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State Dependent Riccati Equation based Nonlinear Controller Design for Ball and Beam System

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Abstract

The design of robust controller for nonlinear uncertain system has three important challenges, namely, nonlinearity, parameter uncertainty and external disturbance. Hence implementing a control algorithm for nonlinear unstable system is always a challenging task. In this paper, we propose a nonlinear controller using State Dependent Riccati Equation (SDRE) technique for a ball and beam system, which is inherently nonlinear and unstable system. Using the first principles,the equation of motion of the ball and beam system is obtained, and the nonlinear dynamics of the system is transformed into State Dependent Coefficient (SDC) matrices using Extended Linearization Technique. One of the key advantages of expressing the dynamics of system in SDC is that it fully captures the nonlinear characteristics of the system. Moreover, in SDRE control, since the weighing matrices are also expressed as a function of state variables, it provides better tracking response than that of its linear counterpart, LQR. Simulations are carried out to assess the trajectory tracking response, disturbance rejection property of controller and robustness of the system against parameter uncertainty. Simulation results prove that the SDRE based control cannot only provide improved transient response but also make the system more robust against disturbance and parameter uncertainty.

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Keywords: SDRE control, Ball and beam system, Optimal control, Weighting matrices, SDC matrices

1. Main text

The ball and beam system is a nonlinear, unstable and under actuated system, which has been widely used in the control laboratories to test the effectiveness of control algorithms. The underlying concept of ball and beam system can be applied to a stabilization problem for various systems such as horizontally stabilizing an airplane during landing and in turbulent airflow, and the balance problem dealing with goods to be carried by robots [1-2]. One interesting property of ball and beam system which motivated much research is that it is a non-regular system, whose relative degree is not well defined. Therefore, conventional exact feedback linearization techniques cannot be

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directly applied. Hauser et al [3] proposed an approximate feedback linearization technique by dropping certain terms which lead to the singularities, but this approach does not result in better performance when the system is away from singularities. Hence, in this paper we propose a nonlinear controller using modified SDRE techniquefor tracking control of ball and beam system.

State dependent Riccati Equation technique, also known as Frozen Riccati equation, is one of the nonlinear control methods proposed by Pearson. SDRE strategy has become very popular in the control community during the last two decades because of its wider non linear applications such as missile guidance and control system [4], autopilot design [5], under actuated robot [6], magnetic levitation system [7], continuously stirred tank reactor [8], and guidance law design [9]. From a computational viewpoint, SDRE control, based on nonlinear parameterization, offers a numerically efficient method that solves only an Algebraic Riccati equation (ARE), which is an appealing alternative to tedious tasks involved with solving two-point boundary value problems or Hamilton-Jacobi-Bellman partial differential equations associated with nonlinear optimal control problems. Moreover, unlike the linear controllers such as PD and LQR, the SDRE technique can change the gain matrix corresponding to states of the controller at each time step so that the actuator can generate proper control signal according to the current state values. An exhaustive survey on SDRE control has been reported in [9]. The goal of this paper is to assess the performance of SDRE based nonlinear controller for three cases namely tracking control, disturbance rejection and robustness against parameter uncertainty for ball and beam system.

The nonlinear dynamics of the ball and beam system obtained from first principles is given in section 2. The SDC matrices representation of the system obtained using extended linearization approach is given in section 3. Nonlinear controller design using SDRE technique is detailed in section 4. The simulation results of trajectory tracking, disturbance rejection and robustness of the controller against parameter uncertainty are illustrated in section 5, and the paper ends with the concluding remarks in section 6.

Nomenclature

x	Ball position	$A(x)$	System matrix
$\dot{x}$	Ball velocity	$B(x)$	Input matrix
θ_l	Servo angle	R_m	Motor armature resistance
$\dot{\theta}_l$	Servo velocity	L_m	Motor armature inductance
ζ	Damping ratio	B_{eq}	Equivalent damping coefficient
ω_n	Natural frequency of oscillation	J_m	Moment of inertia of motor
V_m	Motor voltage	J_l	Moment of inertia of load
K	State feedback gain	t_s	Settling time
Q(x), R(x)	Weighting matrices	α	Beam angle
J	Cost function	P(x)	State transformation matrix

2. System Modeling

The benchmark ball and beam set up consists of a horizontal beam which can pivot about one end; a servo motor whose shaft is attached to the other end of the beam and a ball which can freely roll on top of the beam. The schematic diagram of ball and beam system is illustrated in Fig. 1. The beam rotates in the vertical plane, driven by a torque at the center of rotation created by a dc motor. The ball moves freely along the beam and in contact with the beam. There are two degrees of freedom in ball and beam system. One is the ball moving up and down the beam, and the other one is the beam rotating itself through the connected axis. The control objective is to govern the position of the ball by applying a suitable voltage to the DC servo motor. By adjusting the angle of the beam through the servo motor, the ball can be maintained at the desired position. The position of the ball is determined by measuring the voltage at the steel rod, whereas the angle of the servo motor is measured by the position encoder. The schematic diagram of the ball and beam system is shown in Fig. 2.

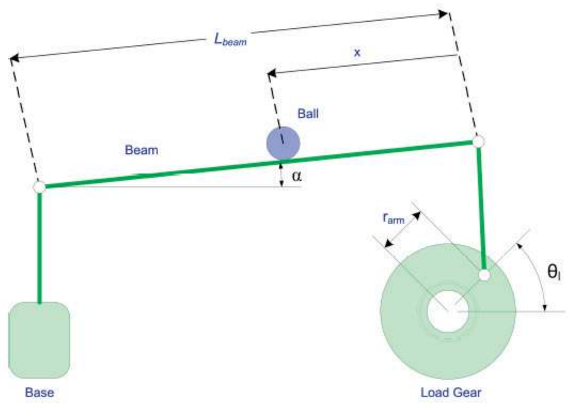

Fig. 1 Schematic diagram of ball and beam system

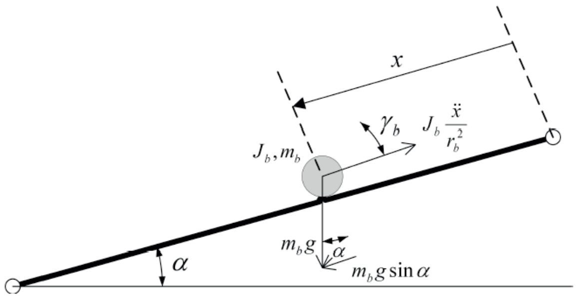

Fig. 2 Rolling ball free-body diagram

Table 1. Parameters of Ball and Beam system

Symbol	Description	Value/Unit
L_{beam}	Beam length	42.55 cm
r_{arm}	Link distance	2.54 cm
r_b	Radius of ball	1.27 cm
m_b	Mass of ball	0.064 kg
M_B	Mass of ball and beam module	0.65 kg
J_b	Moment of inertia of ball	4.129x10^{-6} kg/m^2
J_B	Moment of inertia of beam	0.0172 kg/m^2
g	Acceleration due to gravity	9.81 m/s^2
K_b	Back EMF constant	7.68 x10^{-3}V/rad/s
K_t	Torque constant	7.68 x10^{-3}Nm/A
η_g	Gearbox efficiency	0.90
η_m	Motor efficiency	0.69

The two forces which act on the system are translational force ($F_{x,t}$) acting along the x direction due to gravity and rotational force ($F_{x,r}$) produced by the acceleration of the ball. The translational force acting on the system is governed by the following expression:

$$1.1.\ F_{x,t} = mgsi \tag{1}$$

The force caused by the rotation of the ball is

$$F_{x,r} = \frac{\tau_b}{r_b} \tag{2}$$

where r_b is the radius of the ball and τ_b is the torque which equals

$$\tau_b = J_b\left(\frac{d^2}{dt^2}\gamma_b\right) \tag{3}$$

where γ_b is the beam angle. Using the sector formula $x(t) = \gamma_b r_b$, the angular displacement can be converted into linear displacement. Then, the force acting on the ball in the x direction from its momentum can be represented as

$$F_{x,r} = \frac{J_b\left(\frac{d^2}{dt^2}x(t)\right)}{r_b{}^2} \tag{4}$$

By applying Newton's second law of motion,

$$m_b\left(\frac{d^2}{dt^2}x(t)\right) = F_{x,t} - F_{x,r} \tag{5}$$

Substituting equations (1) and (4)into (5) results in

$$m_b\left(\frac{d^2}{dt^2}x(t)\right) = m_b g \sin\alpha(t) - \frac{J_b\left(\frac{d^2}{dt^2}x(t)\right)}{{r_b}^2} \tag{6}$$

Rearranging the equation (6) yields

$$\frac{d^2}{dt^2}x(t) = \frac{m_b g \sin\alpha(t) {r_b}^2}{(m_b {r_b}^2 + J_b)} \tag{7}$$

From Fig. 2, the beam angle can be written as

$$\sin\alpha(t) = \frac{h}{L_{beam}} \tag{8}$$

The servo angle can be represented as

$$\sin\theta_l(t) = \frac{h}{r_{arm}} \tag{9}$$

Relation between the servo and beam angles is given by

$$\sin\alpha(t) = \frac{\sin\theta_l(t) r_{arm}}{L_{beam}} \tag{10}$$

Substituting (10) into (7), the nonlinear equation of motion of the ball can be obtained as

$$\frac{d^2}{dt^2}x(t) = \frac{m_b g \sin\theta_l(t) r_{arm} {r_b}^2}{(m_b {r_b}^2 + J_b) L_{beam}} \tag{11}$$

Similarly, the dynamic equation of the DC servo can be represented as

$$\frac{d^2}{dt^2}\theta(t) = \left(-\frac{1}{\tau}\right)\frac{d}{dt}\theta(t) + \left(\frac{k}{\tau}\right)V_m(t) \tag{12}$$

where $\tau = \eta_g {K_g}^2 J_m + J_l$ and $k = \frac{(\eta_g K_g \eta_m K_t)}{(B_{eq} R_m + \eta_g K_g^2 \eta_m K_t k_m)}$

In order to obtain the state model of the system, the ball position, ball velocity, servo angle and servo velocity are chosen as state variables. Since the SDRE algorithm requires the dynamics of system to be represented in SDC matrices, in the following section SDC representation is obtained using extended linearization technique.

3. Extended Linearization

Extended linearization, also known as SDC parameterization, is the process of transforming a nonlinear system into linear like structure. By factorizing the nonlinear dynamics into the state vector and matrices which depend on the state itself, the SDC form is formulated. Consider the nonlinear state model of the system as

$$\dot{X} = f(x, u) \tag{13}$$

The SDC parameterization yields the following system in which both the system and input matrices are explicit function of current state variables.

$$\dot{X} = A(x)x + B(x)u \tag{14}$$

Several approaches have been reported in [10] to obtain an optimal parameterization from a number of suboptimal ones. By separating the inputs from states, one possible parameterization can be obtained. For this parameterization, an additional term $\frac{\sin\theta_l}{\theta_l}$ is included in the dynamic equation of ball and beam system such that the new function can be represented as

$$\left.\begin{aligned} sinc(\theta_l) &= \frac{\sin\theta_l}{\theta_l} \; if\,\theta_l \neq 0 \\ sinc(\theta_l) &= 1 \qquad f\theta_l = 0 \end{aligned}\right\} \tag{15}$$

This removes the singularity when θ_l is zero. Thus, thenonlinear equations of the ball and beam system such as (11) and (12) can be rewritten in the form of SDC as

$$\ddot{x}(t) = \frac{m_b g r_{arm} r_b{}^2 \theta_l(t)}{(m_b r_b{}^2 + J_b) L_{beam}} \left(\frac{\sin\theta_l(t)}{\theta_l(t)}\right) \tag{16}$$

$$\ddot{\theta}(t) = \left(-\frac{1}{\tau}\right)\frac{d}{dt}\theta(t) + \left(\frac{k}{\tau}\right) V_m(t) \tag{17}$$

The controllability matrix pair of the parameterized model is given below.

$$A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m_b g r_{arm} r_b{}^2}{(m_b r_b{}^2 + J_b) L_{beam}}\left(\frac{\sin\theta_l(t)}{\theta_l(t)}\right) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{-1}{\tau} \end{bmatrix} B(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{\tau} \end{bmatrix} \tag{18}$$

4. SDRE Controller

SDRE technique is the extension of LQR design to nonlinear systems and it provides a suboptimal solution for the optimal control problem with a linear quadratic cost function. The design involves transforming the system dynamic equations into a pseudo-linear or state dependent coefficient (SDC) form, in which system matrices are represented explicitly as a function of state variables [11]. Assuming the system matrices as constant from the current time step, the optimal state feedback gain is determined for the parameterized system, and this procedure is repeated at each time step.Unlike LQR, in SDRE technique the system matrices A(x),B(x) and the penalty matrices Q(x) and R(x) areexpressed as functions of the states, which enables the design to capture the nonlinearity of the system at each time interval. The general SDRE problem is the infinite-horizon regulation of autonomous non linear input-affine systems. Consider the system equations are

$$\dot{x} = f(x) + g(x)u \tag{19}$$

where $f(x_0) = 0$. The idea behind the method is to extend the applicability of the Algebraic Riccati equation (ARE) for the control design of linear systems to a class of nonlinear systems that can be expressed in a state-dependent linear form as

$$\dot{x} = A(x)x + B(x)u \tag{20}$$

$$y = C(x)x \tag{21}$$

where $f(x) = A(x)x$ (the choice of the matrix A(x) is not unique) and g(x)=B(x). The former parameterization is possible if and only if f(0)=0 and f (x) is continuously differentiable. The goal is to find a state feedback control law that minimizes a cost function given by

$$J(u) = \frac{1}{2}\int_0^\infty [x^T Q(x)x + u^T R(x)u]\, dt \tag{22}$$

where $Q \in \mathcal{R}^{n\times n}$a symmetric positive semi-definite matrix and R is$\in \mathcal{R}^{m\times m}$ is a symmetric positive definite matrix. Moreover,$x^T Q(x)x$ is a measure of control accuracy and $u^T R(x)u$ is a measure of control effort. It should be noted that the SDRE formulation allows one to tradeoff between the control accuracy and control effort, which is a property not generally found in other nonlinear control design methods. To minimize the above cost function, a state feedback control law can be given as

$$u(x) = -K(x)x = -R^{-1}(x)B^T(x)P(x) \tag{23}$$

where P(x) is the unique, symmetric and positive-definite solution of the State-Dependent Riccati equation. The optimal state feedback control gain matrix K of LQR can be determined by solving the following Algebraic Riccati Equation (ARE).

$$A^T(x)P(x) + P(x)A(x) + Q(x) - P(x)B(x)R^{-1}(x)B^T(x)P(x) = 0 \tag{24}$$

This method requires that the full state measurement vector be available and that the pair $(A(x), B($ be pointwise controllable in the linear sense x. This can be checked by forming the controllability matrix as in the linear systems sense and making sure that it has full rank in the domain of interest .This condition simply ensures that the Algebraic Riccati Equation has a solution at the particular state x. The point wise controllability condition is not necessarily equivalent to nonlinear controllability. Due to the non-uniqueness of A(x), different A(x) choices may yield different controllability matrices and thus different controllability characteristics.

5. Results and Discussion

One of the key advantages offered by SDRE is the tradeoff between control effort and state errors, and it can be achieved through tuning of weighting matrices Q(x) and R(x). In addition, these weighting matrices can be chosen to be either constant or function of state variables so as to obtain the desired response. Hence, different modes of behaviour can be imposed in different regions of state space. Global minimization of cost functions relies on the convexity of the Hamiltonian function with respect to both state vector and control input. So, it is desirable to choose *Q(x)* and *R(x)* according to the following proposition.Assume the scalar function $l(x) = X^T Q(x) X$. Let $Q(x) = Q_0 + Q_1(x)$, where $Q_0 = diag(c_{10}, \ldots, c_{n0})$ is any constant symmetric positive definite matrix with $c_{10} > 0$, i=1,2, … , n and $Q_1(x) = diag\{q_1(x_1), \ldots, q_n(x_n)\}$ is such that each q_i takes the following form.

$$q_i(x_i) = c_{i2}x_i^2 + c_{i4}x_i^4 + \cdots + c_{is_i}x_i^{s_i} \quad (25)$$

With $c_{ij} \geq 0$, j=2,4,…s_i. Then the scalar function $l(x)$ is globally convex with respect to state vector x.According to the above proposition, in the proposed modified SDRE (MSDRE), the weighting matrices are chosen as function of state variables, and the performance of the MSDRE are compared with that of the standard LQR and SDRE whose weighting matrices are assumed to be constant. The tracking response of the LQR controller and SDRE controllers whose weighting matrices are chosen to be constant are illustrated in Fig. 3 and 4 respectively. Fig. 5 shows the response of the MSDRE whose weighting matrices are chosen as function of state variables. It can be observed from Table 2, which gives the Q and R matrices of the three controllers along with the corresponding time domain analysis and the tracking error, that the tracking error and overshoot of the MSDRE controller is the least of the three controllers. Reduced tracking error accentuates that if the weighting matrices of SDRE are selected as function of state variables, it will significantly improve the performance of the system because the variations in the state vector is simultaneously updated during the controller gain calculation in each time instant.

Table 2. Weighting matrices and transient performance of LQR, SDRE and MSDRE

Controller	Weighting matrices	Position		Tracking error	
		Settling time (sec)	Over shoot (%)	IAE	ISE
LQR	$Q = diag(1 \quad 1 \quad 1 \quad 1)$ R=1	7.73	8.75	1.06	0.2592
SDRE	$Q = diag(1 + x_1 \quad 1 \quad 1 + x_3 \quad 1)$ R=$1 - x_1$	8.61	3.3	0.519	0.0598
MSDRE	$Q = diag(0.890 + x_1^2 \quad 1 \quad 1 + x_3^2 \quad 1)$ $R = 1 - x_1$	10.6	3.11	0.487	0.0546

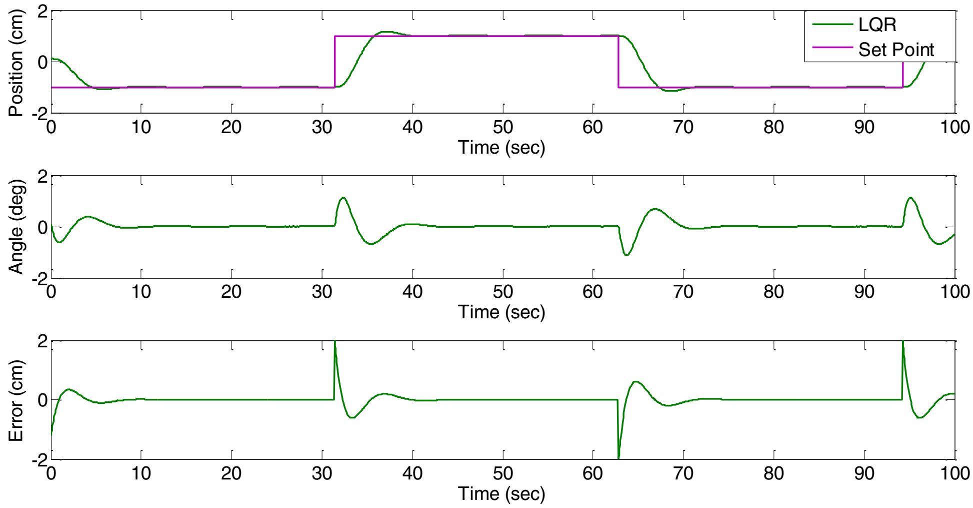

Fig. 3 Response of LQR controller

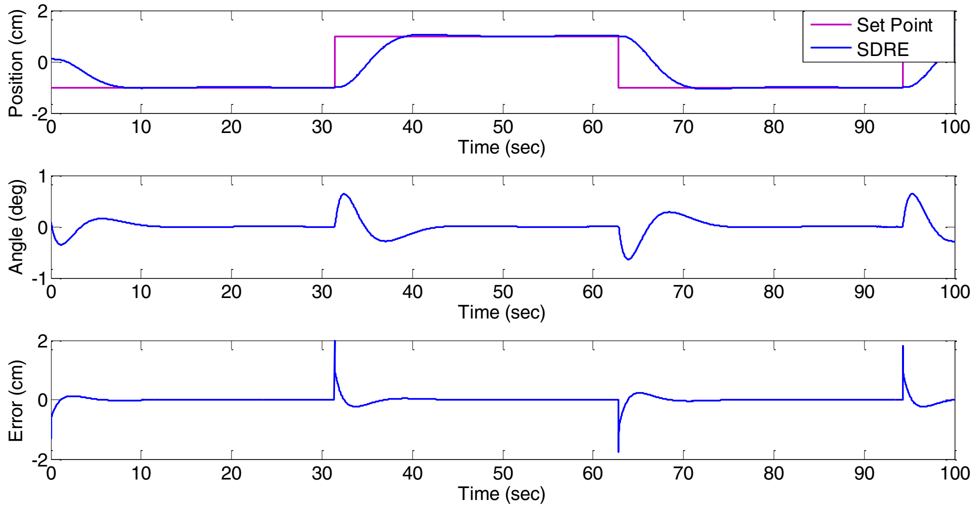

Fig. 4 Response of SDRE controller with constant weighting matrices

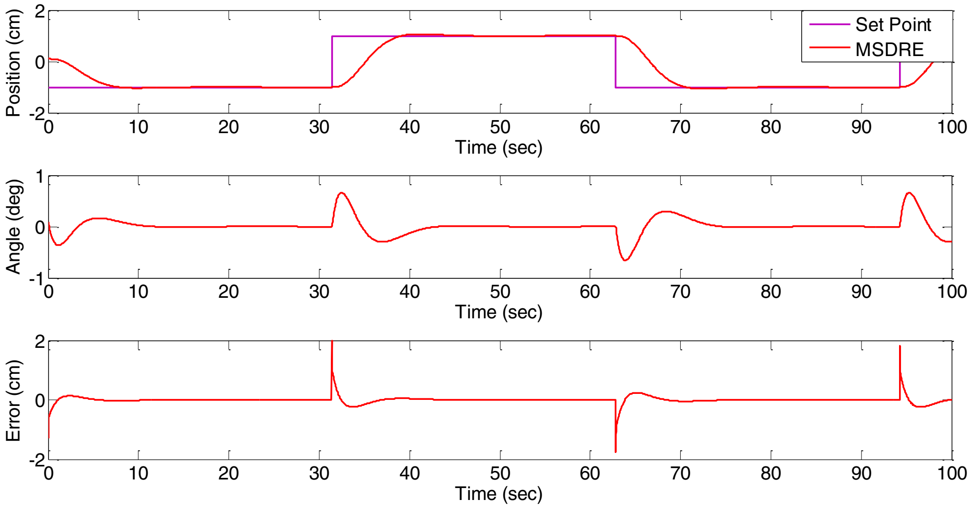

Fig. 5 Response of MSDRE with weighting matrices function of state variables

5.1Disturbance Rejection

To assess the robustness of the controller against disturbance, a harmonic signal with a frequency of f=0.2 Hz is introduced into the system at time t=30sec, and the performances of the three controllers are evaluated using the error performance indices such as IAE, ISE, ITAE, ITSE and RMSE.Fig. 6 shows the disturbance rejection performance of three controllers and the corresponding error performance indices of the LQR, SDRE and MSDRE controllers are given in Table 3. It can be noted that the MSDRE controller takes less time than that of the other two controllers to arrest the deviation from reference trajectory. Moreover, the peak to peak change during disturbance is the least of all three in MSDRE. To highlight the effect of disturbance and the ability of the controllers to arrest the disturbance in short time the, the zoomed view of the ball position response is shown in Fig. 7.

Table 3. Error performance indices during external disturbance

Controller	RMSE	IAE	ISE	ITAE	ITSE	Change In Set point (Peak to Peak)
LQR	$4.641e^{-13}$	4.034	1.524	105.9	43.39	1.04
SDRE	$4.495e^{-14}$	1.243	0.1629	25.92	3.469	0.111
MSDRE	$2.2205e^{-16}$	1.213	0.1561	25.56	3.411	0.099

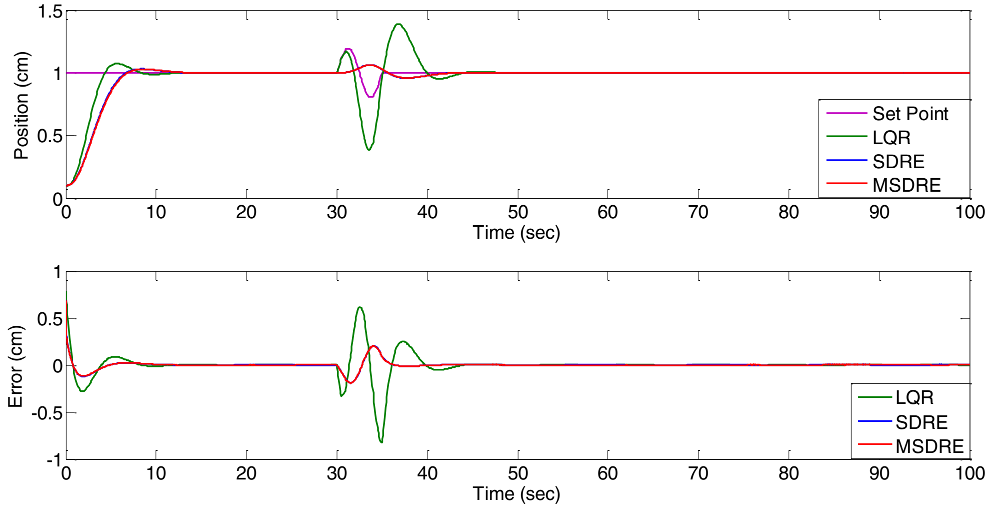

Fig. 6 Response of LQR, SDRE and MSDRE during external disturbance

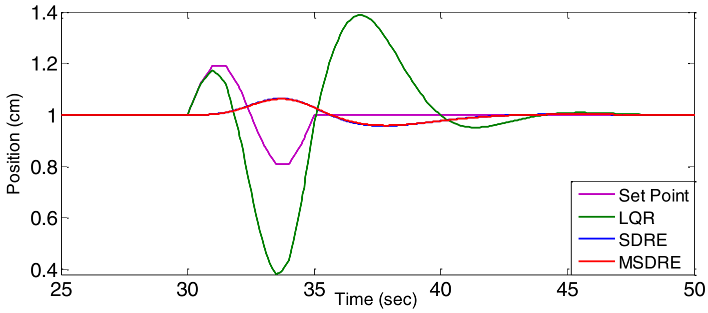

Fig. 7 Zoomed view of position during external disturbance

5.2 Parameter Uncertainty

To evaluate the robustness of the controller against parameter uncertainty, three levels of uncertainty such as 10%, 20% and 30% are considered, and the tracking performance of the controller is plotted and shown in Fig. 8. Table 4 gives the error performance indices of MSDRE for the three levels of parameter uncertainties. It has been found that the controller is able to provide satisfactory response until 30% change in parameter uncertainty, which proves that the proposed controller cannot only provide desired transient performance but it can also significantly improve the robustness of the controller against parameter uncertainty.

Table 4. Error performance indices of MSDRE during parameter uncertainty

Parameter Uncertainty	IAE	ISE	ITAE	ITSE	RMSE
+10%	4.401	1.552	232.5	86.82	0.0548
+20%	4.441	1.624	235.1	90.89	0.0731
+30%	4.492	1.698	238.8	94.64	0.0816

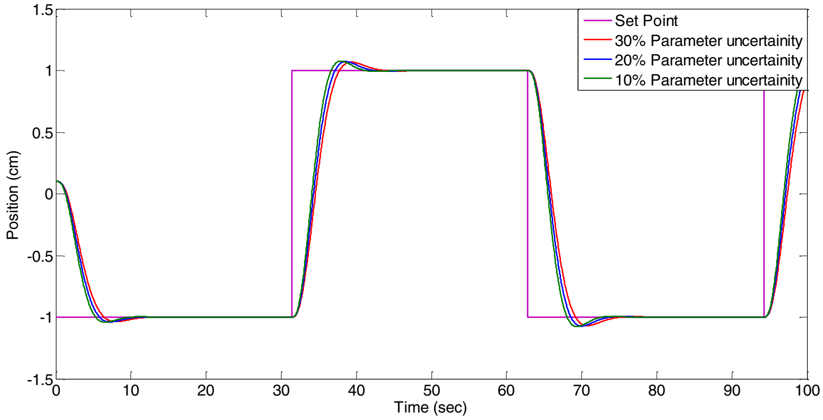

Fig. 8 Response of MSDRE for parameter uncertainty

6. Conclusion

A nonlinear controller, based on modified State Dependent Riccati Equation algorithm, has been proposed for trajectory tracking of ball and beam system. The SDRE controller is the extended form of LQR for nonlinear systems, and it can significantly improve the robustness of the system against disturbance and parameter uncertainty by capturing the nonlinear dynamics of the plant in state dependent coefficient form. The nonlinear equation of motion of ball has been obtained from the first principles and transformed into State dependent coefficient matrices using extended linearization approach. The performance of the controller is validated for three cases, namely, trajectory tracking, disturbance rejection and robustness against parameter uncertainty. The results of modified SDRE have been compared with those of the LQR and standard SDRE. The reduced tracking error and increased robustness of the system for MSDRE suggest that it can significantly improve both the trajectory tracking behaviour and robustness of the system against disturbances and parameter variation.

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