frac{8D}{3E} \quad (40)$$

$$\frac{dE}{d\lambda} = -\frac{10}{3} + \frac{4E}{9D} + \frac{4E}{3F} \quad (41)$$

We numerically solve this system. Fig.2 shows some typical evolution features for F, D and E.

The figures illustrate the fact that there can be a pair among D , E and F which can intersect at some λ . Different pairs intersect at different points. Also, there can also be a situation without any intersection, this seems to be the case when a pair among D , E and F have the same initial value. Furthermore, divergences appear and the flow ends at a finite λ , as was observed in the earlier cases with a product of two manifolds.

It is possible to generalise the above results to products of arbitrary number of manifolds of differing dimensions. One can easily write down the general evolution equations by inspection. The case with products of \mathbb{S}^m and \mathbb{H}^m and also only \mathbb{H}^m can also be discussed. Generic features are possibly not very different. Furthermore, instead of taking the normalisation on the whole manifold, one may consider, following related work in [7], cases with volume normalisation along sections. We will discuss some of these issues in a later article. For the moment, we prefer to switch over and move on to the perhaps more interesting case of warped products.

III. RICCI FLOW OF WARPED PRODUCT MANIFOLDS

In spacetime models with extra dimensions, braneworld scenarios [8] provide many examples of product manifolds. We consider one such case where the manifold is a product of Minkowski spacetime \mathcal{M} and the real line \mathcal{R}^1 . In addition, we have warping in the sense described earlier in the Introduction, which makes the ensuing analysis different from the ones described in the previous sections.

In order to consider the evolution under Ricci flow of the *warped product* $\mathcal{M} \times \mathcal{R}^1$ we assume the line element of the form:

$$ds^2 = e^{2f(\sigma,\lambda)}(-dt^2 + dx^2 + dy^2 + dz^2) + r_c^2(\sigma, \lambda)d\sigma^2 \quad (42)$$

Note that the functions f and r_c are assumed to be functions of σ as well as a non-coordinate parameter λ . We now evolve the metric functions in the above line element according to a un-normalised Ricci flow, with λ as the flow parameter.

The relevant geometric quantities for this metric are –

$$-R_{tt} = R_{xx} = R_{yy} = R_{zz} = -\frac{e^{2f}}{r_c^2} \left(f'' + 4f'^2 - \frac{f'r'_c}{r_c} \right) \quad (43)$$

$$R_{\sigma\sigma} = -4 \left(f'' + f'^2 - \frac{f'r'_c}{r_c} \right) \quad (44)$$

$$R = -\frac{4}{r_c^2} \left(2f'' + 5f'^2 - 2\frac{f'r'_c}{r_c} \right) \quad (45)$$

After some straightforward algebra, the Ricci flow equations become –

$$\dot{f} = \frac{1}{r_c^2} \left(f'' + 4f'^2 - \frac{f'r'_c}{r_c} \right) \quad (46)$$

$$\dot{r}_c = \frac{4}{r_c} \left(f'' + f'^2 - \frac{f'r'_c}{r_c} \right) \quad (47)$$

here “ \dot{f} ” represents $\frac{\partial f}{\partial \lambda}$ and “ f' ” represents $\frac{\partial f}{\partial \sigma}$ and similarly for r_c . The equation for f may be thought of as a certain type of *nonlinear heat equation with the equation for r_c appearing as a constraint*.

We make note of the remarkable fact that, the above Ricci flow equations are independent of the signature of the part of the metric that contains a scaling with e^{2f} .

We now try to understand the behaviour of f (especially as a function of λ) by solving the flow equations. The resulting solutions will tell us how the metric, as well other geometric quantities, evolve along the flow.

A. Separable metric functions: exact results

Let us first assume that

$$r_c(\sigma, \lambda) = r_c(\lambda) \quad ; \quad f(\sigma, \lambda) = f_\sigma(\sigma) + f_\lambda(\lambda) \quad (48)$$

This assumption enables us to write the Eq.46 and Eq.47 in a variable separable form as:

$$r_c^2 \dot{f}_\lambda = (f''_\sigma + 4f'^2_\sigma) = K_1 \quad (49)$$

$$r_c \dot{r}_c = 4 (f''_\sigma + f'^2_\sigma) = -K_2 \quad (50)$$

For the σ -part we get –

$$f'_\sigma = \pm \left(\frac{K_1}{3} + \frac{K_2}{12} \right)^{\frac{1}{2}} ; f''_\sigma = - \left(\frac{K_1 + K_2}{3} \right) \quad (51)$$

which imply $K_1 = -K_2$ and –

$$f_\sigma = \pm \left(\frac{K_1}{4} \right)^{1/2} \sigma \quad (52)$$

Therefore, we can easily solve the λ evolution to get,

$$r_c^2 = e^{2f_\lambda} = 1 + 2K_1\lambda \quad (53)$$

where, we make use of $K_1 = -K_2$ and the initial condition that at $\lambda = 0$ we have $f = 0$ and $r_c = 1$ that is no scaling in the metric.

To simplify, we can further write $K_1 = 1/2\lambda_c$ and thus the final metric turns out to be:

$$ds^2 = \left(1 + \frac{\lambda}{\lambda_c} \right) \left[\exp \left(\pm \frac{\sigma}{\sqrt{2\lambda_c}} \right) (-dt^2 + dx^2 + dy^2 + dz^2) + d\sigma^2 \right] \quad (54)$$

The Ricci scalar for the abovestated metric evolves as,

$$R = -\frac{5/2}{\lambda + \lambda_c} \quad (55)$$

We can prove that the solution obtained above is the only solution of the set Eq.46 and Eq.47 when $r_c = r_c(\lambda)$. Under these assumptions Eq.47 can be written as:

$$r_c \dot{r}_c = 4 (f'' + f'^2) = 4B(\lambda)^2 \quad (56)$$

This readily yields $r_c^2 = 8 \int B(\lambda)^2 d\lambda = \beta$; and then solving for f ,

$$f_1 = \pm B(\lambda)\sigma + C(\lambda)$$

or

$$f_2 = A(\lambda) + \ln [\cosh (B(\lambda)\sigma + C(\lambda))]$$

Using $f = f_1$ in Eq.46, then gives –

$$\beta \times (\pm \dot{B}\sigma + \dot{C}) = 4B^2$$

This implies $B = \text{constant}$ and thus $f = \pm B\sigma + C(\lambda)$. This is just the variable separable case considered earlier.

Similarly, for $f = f_2$ we have –

$$\beta \left[\dot{A} + \tanh (B\sigma + C) \times (\dot{B}\sigma + \dot{C}) \right] = B^2 [1 + 3 \tanh^2 (B\sigma + C)]$$

Since, both sides represent functions analytic in σ , we compare their Taylor series around the point $\sigma = 0$. The comparison of first 5 Taylor coefficients gives –

$$\dot{B} = \frac{3B^3}{\beta} \text{sech}^4 C \quad ; \quad \dot{C} = \frac{3B^2}{\beta} \tanh C (1 + \tanh^2 C)$$

$$-B^2 + \dot{A}\beta + 3B^2 \tanh^4 C = 0$$

$$4B^5 \tanh C \text{sech}^4 C = 0$$

$$B^6 (4 - 3 \cosh 2C) \text{sech}^6 C = 0$$

The only solution to these is $B = 0$ i.e. $f = A(\lambda) + \ln [\cosh C(\lambda)]$ and again f is separable. This is a limiting case of the earlier solution when $\lambda_c \rightarrow \infty$ and leads to a static flat metric.

Thus, Eq.54 is the general (and only) solution of the Ricci flow equations, when $r_c = r_c(\lambda)$.

B. Nonseparable metric functions: numerical results

To deal with a more general situation, we now drop the assumption, $r_c = r_c(\lambda)$ and consider the full non-separable case of Eqs.46 & 47. We observe that if $f(\sigma, \lambda)$ and $r_c(\sigma, \lambda)$ are a solution of the flow then so are $f(\alpha\sigma, \alpha^2\lambda)$ and $r_c(\alpha\sigma, \alpha^2\lambda)$. Thus, we look for *scaling solutions* of the form $f(y)$ and $r_c(y)$ with $y = \frac{\sqrt{\lambda}}{\sigma}$. Note that it is also possible to work with $\frac{1}{y}$ instead of y , though, with such a choice, large y will correspond to small λ and vice

versa. We further assume that either $\sigma > 0$ or $\sigma_0 < \sigma < \sigma_1$, i.e. we consider, as the extra dimensional space the open half line or an open interval.

In terms of the new variable y and defining $u = \frac{1}{r_c}$, the flow equations reduce to the following first order dynamical system,

$$\frac{df}{dy} = A \quad (57)$$

$$\frac{dA}{dy} = \frac{A}{2y^3u^2} - \frac{2A}{y} - 24y^3A^3u^2 \quad (58)$$

$$\frac{du}{dy} = -4Au^3 \left(\frac{1}{u^2} - 6Ay^3 \right) \quad (59)$$

Also, from Eq.45 we can write –

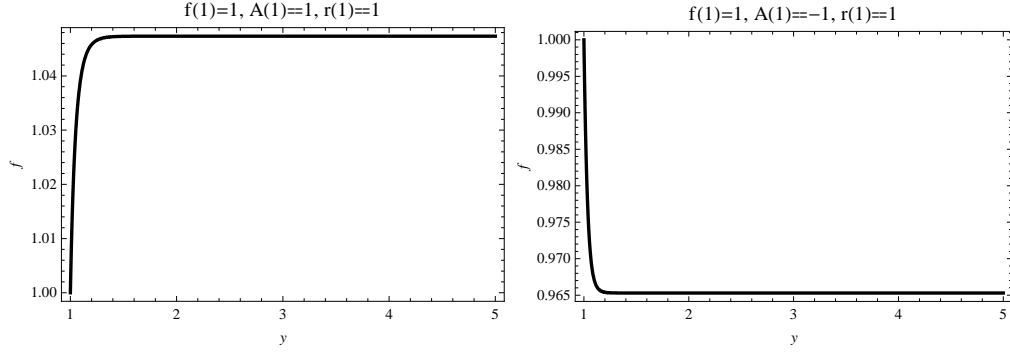
$$\begin{aligned} \sigma^2 R &= -\frac{8y^2}{r_c^2} \left[\frac{d^2 f}{dy^2} + \frac{2}{y} \left(\frac{df}{dy} \right) + \frac{5}{2} \left(\frac{df}{dy} \right)^2 - \frac{1}{r_c} \left(\frac{df}{dy} \right) \left(\frac{dr_c}{dy} \right) \right] \\ &= -\frac{4A}{y} + 12y^2 A^2 u^2 \end{aligned} \quad (60)$$

We shall use this latter expression in our evaluations.

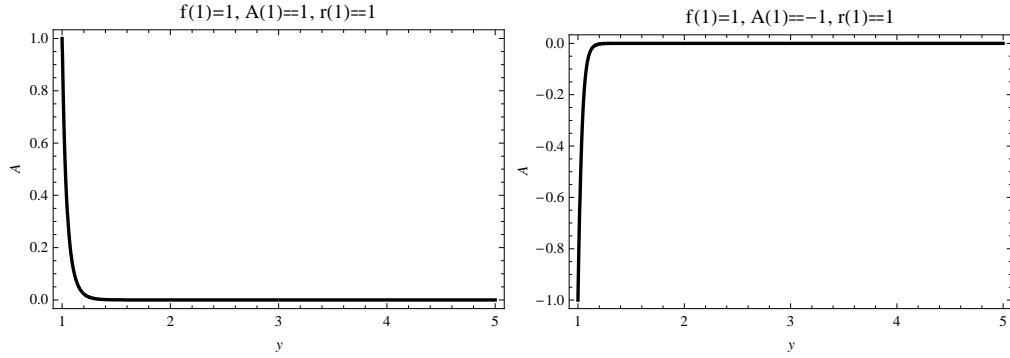
Note that the above dynamical system is invariant under scaling transformations of the type $(y \rightarrow \alpha y, f \rightarrow f, A \rightarrow A/\alpha, r_c \rightarrow \alpha r_c)$. We can, thus, choose the scale by specifying initial conditions at $y = 1$ and solutions for conditions at other values of y can be obtained simply by the appropriate scaling. Hence, starting from $y = 1$ we can study the evolution of the system in the future ($y > 1$).

Further, as all our results will be in terms of $y = \frac{\sqrt{\lambda}}{\sigma}$ we must therefore fix σ (at some $\sigma \neq 0$) in order to understand the evolution in λ at that σ and subsequently repeat things for another σ . Physical considerations dictate that $r_c > 0$ and since the equations are independent of f we can take $f(1) = 1$. We first choose the initial conditions at $y = 1$ as $(f, A, r_c) = (1, \pm 1, 1)$. The results of the numerical integration are plotted in Fig.3(a) - 3(d).

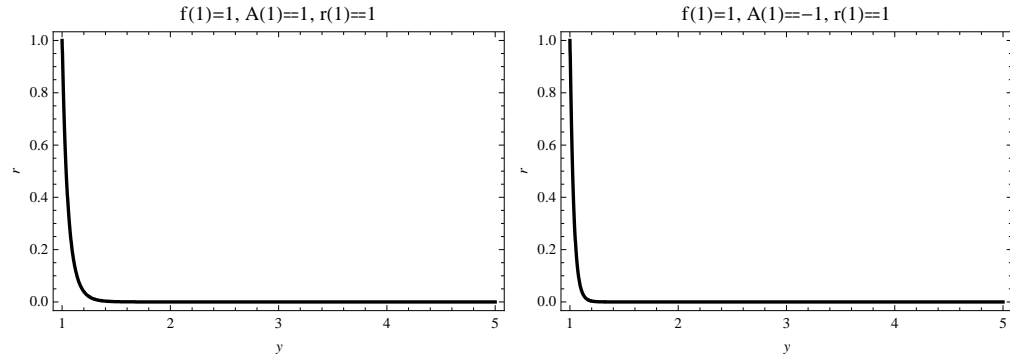
We note that when $A(1) = +1$, f eventually goes to a constant and r_c drops to zero in the future ($y > 1$). Thus the flow exists for arbitrarily large values of y i.e. in the future and



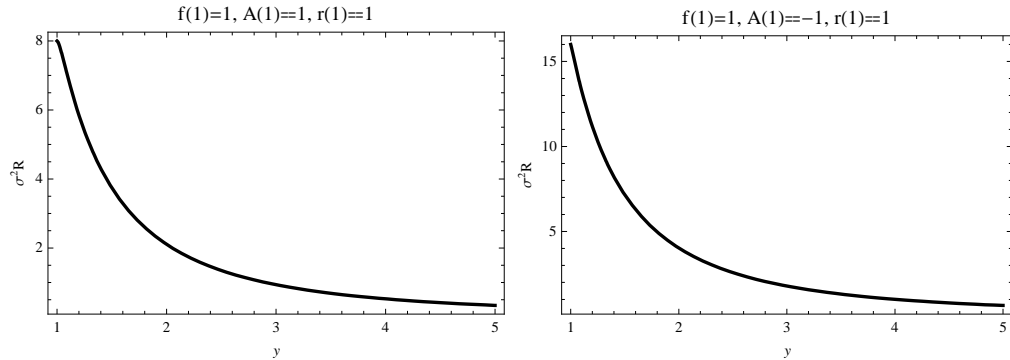
(a) f v/s y



(b) A v/s y



(c) r_c v/s y



(d) $\sigma^2 R$ v/s y

FIG. 3: Results of numerical integration of dynamical system. Figures on left are for $A(1) = +1$ and those on right are for $A(1) = -1$

the geometry tends towards a singular geometry only as y (or λ) becomes infinitely large.

For $A(1) = -1$, as y becomes large f decreases and tends to a constant value, while A beginning at a negative value approaches zero eventually. r_c tend to zero value as $y \rightarrow \infty$. The Ricci scalar R for both initial conditions on A is non-singular at all finite y and decays with y monotonically.

It is necessary to look into the behaviour of the quantity $\sigma^2 R$ as we change the initial values of f , A and r_c . Firstly, let us note that changing the initial value of f does not produce any effect. However, interesting features do arise when we change $A(1)$ or $r_c(1)$. Fig.4 shows this variation for different initial values of A . Starting from the top left, we change $A(1)$ from a small value to successive larger values, while keeping $r_c(1)$ and $f(1)$ fixed. We note that the location of the maximum shifts towards vertical axis and the value of $\sigma^2 R$ at a given y becomes larger with increasing initial A . Also, it is worth noting that the $\sigma^2 R$ is always positive and goes to zero asymptotically.

In Fig.5, we show how a variation of $r_c(1)$ (at fixed $f(1)$ and $A(1)$) changes the behaviour of $\sigma^2 R$. At smaller values of $r_c(1)$ the flow continues indefinitely into the future. But for a critical value of $r_c(1)$, the flow develops a *future singularity*. Moreover, with changing r_c we switch from positive to negative values of $\sigma^2 R$. The flow cannot be extended beyond a certain value of y if the initial r_c is above a certain critical value.

To understand the appearance of genuine singularities, we investigate when such future singularities arise in the flow and their dependence on the initial conditions on r_c and A . We search the range of $1 < y < 100$ with different initial conditions on A and r_c , and look for the appearance of singularities. Since, the flow does not depend on f , we can make a *phase diagram* in the space of $(A(1), r_c(1))$, in which we can represent singular and non-singular flows. Fig.6 shows such a plot. The white (blank) region represents conditions that do not lead to singularities, whereas the darker shades of gray denote flows with value of $y = y_s$ at which the flow becomes singular. Thus, for future singularities, increasing $r_c(1)$ makes y_s smaller, and hence the flow can only be continued to shorter times in the future. These trends can also be seen in Figs.4 and 5. Note that for $A(1) < 0$ the flow has no singularities at all and exists for all time. Also, from the phase diagram in Fig.6 we see that, there are regions of initial conditions $A(1), r_c(1)$ corresponding to both types of flow

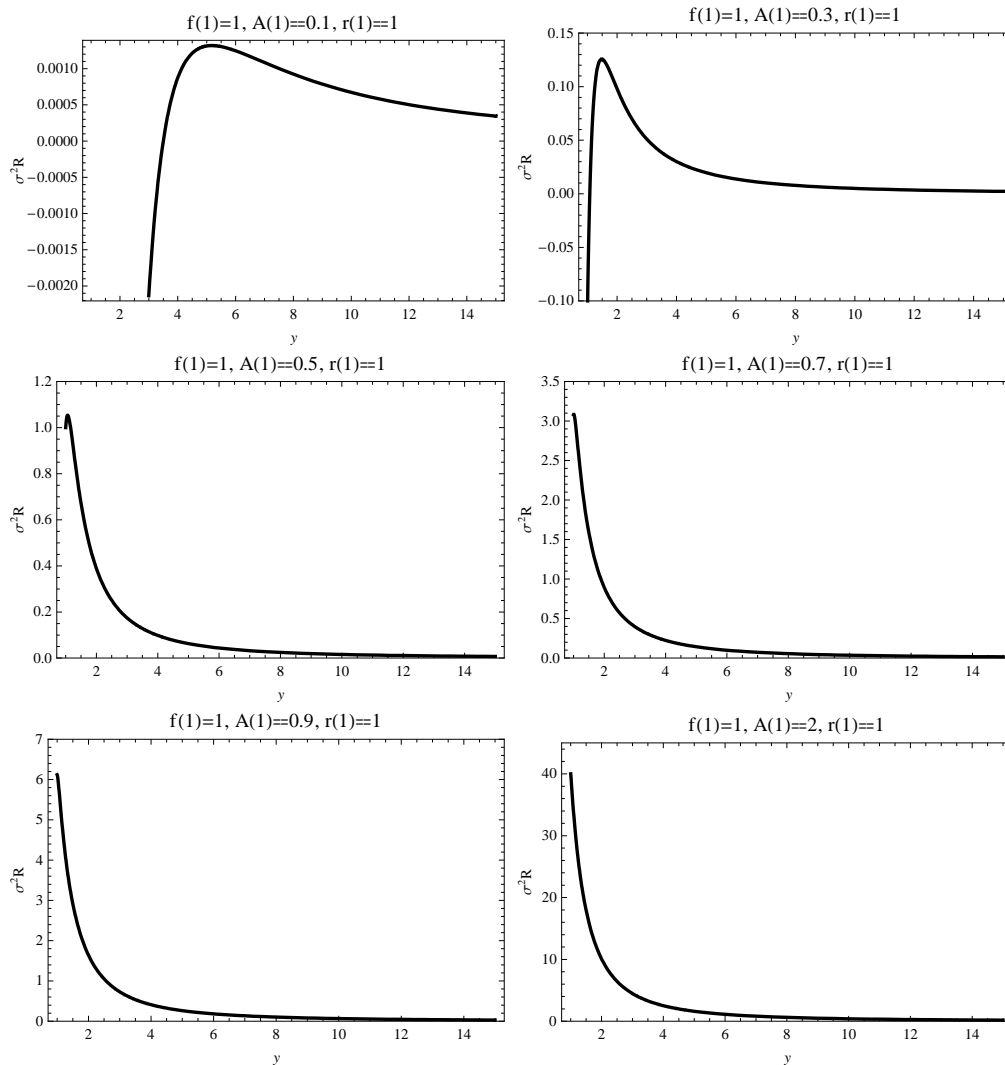


FIG. 4: $\sigma^2 R$ as a function of y for fixed f, r_c and changing A .

— non-singular and only future singular.

Given the above results, let us now try to understand the consequences a little more. From the solutions for f, A, r_c and the expression for $\sigma^2 R$ (all of which are functions of y) we note that for fixed $y = y_0$, all these quantities have a fixed value, i.e. $f(y_0) = f_0, A(y_0) = A_0, r_c(y_0) = r_{c0}$ and $\tilde{R}(y_0) = \sigma^2 R(y_0) = \tilde{R}_0$. However, we also have $y_0 = \frac{\sqrt{\lambda}}{\sigma}$ or $\sigma = \frac{\sqrt{\lambda}}{y_0}$. Hence, for fixed y , the flow along λ is equivalent to moving along the extra dimension. Changing σ (or λ) keeps the obtained f, A, r_c and $\sigma^2 R$ unchanged at fixed y . However, R will change with changing σ and at different σ the value of the Ricci scalar will obviously be different. Thus, the solutions obtained represents Ricci flow along the extra

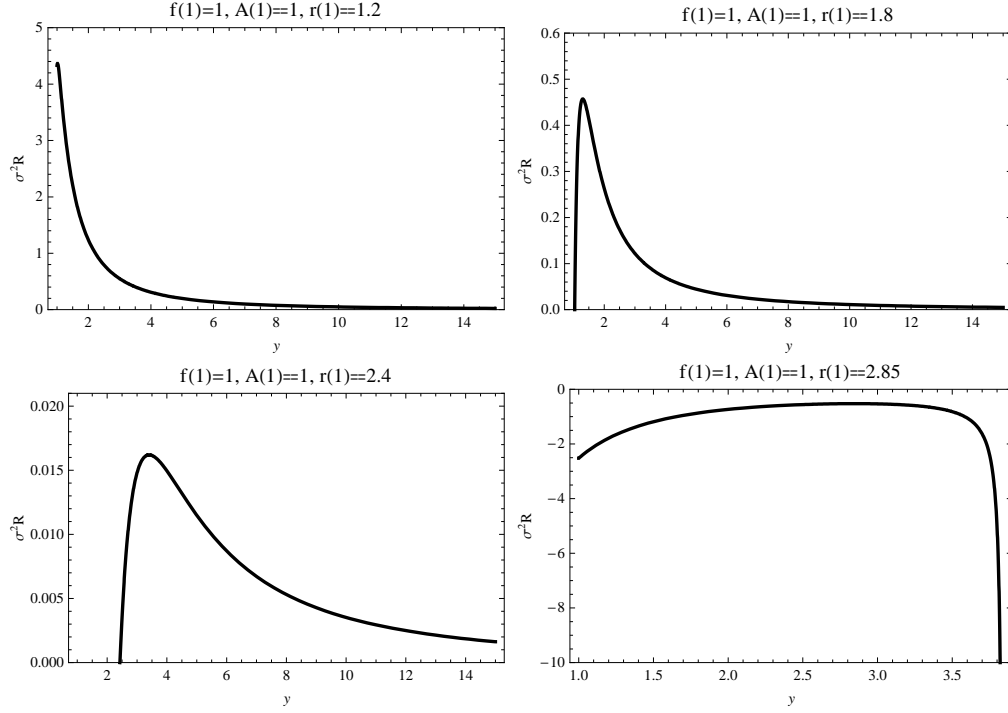
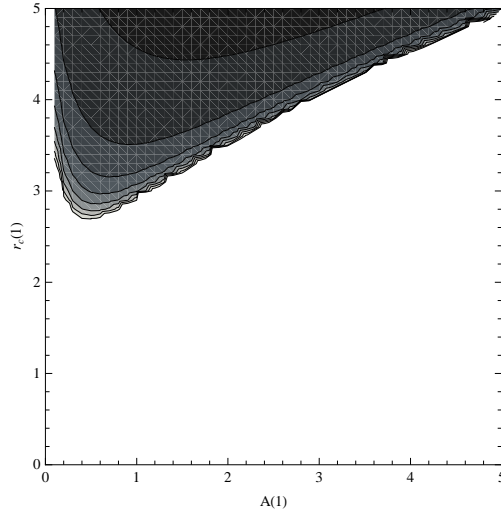


FIG. 5: $\sigma^2 R$ as a function of y for fixed f , A and changing r_c .



(a)future singularities

FIG. 6: phase diagram showing non-singular and singular flows in space of $(A(1), r_c(1))$

dimensional coordinate (proportional to $\sqrt{\lambda}$ at fixed y), such that the value of f , A and r_c , as we move from one location to another in the extra dimension remains unaltered, though the value of the five dimensional Ricci scalar changes.

IV. REMARKS AND CONCLUSIONS

We conclude with a summary and some remarks.

We have looked into Ricci flow of unwarped and warped product manifolds. In the unwarped case we have focused on $\mathbb{S}^n \times \mathbb{S}^m$, $\mathbb{H}^n \times \mathbb{H}^m$ and $\mathbb{S}^n \times \mathbb{H}^m$. Distinguishing features of the Ricci flow of such manifolds have been highlighted with reference to fixed points/curves, singularity formation, intersections which imply isotropisation at some λ and dependence on initial conditions. We have also discussed multiple products and provided illustrative examples for the case $\mathbb{S}^p \times \mathbb{S}^n \times \mathbb{S}^m$.

Further, inspired by warped braneworlds, we have looked into Ricci flow of such manifolds with arbitrary functional forms of the warp factor and the scale of the extra dimensions. We discuss two separate cases—one where these functions are separable (this being analytically solvable) and the other, the non-separable case where we use numerical methods to find the evolution. In the former case, we are able to find the line element exactly and also compute its curvature. We note that the spacetime is necessarily AdS for all λ and this λ can, in some way, be associated with the k (AdS curvature scale) in the warped braneworld scenario. Thus Ricci flow seems to lead to a RS type warp factor and the curvature of the spacetime remains AdS during evolution. In the more general case, we have analysed numerically the particular case of scaling solutions. We find that, for certain initial conditions, the evolution is regular throughout. We have shown that Ricci flow in λ turns out to be equivalent to a flow along σ (at fixed y) and the f , A , r_c and $\sigma^2 R$ at that value of y do not change with varying λ (or σ), though the value of R does evolve. For other initial conditions, we note that genuine singularities may appear at finite y . We also observe a transition from non-singular to singular solutions as we tune the initial conditions.

Finally, we mention that we hope to look into more complex and realistic scenarios in the warped context, in future. In the unwarped case, we would like to arrive at more general conclusions for arbitrary multiple products by a more detailed analysis of the associated dynamical system.

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