

Research Article

Some Properties of Certain Analytic Functions

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Let \mathcal{A}_p be the class of certain analytic functions $f(z)$ in the open unit disk \mathbb{U} . For $f(z) \in \mathcal{A}_p$, a subclass $\mathcal{A}_p(\alpha, \beta, \gamma, j)$ of \mathcal{A}_p is introduced. The object of the present paper is to discuss some properties of functions $f(z)$ belonging to class $\mathcal{A}_p(\alpha, \beta, \gamma; j)$.

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1. Introduction

Let \mathcal{A}_p denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}_p(\alpha, \beta, \gamma; j)$ be the subclass of \mathcal{A}_p consisting of functions $f(z)$ which satisfy

$$\operatorname{Re} \left\{ \alpha \frac{f^{(j)}(z)}{z^{p-j}} + \beta \frac{f^{(j+1)}(z)}{z^{p-j-1}} \right\} > \gamma \quad (z \in \mathbb{U}), \tag{1.2}$$

for some α ($\alpha > 0$), β ($\beta > 0$), and γ ($0 \leq \gamma < p! \{\alpha + (p-j)\beta\} / (p-j)!$), where $j = 0, 1, 2, \dots, p-1$. If $p = 1$ and $j = 0$, then the class $\mathcal{A}_1(\alpha, \beta, \gamma; 0)$ is defined by

$$\operatorname{Re} \left\{ \alpha \frac{f(z)}{z} + \beta f'(z) \right\} > \gamma \quad (z \in \mathbb{U}) \tag{1.3}$$

for some α ($\alpha > 0$), β ($\beta > 0$), and γ ($0 \leq \gamma < \alpha + \beta$). This class was studied by Wang et al. [1].

From the definition for the class $\mathcal{A}_p(\alpha, \beta, \gamma; j)$, we see the following.

Remark 1.1. $\mathcal{A}_p(\alpha, \beta, \gamma; j)$ is convex.

Proof. For $f(z) \in \mathcal{A}_p(\alpha, \beta, \gamma; j)$ and $g(z) \in \mathcal{A}_p(\alpha, \beta, \gamma; j)$, we define

$$F(z) = (1 - t)f(z) + tg(z) \quad (0 \leq t \leq 1). \tag{1.4}$$

Then,

$$\begin{aligned} & \operatorname{Re} \left\{ \alpha \frac{F^{(j)}(z)}{z^{p-j}} + \beta \frac{F^{(j+1)}(z)}{z^{p-j-1}} \right\} \\ &= \operatorname{Re} \left\{ \alpha \frac{(1-t)f^{(j)}(z) + tg^{(j)}(z)}{z^{p-j}} + \beta \frac{(1-t)f^{(j+1)}(z) + tg^{(j+1)}(z)}{z^{p-j-1}} \right\} \\ &= (1-t) \operatorname{Re} \left\{ \alpha \frac{f^{(j)}(z)}{z^{p-j}} + \beta \frac{f^{(j+1)}(z)}{z^{p-j-1}} \right\} \\ & \quad + t \operatorname{Re} \left\{ \alpha \frac{g^{(j)}(z)}{z^{p-j}} + \beta \frac{g^{(j+1)}(z)}{z^{p-j-1}} \right\} > (1-t)\gamma + t\gamma = \gamma; \end{aligned} \tag{1.5}$$

therefore, $F(z) \in \mathcal{A}_p(\alpha, \beta, \gamma; j)$, that is, $\mathcal{A}_p(\alpha, \beta, \gamma; j)$ is convex. □

In the present paper, we consider some properties of functions $f(z)$ belonging to class $\mathcal{A}_p(\alpha, \beta, \gamma; j)$.

2. Properties of class $\mathcal{A}_p(\alpha, \beta, \gamma; j)$

We begin with the statement and the proof of the following result.

THEOREM 2.1. *A function $f(z) \in \mathcal{A}_p$ is in the class of $\mathcal{A}_p(\alpha, \beta, \gamma; j)$ if and only if*

$$f(z) = z^p + 2(\delta - \gamma) \int_{|x|=1} \left(\sum_{k=p+1}^{\infty} \frac{(k-j)!}{k! \{\alpha + (k-j)\beta\}} x^{k-p} z^k \right) d\mu(x), \tag{2.1}$$

where $\mu(x)$ is the probability measure on $X = \{x \in \mathbb{C} : |x| = 1\}$ and $\delta = p! \{\alpha + (p-j)\beta\} / (p-j)!$.

Proof. For $f(z) \in \mathcal{A}_p(\alpha, \beta, \gamma; j)$, we define

$$F(z) = \frac{\alpha(f^{(j)}(z)/z^{p-j}) + \beta(f^{(j+1)}(z)/z^{p-j-1}) - \gamma}{\delta - \gamma}, \tag{2.2}$$

where $\delta = p! \{\alpha + (p-j)\beta\} / (p-j)!$. Then, $F(z)$ is the Carathéodory function, since $F(0) = 1$ and $\operatorname{Re} F(z) > 0$. Hence, we can write (see [2])

$$F(z) = \frac{\alpha(f^{(j)}(z)/z^{p-j}) + \beta(f^{(j+1)}(z)/z^{p-j-1}) - \gamma}{\delta - \gamma} = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x). \tag{2.3}$$

Since (2.3) is equivalent to

$$\beta \left(\frac{\alpha}{\beta} f^{(j)}(z) + z f^{(j+1)}(z) \right) = \gamma z^{p-j} + (\delta - \gamma) z^{p-j} \int_{|x|=1} \left(1 + \sum_{k=1}^{\infty} 2x^k z^k \right) d\mu(x), \quad (2.4)$$

we have that

$$z^{\alpha/\beta-1} \left(\frac{\alpha}{\beta} f^{(j)}(z) + z f^{(j+1)}(z) \right) = \frac{1}{\beta} z^{\alpha/\beta-1} \left\{ \delta z^{p-j} + (\delta - \gamma) \int_{|x|=1} \left(\sum_{k=1}^{\infty} 2x^k z^{k+p-j} \right) d\mu(x) \right\}. \quad (2.5)$$

Integrating both sides of (2.5), we know that

$$\begin{aligned} & \int_0^z \zeta^{\alpha/\beta-1} \left(\frac{\alpha}{\beta} f^{(j)}(\zeta) + \zeta f^{(j+1)}(\zeta) \right) d\zeta \\ &= \frac{1}{\beta} \int_{|x|=1} \left\{ \int_0^z \left(\delta \zeta^{p-j+\alpha/\beta-1} + 2(\delta - \gamma) \left(\sum_{k=1}^{\infty} x^k \zeta^{k+p-j+\alpha/\beta-1} \right) \right) d\zeta \right\} d\mu(x), \end{aligned} \quad (2.6)$$

that is,

$$\begin{aligned} z^{\alpha/\beta} f^{(j)}(z) &= \frac{\delta}{\alpha + (p-j)\beta} z^{p-j+\alpha/\beta} + 2(\delta - \gamma) \\ &\quad \times \int_{|x|=1} \left(\sum_{k=1}^{\infty} \frac{x^k}{\alpha + (k+p-j)\beta} z^{k+p-j+\alpha/\beta} \right) d\mu(x). \end{aligned} \quad (2.7)$$

This implies that

$$f^{(j)}(z) = \frac{p!}{(p-j)!} z^{p-j} + 2(\delta - \gamma) \int_{|x|=1} \left(\sum_{k=1}^{\infty} \frac{x^k}{\alpha + (k+p-j)\beta} z^{k+p-j} \right) d\mu(x). \quad (2.8)$$

An integration of both sides in (2.8) gives us that

$$\begin{aligned} \int_0^z f^{(j)}(\zeta) d\zeta &= \frac{p!}{(p-j)!} \int_0^z \zeta^{p-j} d\zeta + 2(\delta - \gamma) \\ &\quad \times \int_{|x|=1} \left\{ \int_0^z \left(\sum_{k=1}^{\infty} \frac{x^k}{\alpha + (k+p-j)\beta} \zeta^{k+p-j} \right) d\zeta \right\} d\mu(x), \end{aligned} \quad (2.9)$$

or

$$\begin{aligned} f^{(j-1)}(z) - f^{(j-1)}(0) &= \frac{p!}{(p-j+1)!} z^{p-j+1} + 2(\delta - \gamma) \\ &\quad \times \int_{|x|=1} \left(\sum_{k=1}^{\infty} \frac{x^k}{\{\alpha + (k+p-j)\beta\} (k+p-j+1)} z^{k+p-j+1} \right) d\mu(x). \end{aligned} \quad (2.10)$$

Therefore, we obtain that

$$f^{(j-1)}(z) = \frac{p!}{(p-j+1)!} z^{p-j+1} + 2(\delta - \gamma) \times \int_{|x|=1} \left(\sum_{k=p+1}^{\infty} \frac{x^{k-p}}{\{\alpha + (k-j)\beta\}(k-j+1)} z^{k-j+1} \right) d\mu(x). \tag{2.11}$$

Applying the same method for (2.11), we see that

$$f^{(j-2)}(z) = \frac{p!}{(p-j+2)!} z^{p-j+2} + 2(\delta - \gamma) \times \int_{|x|=1} \left(\sum_{k=p+1}^{\infty} \frac{x^{k-p}}{\{\alpha + (k-j)\beta\}(k-j+1)(k-j+2)} z^{k-j+2} \right) d\mu(x). \tag{2.12}$$

Furthermore, integrating $(j - 2)$ times both sides in (2.12) and noting that $f^{(j)}(0) = 0$ ($j = 0, 1, 2, \dots, p - 1$), we conclude that

$$f(z) = z^p + 2(\delta - \gamma) \int_{|x|=1} \left(\sum_{k=p+1}^{\infty} \frac{(k-j)!}{k! \{\alpha + (k-j)\beta\}} x^{k-p} z^k \right) d\mu(x). \tag{2.13}$$

This completes the proof of Theorem 2.1. □

Taking $p = 1$ and $j = 0$ in Theorem 2.1, we have the following.

COROLLARY 2.2. *A function $f(z) \in \mathcal{A}_1$ is in the class of $\mathcal{A}_1(\alpha, \beta, \gamma; 0)$ if and only if*

$$f(z) = z + 2(\alpha + \beta - \gamma) \int_{|x|=1} \left(\sum_{k=2}^{\infty} \frac{1}{\alpha + k\beta} x^{k-1} z^k \right) d\mu(x), \tag{2.14}$$

where $\mu(x)$ is the probability measure on $X = \{x \in \mathbb{C} : |x| = 1\}$ and $0 \leq \gamma < \alpha + \beta$.

In view of Theorem 2.1, we have following corollary for a_k .

COROLLARY 2.3. *If $f(z)$ is in the class $\mathcal{A}_p(\alpha, \beta, \gamma; j)$, then*

$$|a_k| \leq \frac{2(\delta - \gamma)(k-j)!}{k! \{\alpha + (k-j)\beta\}} \quad (k \geq p+1), \tag{2.15}$$

where $\delta = p! \{\alpha + (p-j)\beta\} / (p-j)!$. Equality holds for the function $f(z)$ given by

$$f(z) = z^p + 2(\delta - \gamma) \left(\sum_{k=p+1}^{\infty} \frac{(k-j)!}{k! \{\alpha + (k-j)\beta\}} z^k \right). \tag{2.16}$$

Further, the following distortion inequality follows from Theorem 2.1.

COROLLARY 2.4. If $f(z)$ is in class $\mathcal{A}_p(\alpha, \beta, \gamma; j)$, then

$$\begin{aligned} & \max \left\{ 0, \frac{p!}{(p-j)!} |z|^{p-j} - 2(\delta - \gamma) \left(\sum_{k=1}^{\infty} \frac{1}{\alpha + (k+p-j)\beta} |z|^{k+p-j} \right) \right\} \\ & \leq |f^{(j)}(z)| \leq \frac{p!}{(p-j)!} |z|^{p-j} + 2(\delta - \gamma) \left(\sum_{k=1}^{\infty} \frac{1}{\alpha + (k+p-j)\beta} |z|^{k+p-j} \right) \quad (z \in \mathbb{U}), \end{aligned} \tag{2.17}$$

where $j = 0, 1, 2, \dots, p-1$.

Next we derive the following.

THEOREM 2.5. A function $f(z) \in \mathcal{A}_p(\alpha, \beta, \gamma; j)$ satisfies

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p - \mu \tag{2.18}$$

for $|z| < r_0$, where

$$r_0 = \inf_{k \geq p+1} \left(\frac{(k-2)!p(p-\mu)\{\alpha + (k-j)\beta\}}{(k-j)!2(\delta-\gamma)(k-\mu)} \right)^{1/(k-p)}, \tag{2.19}$$

$0 \leq \mu < p$, and $0 \leq \gamma < p! \{\alpha + (p-j)\beta\} / (p-j)!$. Therefore, $f(z)$ is p -valently starlike of order μ for $|z| < r_0$.

Proof. Note that

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - p \right| &= \left| \frac{zf'(z) - pf(z)}{f(z)} \right| = \left| \frac{\sum_{k=p+1}^{\infty} (k-p)a_k z^k}{z^p + \sum_{k=p+1}^{\infty} a_k z^k} \right| \\ &\leq \frac{\sum_{k=p+1}^{\infty} (k-p) |a_k| |z|^k}{|z|^p - \sum_{k=p+1}^{\infty} |a_k| |z|^k} = \frac{\sum_{k=p+1}^{\infty} (k-p) |a_k| |z|^{k-p}}{1 - \sum_{k=p+1}^{\infty} |a_k| |z|^{k-p}}. \end{aligned} \tag{2.20}$$

Now, if

$$\sum_{k=p+1}^{\infty} (k-\mu) |a_k| |z|^{k-p} < p - \mu, \tag{2.21}$$

that is, using Corollary 2.3, if

$$\sum_{k=p+1}^{\infty} \frac{2(\delta-\gamma)(k-\mu)(k-j)!}{k! \{\alpha + (k-j)\beta\}} |z|^{k-p} < p - \mu, \tag{2.22}$$

then we see that

$$\frac{\sum_{k=p+1}^{\infty} (k-p) |a_k| |z|^{k-p}}{1 - \sum_{k=p+1}^{\infty} |a_k| |z|^{k-p}} < p - \mu. \tag{2.23}$$

Furthermore, considering that

$$p - \mu = p(p - \mu) \left(\sum_{k=p+1}^{\infty} \frac{1}{k(k-1)} \right), \tag{2.24}$$

we know that the inequality (2.23) can be written by

$$\sum_{k=p+1}^{\infty} \frac{2(\delta - \gamma)(k - \mu)(k - j)!}{k! \{ \alpha + (k - j)\beta \}} |z|^{k-p} < p(p - \mu) \left(\sum_{k=p+1}^{\infty} \frac{1}{k(k-1)} \right). \tag{2.25}$$

Thus, if

$$|z|^{k-p} < \frac{(k-2)! p(p - \mu) \{ \alpha + (k - j)\beta \}}{(k - j)! 2(\delta - \gamma)(k - \mu)} \tag{2.26}$$

for all $k \geq p + 1$, then

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p - \mu. \tag{2.27}$$

Therefore, we obtain

$$|z| < \inf_{k \geq p+1} \left(\frac{(k-2)! p(p - \mu) \{ \alpha + (k - j)\beta \}}{(k - j)! 2(\delta - \gamma)(k - \mu)} \right)^{1/(k-p)}. \tag{2.28}$$

□

Letting $p = 1$ and $j = 0$ in Theorem 2.5, we have the following.

COROLLARY 2.6. *If $f(z)$ is in class $\mathcal{A}_1(\alpha, \beta, \gamma; 0)$, then*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \mu \tag{2.29}$$

for

$$|z| < \inf_{k \geq 2} \left(\frac{(\alpha + k\beta)(1 - \mu)}{2k(k-1)(k - \mu)(\alpha + \beta - \gamma)} \right)^{1/(k-1)}, \tag{2.30}$$

where $0 \leq \mu < 1$ and $0 \leq \gamma < \alpha + \beta$.

3. Application of Jack's lemma

We give an application of Jack's lemma for class $\mathcal{A}_p(\alpha, \beta, \gamma; j)$. The next lemma was given by Jack [3].

LEMMA 3.1. Let $w(z)$ be analytic in \mathbb{U} with $w(0) = 0$. If there are $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)|, \tag{3.1}$$

then

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1). \tag{3.2}$$

Now we show the following.

THEOREM 3.2. If $f(z)$ belongs to class $\mathcal{A}_p(\alpha, \beta, \gamma; j+1)$ for $j = 0, 1, 2, \dots, p-1$, then

$$f(z) \in \mathcal{A}_p\left(\alpha - \beta, \beta, \frac{1+4\gamma}{4(p-j)}; j\right), \tag{3.3}$$

where $0 < \beta < \alpha$ and $0 \leq \gamma < p! \{\alpha + (p-j-1)\beta\} / (p-j-1)!$.

Proof. For $f(z) \in \mathcal{A}_p(\alpha, \beta, \gamma; j+1)$ and $A = 1/2 + (1+4\gamma)/4(p-j)$, we define $w(z)$ by

$$(\alpha - \beta) \frac{f^{(j)}(z)}{z^{p-j}} + \beta \frac{f^{(j+1)}(z)}{z^{p-j-1}} = \frac{w(z)}{1-w(z)} + A \quad (w(z) \neq 1). \tag{3.4}$$

Then, we have that

$$(\alpha - \beta) f^{(j)}(z) + \beta z f^{(j+1)}(z) = \frac{z^{p-j} w(z)}{1-w(z)} + Az^{p-j}. \tag{3.5}$$

It follows from (3.5) that

$$\alpha f^{(j+1)}(z) + \beta z f^{(j+2)}(z) = (p-j)Az^{p-j-1} + \frac{(p-j)z^{p-j-1}w(z)}{1-w(z)} + \frac{z^{p-j}w'(z)}{(1-w(z))^2}, \tag{3.6}$$

or

$$\alpha \frac{f^{(j+1)}(z)}{z^{p-j-1}} + \beta \frac{f^{(j+2)}(z)}{z^{p-j-2}} = (p-j)A + \frac{(p-j)w(z)}{1-w(z)} + \frac{zw'(z)}{(1-w(z))^2}. \tag{3.7}$$

Therefore, $f(z) \in \mathcal{A}_p(\alpha, \beta, \gamma; j+1)$ satisfies

$$\begin{aligned} & \operatorname{Re} \left\{ \alpha \frac{f^{(j+1)}(z)}{z^{p-j-1}} + \beta \frac{f^{(j+2)}(z)}{z^{p-j-2}} \right\} \\ &= (p-j)A + (p-j) \operatorname{Re} \left(\frac{w(z)}{1-w(z)} \right) + \operatorname{Re} \left\{ \frac{zw'(z)}{(1-w(z))^2} \right\} > \gamma \end{aligned} \tag{3.8}$$

for $z \in \mathbb{U}$. Since $w(z)$ is analytic in \mathbb{U} and $w(0) = 0$, if there are $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1, \tag{3.9}$$

then, we can write

$$w(z_0) = e^{i\theta}, \quad z_0 w'(z_0) = k w(z_0) = k e^{i\theta} \quad (k \geq 1) \tag{3.10}$$

by Lemma 3.1. For such a point $z_0 \in \mathbb{U}$, we obtain that

$$\begin{aligned} \operatorname{Re} \left\{ \alpha \frac{f^{(j+1)}(z_0)}{z_0^{p-j-1}} + \beta \frac{f^{(j+2)}(z_0)}{z_0^{p-j-2}} \right\} &= (p-j)A + (p-j) \operatorname{Re} \left(\frac{e^{i\theta}}{1-e^{i\theta}} \right) + \operatorname{Re} \left(\frac{k e^{i\theta}}{(1-e^{i\theta})^2} \right) \\ &= (p-j)A - \frac{p-j}{2} + \frac{k}{2(\cos\theta-1)} \\ &\leq (p-j)A - \frac{p-j}{2} + \frac{1}{2(\cos\theta-1)} \\ &\leq (p-j)A - \frac{p-j}{2} - \frac{1}{4} = \gamma, \end{aligned} \tag{3.11}$$

which contradicts our assumption. Hence there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ for all $z \in \mathbb{U}$. Noting that

$$\operatorname{Re} \left(\frac{w(z)}{1-w(z)} \right) > -\frac{1}{2} \quad (z \in \mathbb{U}) \tag{3.12}$$

for $|w(z)| < 1$, we have

$$\operatorname{Re} \left\{ (\alpha - \beta) \frac{f^{(j)}(z)}{z^{p-j}} + \beta \frac{f^{(j+1)}(z)}{z^{p-j-1}} \right\} > -\frac{1}{2} + \frac{1}{2} + \frac{1+4\gamma}{4(p-j)} = \frac{1+4\gamma}{4(p-j)}, \tag{3.13}$$

which shows that

$$f(z) \in \mathcal{A}_p \left(\alpha - \beta, \beta, \frac{1+4\gamma}{4(p-j)}; j \right). \tag{3.14}$$

□

Letting $p = 1$ and $j = 0$ in Theorem 3.2, we see the following.

COROLLARY 3.3. *If $f(z)$ belongs to class $\mathcal{A}_1(\alpha, \beta, \gamma; 1)$, then*

$$f(z) \in \mathcal{A}_1 \left(\alpha - \beta, \beta, \frac{1+4\gamma}{4}; 0 \right), \tag{3.15}$$

where $0 < \beta < \alpha$ and $0 \leq \gamma < \alpha$.

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