Journal of Mathematical Physics

Strongly asymmetric discrete Painlevé equations: The multiplicative case

Cite as: J. Math. Phys. **57**, 043506 (2016); https://doi.org/10.1063/1.4947061
Submitted: 15 October 2015 . Accepted: 05 April 2016 . Published Online: 21 April 2016

B. Grammaticos, A. Ramani, K. M. Tamizhmani, T. Tamizhmani, and J. Satsuma

JOURNAL OF MATHEMATICAL PHYSICS **57**, 043506 (2016)

Strongly asymmetric discrete Painlevé equations: The multiplicative case

B. Grammaticos,[1] A. Ramani,[2] K. M. Tamizhmani,[3] T. Tamizhmani,[4] and J. Satsuma[5]
[1]*IMNC, Université Paris VII & XI, CNRS, UMR 8165, Bât. 440, 91406 Orsay, France*
[2]*Centre de Physique Théorique, Ecole Polytechnique, CNRS, Université Paris-Saclay, F-91128 Palaiseau, France*
[3]*Department of Mathematics, Pondicherry University, Kalapet, 605014 Puducherry, India*
[4]*K.M. Centre for Postgraduate Studies, 605008 Puducherry, India*
[5]*Department of Mathematical Engineering, Musashino University, 3-3-3 Ariake, Koto-ku, 135-8181 Tokyo, Japan*

(Received 15 October 2015; accepted 5 April 2016; published online 21 April 2016)

We examine a class of multiplicative discrete Painlevé equations which may possess a strongly asymmetric form. When the latter occurs, the equation is written as a system of two equations the right hand sides of which have different functional forms. The present investigation focuses upon two canonical families of the Quispel-Roberts-Thompson classification which contain equations associated with the affine Weyl groups $D_5^{(1)}$ and $E_6^{(1)}$ (or groups appearing lower in the degeneration cascade of these two). Many new discrete Painlevé equations with strongly asymmetric forms are obtained. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4947061]

I. INTRODUCTION

Discrete Painlevé equations have been derived through a variety of methods.[1] However, the main bulk of discrete Painlevé equations known to date were obtained by the method known as deautonomisation.[2] The procedure is simple (in principle). One starts from a given autonomous integrable mapping which contains free parameters. Usually, for this starting point, a QRT[3] mapping is chosen. Next, one assumes that the coefficients of the mapping are functions of the independent variable and uses an integrability criterion in order to fix the form of these functions. The two integrability criteria largely used, and which will guide the derivations presented in this paper, are singularity confinement[4] and algebraic entropy.[5] The way these two criteria are used for deautonomisation is the following. One starts from the autonomous case and obtains all the singularity patterns or, in the case of algebraic entropy, the degree growth of some initial condition expressed in homogeneous coordinates. Then one introduces non-autonomous coefficients and requires that the singularity pattern and/or the degree growth be identical to the ones obtained before deautonomisation. This introduces constraints on the coefficients which, in principle, allow to obtain the precise dependence of the latter on the independent variable.

The method sketched in the previous paragraph has been extensively used for the derivation of discrete Painlevé equations. However, the studies in question have been systematically ignoring a subclass of equations. In order to elucidate this omission we must go back to the derivation method. As explained, the customary starting point for the derivation of discrete Painlevé equations is the QRT mapping, which, as is well known, exists in two variants, the symmetric

$$x_{m+1} = \frac{f_1(x_m) - x_{m-1}f_2(x_m)}{f_2(x_m) - x_{m-1}f_3(x_m)} \tag{1}$$

and the asymmetric

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)}, \tag{2a}$$

$$y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})} \tag{2b}$$

ones. In most derivations by deautonomisation the starting point has been a *symmetric* form like (1). The rationale behind this choice is that (1) does encompass (2) if one allows the coefficients to become periodic functions of the independent variable with a period 2. (Higher periodicities may, and in fact do, exist). However a pitfall is present here. In most derivations, be it with singularity confinement or algebraic entropy, one implicitly assumes that all coefficients are always non-zero (the opposite assumption would make the analysis particularly awkward). Thus one misses cases where in an asymmetric QRT mapping the right hand sides of the asymmetric equations do not have the same functional form. These are the equations we have dubbed "strongly asymmetric," in contrast to the weakly asymmetric ones where the right hand sides have the same functional form and only the values of the coefficients may differ. Let us give an example of weakly and strongly asymmetric discrete Painlevé equations,

$$x_n + x_{n+1} = \frac{z_{n+1/2}y_n + a}{y_n^2 - c^2}, \tag{3a}$$

$$y_n + y_{n-1} = \frac{z_n x + b}{x_n^2 - d^2}, \tag{3b}$$

where $z_n = pn + q$ is a discrete form of Painlevé III, as shown in Ref. 6. This is a weakly asymmetric form. However the one-parameter discrete Painlevé III,[7]

$$x_n + x_{n+1} = \frac{z_{n+1/2}}{y_n} + \frac{a}{y_n^2}, \tag{4a}$$

$$y_n + y_{n-1} = \frac{z_n x + b}{x_n^2 - d^2}, \tag{4b}$$

is a strongly asymmetric one, since the right hand sides of (4a) and (4b) do not have the same form. Equation (4) cannot be obtained by a casual deautonomisation of a symmetric mapping.

In Ref. 8 we presented results on strongly asymmetric discrete Painlevé which are difference equations, i.e., equations where the dependence of the coefficients on the independent variable n is linear. We shall refer to such a dependence as "secular," in contrast to any dependence which may enter through a periodic function. Here we shall extend these results to the multiplicative case, i.e., to equations where the independent variable enters through an exponential and thus the moniker secular here alludes to a linear dependence of the logarithm of the coefficients on n. Two families, corresponding to two different QRT canonical forms, will be studied and all the associated discrete Painlevé equations will be derived.

Our main claim is that strongly asymmetric discrete Painlevé equations not only do exist but are quite frequent. Another important finding, materialised through several examples, is the fact that the existence of terms which appear in powers higher than one in the right hand side of the equation may lead, after deautonomisation, to results different from the ones obtained when all terms enter linearly. In particular it is the deautonomisation of cases with the highest power that produces discrete Painlevé equations with coefficients of maximal periodicities. Finally, this extensive study constitutes an excellent testing ground for the comparison of the predictions of the two discrete integrability criteria: singularity confinement and algebraic entropy. It turns out that the agreement of the two methods is perfect throughout the present work.

II. DERIVATION OF THE ASYMMETRIC DISCRETE PAINLEVÉ EQUATIONS

In order to derive strongly asymmetric forms for discrete Painlevé equations we will deautonomise a selection of QRT mappings, with the help of some discrete integrability criterion. We shall work within a given family of QRT mappings, based on our classification[9] of canonical forms. In this paper we shall consider only multiplicative, or q, cases and study two families related to A_1 QRT matrices of the form

(II)
$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

(IV)
$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

the corresponding forms of the mappings being

(II)
$$x_{n+1}x_{n-1} = F(x_n)$$

and

(IV)
$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = F(x_n)$$

where $F(x_n)$ is a rational expression of x_n. All possible limiting cases of the mappings of the form (II) and (IV) will be considered but also the degenerate forms, obtained by a simplification of $F(x)$.

In Ref. 2 we have presented the derivation of the symmetric discrete Painlevé equations of these two families. This concerned mainly purely symmetric or weakly asymmetric cases but no strongly asymmetric equations were derived. In Secs. III and IV we shall analyse each limit and degenerate subcase and strive to identify all strongly asymmetric cases. Our starting point will be the general A_0 QRT matrix of the form

$$A_0 = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \\ \kappa & \lambda & \mu \end{pmatrix}.$$

Since our procedure is systematic, all symmetric and weakly asymmetric cases which exist will also be identified.

Given that all the equations in this paper are of multiplicative type, we shall express the dependence of the various parameters on the independent variable in terms of logarithms, in order to improve readability. Moreover, given that the various coefficients have a periodic dependence on the independent variable we shall introduce the periodic function $\phi_k(n)$ expressed in terms of the appropriate roots of unity as ϕ_k as $\phi_k(n) = \sum_{m=1}^{k-1} r_m e^{2nim\pi/k}$, which satisfies the relation $\phi_k(n) = \phi_k(n+k)$. Notice that since the sum starts at 1 the constant component is absent. In what follows, instead of using $\phi_2(n)$ we shall, whenever necessary, introduce simply a term proportional to $(-1)^n$.

Before proceeding to the derivation of the various discrete Painlevé equations a remark is in order. Typically the right hand side of a mapping which constitutes our starting point for deautonomisation is given as a ratio of two polynomials, conveniently written as a product of factors. Clearly the cases obtained by all possible simplifications must be examined separately since they lead, in principle, to different deautonomisations. (In Ref. 2 we dubbed this process of simplification cum deautonomisation "degeneracy," a term admittedly not optimally chosen). The generic case corresponds to all factors of both the numerator and of the denominator being different. However it may turn out that a factor may appear in a power two, three, or even four. This does play an important role when one considers simplifications. Simplifying by a factor which appears in a square (or higher power) is not the same as simplifying by a factor that enters linearly. The best way to see this is to use algebraic entropy techniques and compute the homogeneous degree growth. When the simplification involves a factor appearing in a square, or higher power, one obtains a growth slower than in the case of a simplification by a factor entering linearly. Thus these various cases must be studied separately.

Since the structure of the paper is rather complicated we present here a guide to the classification we are using. Sections III and IV are devoted to the two families (II) and (IV).

Within each family we distinguish the major cases using capital letters: A to E for family (II) and A to F for family (IV). The various letters correspond to the number and position of zeros in the relevant (in as much as they put to zero a coefficient of the equation) corners of the matrix A_0, perhaps up to a trivial change of variables.

Arabic numerals following the capital letter are related to the fact that some more coefficients of the equation have been put to zero. These zeros acquire an importance *only* after some corner of A_0 is chosen to be zero. The numeral 1 is reserved to the case where nothing is put to zero. Thus case A, with all corners of A_0 being nonzero, has the unique subcase A1. The remaining cases B to F have several subcases which, for the E case of family (IV) span the range E1 to E8.

A lower case letter following the arabic numeral is associated to the simplifications in the right hand side of the equation. The letter a is reserved to the case where no simplification occurs. In the case C1 of family (IV) we have several subcases labelled by a lower case letter: from C1a to C1f.

A lower case roman numeral indicates the existence of factors that appear in a power higher than one before simplification, *provided this power is reduced by the simplification*. Thus a roman numeral may never appear after a lower case a since this letter is reserved to the case without simplifications. For family (IV) we have lower case roman numerals in subcase A1c covering the range A1ci to A1cx.

Finally, another symbol, a "prime," is being used in order to distinguish subcases. We use a prime whenever the vanishing of a coefficient of the A_0 matrix leads to a ratio of coefficients in the equation becoming exactly 1. Typically we have just one prime except for the case E1 of family (IV) which has two primes.

III. THE DISCRETE EQUATIONS ASSOCIATED TO FAMILY (II)

The general form of the family (II) asymmetric QRT mapping is

$$x_{n+1}x_n = \frac{\kappa y_n^2 + \lambda y_n + \mu}{\alpha y_n^2 + \beta y_n + \gamma}, \tag{5a}$$

$$y_n y_{n-1} = \frac{\gamma x_n^2 + \zeta x_n + \mu}{\alpha x_n^2 + \delta x_n + \kappa}. \tag{5b}$$

In order to explore the possible branches of limiting and degenerate forms of (5) we start by classifying the cases according to the number of corners of the A_0 QRT matrix which are put to 0.

A. Case A

The general case (A1) of the mapping (5) is obtained for $\alpha\kappa\gamma\mu \neq 0$. Before proceeding we introduce a more convenient autonomous form of (5),

$$x_{n+1}x_n = hk\frac{(y_n - a)(y_n - b)}{(y_n - c)(y_n - d)}, \tag{6a}$$

$$y_n y_{n-1} = cd\frac{(x_n - f)(x_n - g)}{(x_n - h)(x_n - k)}, \tag{6b}$$

where $abhk = cdfg$. Case (A1a) corresponds to the absence of any simplification in (5). By deautonomising we obtain, after the proper gauge choice (where c, d, h, k are constant with $cd = 1$, $hk = 1$), the asymmetric q-Painlevé III,

$$x_{n+1}x_n = \frac{(y_n - a_n)(y_n - b_n)}{(y_n - 1/c)(y_n - c)}, \tag{7a}$$

$$y_n y_{n-1} = \frac{(x_n - f_n)(x_n - g_n)}{(x_n - 1/h)(x_n - h)}, \tag{7b}$$

where $\log a_n = 2pn + q + r$, $\log b_n = 2pn + q - r$ and $\log f_n = p(2n - 1) + q + s$ and $\log g_n = p(2n - 1) + q - s$. This equation was first obtained in Ref. 10. Jimbo and Sakai have shown in Ref. 11 that it is indeed a discrete form of the continuous Painlevé VI. The geometry of its transformations[12] is described by the affine Weyl group $D_5^{(1)}$. In fact all subcases of case (A1) below, obtained by successive simplifications of the right hand side of (5) and deautonomisation, will be associated to the same Weyl group $D_5^{(1)}$.

Case (A1b) corresponds to one simplification occurring in the right hand side of (6a), $b = d$ so we have $ahk = cfg$. Assuming that there are no squares, case (A1bi), using the gauge h, k constant with $hk = 1$ and $c = 1$, we can deautonomise this equation to

$$x_{n+1}x_n = \frac{y_n - a_n}{y_n - 1}, \tag{8a}$$

$$y_n y_{n-1} = d_n \frac{(x_n - f_n)(x_n - g_n)}{(x_n - h)(x_n - 1/h)}, \tag{8b}$$

where $\log d_n = pn + q + r(-1)^n$, $\log f_n = pn + s - r(-1)^n$, $\log g_n = pn + u - r(-1)^n$, and $\log a_n = p(2n + 1) + s + u$.

Next we examine the case where a square is present before simplification, case (A1bii). The square can be present either in the numerator, $a = b$, or the denominator, $c = d$. However these two cases are dual upon inversion of x (and rescaling) and it suffices to deal with one of them. Taking $c = d$ and choosing the gauge so as to have $c = 1$ leads to $d = 1$ and thus the first equation is (8a) with a_n given by $\log a_n = p(2n + 1) + q + r - \phi_3(n + 2)$ while the second equation is (8b) with $d_n \equiv 1$, $\log f_n = pn + q + \phi_3(n)$, $\log g_n = pn + r + \phi_3(n)$, where h is again a constant.

Case (A1c) corresponds to one simplification occurring in each of the right hand sides of (7), $b = d$ and $g = k$, where we can take $c = h = 1$ by the appropriate gauge,

$$x_{n+1}x_n = k_n \frac{y_n - a_n}{y_n - 1}, \tag{9a}$$

$$y_n y_{n-1} = d_n \frac{x_n - f_n}{x_n - 1}. \tag{9b}$$

In the absence of squares, case (A1ci), we find, after deautonomisation, $\log a_n = 2pn + q + \phi_3(n)$, $\log f_n = p(2n - 1) + q + \phi_3(n + 1)$, $\log k_n = pn + r - \phi_3(n)$, and $\log d_n = pn + s - \phi_3(n + 1)$. We remark that this form is a weakly asymmetric one.

Next we examine the case when a square is present in one of the equations, (A1cii). We choose arbitrarily the case of a square in the denominator of the first equation, the case of a square in the numerator being just its dual. Starting with $c = d$ we find readily, after simplification and with the appropriate gauge choice $c = d = 1$, so $d_n \equiv 1$ in (9b). After deautonomisation we find $\log a_n = 2pn + q + \phi_4(n + 1) + \phi_4(n - 1)$, $\log f_n = p(2n - 1) + q + \phi_4(n) + \phi_4(n - 1)$, and $\log k_n = pn + r + \phi_4(n)$.

When two squares are present (one in each of the equations of the system) we have, case (A1ciii), $d_n = k_n = 1$ and we obtain $\log a_n = 2pn + q + \phi_5(n)$ and $\log f_n = p(2n - 1) + q + \phi_5(n + 2)$, an equation of weakly asymmetric form already identified in Ref. 13.

B. Case B

We take now $\alpha = 0$ and assuming that $\kappa\gamma\mu \neq 0$ we can take $\gamma = \kappa = \mu = 1$. Case (B1) corresponds to $\beta\delta \neq 0$,

$$x_{n+1}x_n = \frac{y_n^2 + \lambda y_n + 1}{\beta y_n + 1}, \tag{10a}$$

$$y_n y_{n-1} = \frac{x_n^2 + \zeta x_n + 1}{\delta x_n + 1}, \tag{10b}$$

which can be more conveniently written as

$$x_{n+1}x_n = \frac{(1 - a y_n)(1 - y_n/a)}{f y_n + 1}, \tag{11a}$$

$$y_n y_{n-1} = \frac{(1 - c x_n)(1 - x_n/c)}{g x_n + 1}. \tag{11b}$$

When there is no simplification in the right hand side, case (B1a), we can deautonomise (10) obtaining a and c constants, $\log f_n = pn + q$ and $\log g_n = pn + r$. This equation was first identified in Ref. 13 as a q-discrete form of Painlevé V, albeit in a different gauge. The geometry of its transformations is described by the affine Weyl group $A_4^{(1)}$ (and the same applies to cases (B1b) and (B1c) below). If we assume, case (B1b), that one simplification is possible by taking, say, $f = -1/a$, we find

$$x_{n+1}x_n = 1 - ay_n, \tag{12a}$$

$$y_n y_{n-1} = \frac{(1-cx_n)(1-x_n/c)}{gx_n+1}. \tag{12b}$$

While (12) looks strongly asymmetric it is possible to solve (12a) for y and obtain a single equation for x. When there is no square its non-autonomous form (B1bi) is

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = b_n \frac{(x_n - c)(x_n - 1/c)}{1 + g_n x_n}, \tag{13}$$

where c is a constant, $\log g_n = pn + q + r(-1)^n$ and $\log b_n = -pn + s + r(-1)^n$ (and b_n is related to a_n through $b_n = a_n a_{n-1}$). Equation (13) was studied in Ref. 14 where it was shown that it is a q-discrete form of Painlevé IV.

However, when $a^2 = 1$ we have a square in the numerator of (11a) and the equation may be written in the form of (12a) with $a = \pm 1$. Its deautonomisation, case (B1bii), leads to the same form as (13) but now with $b_n \equiv a_n a_{n-1} = 1$ and $\log g_n = pn + q + \phi_3(n)$ while c is always a constant.

Case (B1c) corresponds to the situation where both right hand sides of (10) can be simplified, i.e., $f = -1/a$ and $g = -1/c$. When there is no square in the numerators we obtain the system

$$x_{n+1}x_n = 1 - ay_n, \tag{14a}$$

$$y_n y_{n-1} = 1 - cx_n. \tag{14b}$$

The deautonomisation of (14) leads to, case (B1ci), $\log a_n = 2pn + q + \phi_3(n)$ and $\log c_n = p(2n - 1) + r + \phi_3(n-2)$. This equation was first obtained in Ref. 13. However another interesting situation exists when a square is present in one of the numerators before simplification, so $a = b = 1$. In this case, with a square present in the first numerator, the system has the form

$$x_{n+1}x_n = 1 - y_n, \tag{15a}$$

$$y_n y_{n-1} = 1 - cx_n. \tag{15b}$$

Eliminating x we obtain for y an equation which can be deautonomised to, case (B1cii),

$$(y_{n+1}y_n - 1)(y_n y_{n-1} - 1) = b_n(1 - y_n), \tag{16}$$

where $\log b_n = pn + q + \phi_4(n)$. Finally when we have squares in both numerators, i.e., $a = c = 1$, we obtain the mapping $x_{n+1}x_n x_{n-1} - x_{n+1} - x_{n-1} + 1 = 0$ the solutions of which are periodic with period 5.

Next we consider the case where $\delta = 0$ and $\beta \neq 0$, case (B2). The deautonomisation of this case leads to the strongly asymmetric Painlevé equation, case (B2a),

$$x_{n+1}x_n = \frac{(y_n - a)(y_n - 1/a)}{c_n y_n + 1}, \tag{17a}$$

$$y_n y_{n-1} = (x_n - b)(x_n - 1/b), \tag{17b}$$

where a, b are constant and $\log c_n = pn + q$. The geometry of Equation (17) is described by the affine Weyl group $A_2^{(1)} + A_1^{(1)}$. Case (B2b) corresponds to one simplification in the right hand side of the first equation. When no square is present in the numerator of (17a) we find, case (B2bi),

$$x_{n+1}x_n = 1 - ay_n, \tag{18a}$$

$$y_n y_{n-1} = (x_n - b)(x_n - 1/b). \tag{18b}$$

We can now solve (18) for y and obtain for x the weakly asymmetric Painlevé equation

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = k_n(x_n - b)(x_n - 1/b), \tag{19}$$

where $\log k_n = pn + q + r(-1)^n$, related to the same affine Weyl group as (17). When a square is present, case (B2bii), we obtain

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = (x_n - b_n)(x_n - 1/b_n). \tag{20}$$

We find that $\log b_n = q + \phi_4(n)$ and no extension including secular terms is possible. When $\beta = \delta = 0$ we obtain a mapping

$$x_{n+1}x_n = y_n^2 + \lambda y_n + 1, \tag{21a}$$

$$y_n y_{n-1} = x_n^2 + \zeta x_n + 1, \tag{21b}$$

which cannot be deautonomised and which moreover is linearisable.

C. Case C

We could now take $\alpha = \mu = 0$ and assume that $\kappa\gamma \neq 0$. But we can just as well choose $\gamma = \kappa = 0$ and $\alpha\mu \neq 0$ with $\alpha = 1$ without loss of generality and, though the gauge $\mu = 1$ is possible, it is not convenient. So we forgo this gauge and obtain a more familiar form,

$$x_{n+1}x_n = \frac{\lambda y_n + \mu}{y_n(y_n + \beta)}, \tag{22a}$$

$$y_n y_{n-1} = \frac{\zeta x_n + \mu}{x_n(x_n + \delta)}. \tag{22b}$$

Case (C1) corresponds to $\beta\delta\zeta\lambda \neq 0$. First we consider the case where no simplification is possible, case (C1a). In the proper gauge $\beta = \delta = -1$,

$$x_{n+1}x_n = \frac{\lambda y_n + \mu}{y_n(y_n - 1)}, \tag{23a}$$

$$y_n y_{n-1} = \frac{\zeta x_n + \mu}{x_n(x_n - 1)} \tag{23b}$$

and upon deautonomisation we obtain

$$x_{n+1}x_n = \frac{a_n y_n + b_n}{y_n(y_n - 1)}, \tag{24a}$$

$$y_n y_{n-1} = \frac{c_n x_n + d_n}{x_n(x_n - 1)}, \tag{24b}$$

where $\log a_n = pn + q$, $\log b_n = 2pn + r$, $\log c_n = pn + s$, and $\log d_n = p(2n - 1) + r$. Equation (24) is a q-discrete form of Painlevé III, and was first identified in Ref. 13. The geometry of its transformations is described by the affine Weyl group $A_2^{(1)} + A_1^{(1)}$. Case (C1b) is obtained if we assume that a simplification occurs in the right hand side of (22a). In this case we can solve for y and obtain in terms of x, upon deautonomisation and in the proper gauge,

$$x_{n+1}x_{n-1} = \frac{f_n x_n + g_n}{x_n(x_n - 1)}, \tag{25}$$

where $\log f_n = pn + q + r(-1)^n$ and $\log g_n = 2pn + s$ which is nothing but the rewriting of (24), taking into account the weak-asymmetric character of the latter. If there are two simplifications one gets a trivial linearisable equation $x_{n+1}x_n x_{n-1} = f_n$, where f_n is free.

Case (C2) corresponds to putting $\delta = 0$, $\beta\zeta\lambda \neq 0$ in (22). (Notice that we could have taken $\zeta = 0$ instead, but this choice is equivalent to $\delta = 0$ up to an inversion of x.) If there is no simplification, deautonomising and introducing the adequate gauge into the equation we have, case (C2a),

$$x_{n+1}x_n = \frac{b_n y_n + 1}{y_n(y_n + a)}, \tag{26a}$$

$$y_n y_{n-1} = \frac{x_n + 1}{x_n^2}, \tag{26b}$$

with a constant and $\log b_n = pn + q$, which is the one-parameter q-$\mathrm{P_{III}}$, introduced in Ref. 7, and which is associated to the affine Weyl group $A_1^{(1)} + A_1^{(1)}$. Case (C2b) is obtained by assuming that one simplification occurs in the right hand side of (22a) and taking $\delta = 0$ in (22b). In this case we can solve for y in terms of x and obtain an equation of the form

$$x_{n+1}x_{n-1} = \frac{c_n}{1 + x_n}, \tag{27}$$

where $\log c_n = pn + q + r(-1)^n$.

Case (C3) is obtained by putting $\delta = \zeta = 0$ in (22b) with $\beta\lambda \neq 0$. In this case we have

$$x_{n+1}x_n = \frac{\lambda y_n + \mu}{y_n(y_n + \beta)}, \tag{28a}$$

$$y_n y_{n-1} = \frac{\mu}{x_n^2} \tag{28b}$$

and we can eliminate x between the two equations, provided we square the first one. This is a transformation known as folding (the terminology is due to Okamoto, Sakai, and Tsuda[15]) and was introduced for the equation at hand in Ref. 16. The non-autonomous result, after choosing the proper gauge, is

$$y_{n+1}y_{n-1} = \left(\frac{y_n + a_n}{y_n + 1}\right)^2, \tag{29}$$

where $\log a_n = pn + q$, which is just a special case of the generic q-$\mathrm{P_{III}}$, Equation (7).

Case (C4) corresponds to taking $\delta = \beta = 0$, $\lambda\zeta \neq 0$ in (22). In this case we find a weakly asymmetric form which the proper choice of gauge reduces to the symmetric mapping

$$w_{n+1}w_{n-1} = \frac{1}{w_n^2} + \frac{a_n}{w_n} \tag{30}$$

with $\log a_n = pn + q$, which is the well-known q-Painlevé I.

Finally if we take $\delta = \lambda = 0$ in (22) with $\beta\zeta \neq 0$, we find

$$x_{n+1}x_n = \frac{\mu}{y_n(y_n + \beta)}, \tag{31a}$$

$$y_n y_{n-1} = \frac{\zeta x_n + \mu}{x_n^2}. \tag{31b}$$

Inverting x we obtain a linearisable mapping

$$x_{n+1}x_n = y_n^2 + \beta y_n, \tag{32a}$$

$$y_n y_{n-1} = x_n^2 + \zeta x_n, \tag{32b}$$

and which is a limiting case of (21). We would have obtained the same result by taking $\beta = \zeta = 0$ up to an inversion of y in this case.

D. Case D

Next we could choose $\alpha = \gamma = 0$ in which case we can take $\mu = \kappa = 1$ (assuming that none of them vanish). Instead, we can take equivalently $\alpha = \kappa = 0$ and normalise to $\mu = \gamma = 1$. Working within the latter parametrisation we have

$$x_{n+1}x_n = \frac{\lambda y_n + 1}{\beta y_n + 1}, \tag{33a}$$

$$y_n y_{n-1} = \frac{x_n^2 + \zeta x_n + 1}{\delta x_n}. \tag{33b}$$

Assuming $\beta \neq 0$, whereupon we can fix its value to 1 by gauge, we have after deautonomisation, case (D1),

$$x_{n+1}x_n = \frac{a_n y_n + 1}{y_n + 1}, \tag{34a}$$

$$y_n y_{n-1} = \frac{(x_n + c)(x_n + 1/c)}{d_n x_n}, \tag{34b}$$

with c a constant, $\log a_n = pn + q$ and $\log d_n = pn + r$. Solving (34a) for y in terms of x and obtain a single equation for x which is a discrete analogue of the Painlevé IV, first derived by Kajiwara and collaborators[17] (see also Ref. 18) who have shown that the geometry of its transformations is governed by the affine Weyl group $A_2^{(1)} + A_1^{(1)}$.

Case (D2) corresponds to taking $\beta = 0$. (Taking $\lambda = 0$ does not lead to a new case but rather to a dual of (D2) under the transformation $x \to 1/x$.) Again we can eliminate y and obtain a mapping for x, which upon deautonomisation becomes

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = b_n \frac{(x_n + a)(x_n + 1/a)}{x_n}, \tag{35}$$

with $\log b_n = pn + q$. Equation (35) is a discrete analogue of the Equation (34) in the Painlevé-Gambier list. It was first derived in Ref. 7 and it is associated to the affine Weyl group $A_1^{(1)} + A_1^{(1)}$.

E. Case E

This is the final case of family (II). We take $\alpha = \gamma = \kappa = 0$ and normalise $\mu = 1$,

$$x_{n+1}x_n = \frac{1 + \lambda y_n}{\beta y_n}, \tag{36a}$$

$$y_n y_{n-1} = \frac{1 + \zeta x_n}{\delta x_n}. \tag{36b}$$

Equation (36) was deautonomised, case (E1), in Ref. 13. Taking $\lambda = \zeta = 1$ we find

$$x_{n+1}x_n = a_n \frac{1 + y_n}{y_n}, \tag{37a}$$

$$y_n y_{n-1} = b_n \frac{1 + x_n}{x_n}, \tag{37b}$$

where $\log a_n = pn + q$, $\log b_n = pn + r$, i.e., a weakly asymmetric equation. A different representation of the same equation can be obtained if one eliminates y leading to an equation in terms of x the non-autonomous form of which is

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{a b_n^2 x_n}{x_n + b_n} \tag{38}$$

with $\log b_n = pn + q$ and constant a, first derived in Ref. 7, where it was shown that it is a discrete form of Painlevé II. The geometry of the transformations of both (37) and (38) is described by $A_1^{(1)} + A_1^{(1)}$.

Case (E2) corresponds to taking $\zeta = 0$. Eliminating y we write a single equation in terms of x and deautonomise it, obtaining

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = b_n x_n \tag{39}$$

with $\log b_n = pn + q$, identified in Ref. 2 as a q-discrete form of Painlevé I. But we could equivalently have eliminated x in terms of y and obtain in terms of $1/y$ the better known q-discrete form of P_{I}, i.e., Equation (30).

This completes the exploration of limits and degenerate cases of family (II). Only two genuinely strongly asymmetric cases have been obtained here: (17) and (26). There are also cases like (8), (12), (18), (31), and (33) which look strongly asymmetric but which can be cast into a weakly asymmetric form.

IV. THE DISCRETE EQUATIONS ASSOCIATED TO FAMILY (IV)

The general form of the family (IV) asymmetric QRT mapping is

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{\kappa y_n^4 + (\delta+\lambda)y_n^3 + (\mu+\epsilon+\alpha)y_n^2 + (\beta+\zeta)y_n + \gamma}{\alpha y_n^2 + \beta y_n + \gamma}, \tag{40a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{\gamma x_n^4 + (\beta+\zeta)x_n^3 + (\mu+\epsilon+\alpha)x_n^2 + (\delta+\lambda)x_n + \kappa}{\alpha x_n^2 + \delta x_n + \kappa}. \tag{40b}$$

Again, we classify the cases according to the number of corners of the A_0 QRT matrix which are put to 0 and obtain thus the possible branches of limiting and degenerate forms of (40).

A. Case A

The general case (A1) is obtained when $\alpha\gamma\kappa \neq 0$ and without loss of generality we can take $\gamma = \kappa = 1$. The case (A1a) corresponds to absence of simplification in the right hand side of (40). In order to facilitate the simplification process we introduce a more convenient form for the generic autonomous family (IV) mapping,

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1-y_n/a)(1-y_n/b)(1-y_n/c)(1-y_n/d)}{(1-f y_n)(1-g y_n)}, \tag{41a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1-a x_n)(1-b x_n)(1-c x_n)(1-d x_n)}{(1-h x_n)(1-k x_n)}, \tag{41b}$$

where the parameters satisfy the constraint $abcdfg = hk$ at the autonomous limit. Its deautonomisation leads to

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1-y_n/a)(1-y_n/b)(1-y_n/c)(1-y_n/d)}{(1-f_n y_n)(1-g_n y_n)}, \tag{42a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1-a x_n)(1-b x_n)(1-c x_n)(1-d x_n)}{(1-h_n x_n)(1-k_n x_n)}, \tag{42b}$$

where a,b,c,d are constant, which allows a gauge $abcd = 1$, and $\log f_n = 2pn + q + r$, $\log g_n = 2pn + q - r$, $\log h_n = p(2n-1) + q + s$, and $\log k_n = p(2n-1) + q - s$. As shown in Ref. 19 the geometry of this equation is associated to the affine Weyl group $E_6^{(1)}$. In fact all equations of case (A1) obtained by successive simplifications will be associated to the same affine Weyl group.

The case (A1b) corresponds to one simplification in one of the equations of (41) taking for instance $d = k$. Deautonomising the generic case, (A1bi), where d is not equal to any of the a,b,c,h, we obtain

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1-y_n/a)(1-y_n/b)(1-y_n/c)(1-y_n/d_n)}{(1-f_n y_n)(1-g_n y_n)}, \tag{43a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1-a x_n)(1-b x_n)(1-c x_n)}{1-h_n x_n}, \tag{43b}$$

where a,b,c are constant and, with the appropriate choice of gauge, we can take $abc = 1$, leading to $\log f_n = pn + q + r(-1)^n + s$, $\log g_n = pn + q + r(-1)^n - s$, $\log h_n = p(2n-1) + 2q$, $\log d_n = pn + u - r(-1)^n$.

Next we consider the case (A1bii) where $d = h = k$, different from a,b,c lest another simplification occur. The appropriate gauge here leads again to $abc = 1$ and the deautonomisation results to $\log f_n = pn + q + \phi_3(n)$, $\log g_n = pn + r + \phi_3(n)$, $\log d_n = 2pn + q + r - \phi_3(n)$ and $\log h_n = p(2n-1) + q + r - \phi_3(n+1)$ (which is equal to d at the autonomous limit).

Now we turn to the cases where a higher power appears in the numerator. The case (A1biii) corresponds to $d = c = k$, different from a,b. We find, after deautonomisation

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1-y_n/a)(1-y_n/b)(1-y_n/c_n)(1-y_n/c_{n+1})}{(1-f_n y_n)(1-g_n y_n)}, \tag{44a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1 - a x_n)(1 - b x_n)(1 - c_n x_n)}{1 - h_n x_n} \tag{44b}$$

with $ab = 1$ by a gauge and $\log f_n = pn + q + \phi_3(n)$, $\log g_n = pn + r + \phi_3(n)$, $\log c_n = pn + s + \phi_3(n+1)$, and $\log h_n = p(3n - 1) + q + r + s$. In the case, (A1biv), $d = c = b = k$, different from a we find

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/a)(1 - y_n/c_{n-1})(1 - y_n/c_n)(1 - y_n/c_{n+1})}{(1 - f_n y_n)(1 - g_n y_n)}, \tag{45a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1 - a x_n)(1 - c_{n-1} x_n)(1 - c_n x_n)}{1 - h_n x_n} \tag{45b}$$

with $a = 1$ by a gauge and $\log f_n = pn + q + \phi_4(n)$, $\log g_n = pn + r + \phi_4(n)$, $\log c_n = pn + s + \phi_4(n+2)$, $\log h_n = p(4n - 2) + q + r + 2s$. Finally when $d = c = b = a = k$, case (A1bv), we have

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/c_{n-1})(1 - y_n/c_n)(1 - y_n/c_{n+1})(1 - y_n/c_{n+2})}{(1 - f_n y_n)(1 - g_n y_n)}, \tag{46a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1 - c_{n-1} x_n)(1 - c_n x_n)(1 - y_n/c_{n+1})}{1 - h_n x_n}, \tag{46b}$$

with $\log f_n = pn + q + \phi_5(n)$, $\log g_n = pn + r + \phi_5(n)$, $\log c_n = pn + s + \phi_5(n+2)$, and $\log h_n = p(5n - 1) + q + r + 3s$.

Case (A1c) corresponds to simplifications occurring in both equations of (41) which can be realised either by $d = k$ and $c = 1/g$ or by $d = k = 1/g$. We begin by the first possibility and we have

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/a)(1 - y_n/b)(1 - y_n/d)}{1 - f y_n}, \tag{47a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1 - a x_n)(1 - b x_n)(1 - c x_n)}{1 - h x_n}. \tag{47b}$$

The generic case, (A1ci), corresponds to all remaining parameters being distinct,

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/a)(1 - y_n/b)(1 - y_n/d_n)}{1 - f_n y_n}, \tag{48a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1 - a x_n)(1 - b x_n)(1 - c_n x_n)}{1 - h_n x_n}. \tag{48b}$$

By the appropriate gauge we can take $ab = 1$. The deautonomisation of (48) gives $\log d_n = pn + q + \phi_3(n)$, $\log c_n = -pn + r - \phi_3(n+1)$, $\log f_n = 2pn + s - \phi_3(n)$, and $\log h_n = p(2n - 1) + s - \phi_3(n+1)$. The case (A1cii) corresponds to a single square in one numerator, for instance $d = b$. We have

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/a)(1 - y_n/b_n)(1 - y_n/b_{n+1})}{1 - f_n y_n}, \tag{49a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1 - a x_n)(1 - b_n x_n)(1 - c_n x_n)}{1 - h_n x_n}. \tag{49b}$$

With a gauge $a = 1$ the deautonomisation gives $\log b_n = pn + q + \phi_4(n)$, $\log c_n = -pn + r - \phi_4(n+2)$, $\log f_n = p(2n + 1) + s - \phi_4(n) - \phi_4(n+1)$, and $\log h_n = 3pn + q + s - \phi_4(n+2)$. The case (A1ciii) corresponds to a cube in one numerator $d = b = a$ leading to

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/b_{n-1})(1 - y_n/b_n)(1 - y_n/b_{n+1})}{1 - f_n y_n}, \tag{50a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1 - b_{n-1} x_n)(1 - b_n x_n)(1 - c_n x_n)}{1 - h_n x_n} \tag{50b}$$

which is deautonomised to $\log b_n = pn + q + \phi_5(n)$, $\log c_n = -pn + r - \phi_5(n+2)$, $\log f_n = 2pn + s + \phi_5(n+2) + \phi_5(n-2)$, and $\log h_n = p(4n - 2) + 2q + s - \phi_5(n+2)$. The gauge freedom can be used in order to reduce the number of parameters by one, and bring them down to the expected

6, but no optimal choice seems to exist. The next case (A1civ) corresponds to two squares in the numerators, $d = b$, $c = a$. The deautonomised equation has the form

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{(1 - y_n/c_n)(1 - y_n/b_n)(1 - y_n/b_{n+1})}{1 - f_ny_n}, \tag{51a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{(1 - c_{n-1}x_n)(1 - b_nx_n)(1 - c_nx_n)}{1 - h_nx_n}, \tag{51b}$$

with $\log b_n = pn + q + \phi_5(n)$, $\log c_n = -pn + r - \phi_5(n-2)$, $\log f_n = p(3n+1) + s - \phi_5(n) - \phi_5(n+1)$, and $\log h_n = 3pn + q + r + s - \phi_5(n+2) - \phi_5(n-2)$. A gauge freedom does exist here also, allowing to reduce the number of parameters by one.

Next we examine the cases where one square exists in one denominator, before simplification, for instance $h = k$. The case (A1cv) corresponds to $d = h = k$ before deautonomisation after which we obtain an equation of exactly the same form as (48) but the n dependence of the parameters is now different. Again we choose a gauge $ab = 1$ and obtain $\log c_n = -pn + q + \phi_4(n)$, $\log d_n = 2pn + r + \phi_4(n) + \phi_4(n+1)$, $\log f_n = 2pn + r - \phi_4(n) - \phi_4(n+1)$ and $\log h_n = p(2n-1) + r - \phi_4(n+1) - \phi_4(n-1)$. The case (A1cvi) corresponds to one square in the numerator which cannot involve d, lest a further simplification appear. We take $b = c$ and obtain after deautonomisation

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{(1 - y_n/a)(1 - y_n/b_n)(1 - y_n/d_n)}{1 - f_ny_n}, \tag{52a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{(1 - ax_n)(1 - b_nx_n)(1 - b_{n-1}x_n)}{1 - h_nx_n}, \tag{52b}$$

where, with the proper gauge we can take $a = 1$, and find $\log b_n = -pn + q + \phi_5(n)$, $\log d_n = 2pn + r - \phi_5(n+2) - \phi_5(n-2)$, $\log f_n = 3pn + q + r + \phi_5(n+2) + \phi_5(n-2)$, and $\log h_n = p(2n-1) + r - \phi_5(n+1) - \phi_5(n-2)$. The case (A1cvii) corresponds to a cube in the numerator which again cannot involve d. Taking $a = b = c$ we deautonomise and obtain

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{(1 - y_n/b_{n+1})(1 - y_n/b_n)(1 - y_n/d_n)}{1 - f_ny_n}, \tag{53a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{(1 - b_{n+1}x_n)(1 - b_nx_n)(1 - b_{n-1}x_n)}{1 - h_nx_n} \tag{53b}$$

with a genuine periodicity of 6. We have $\log b_n = pn + q + \phi_6(n)$ and express the remaining parameters in terms of b_n as follows: $d_n = (b_{n-2}b_{n+3})^{-1}$, $h_n = (b_{n-2}b_{n+2})^{-1}$, and $f_n = (b_{n-1}b_nb_{n+1}b_{n+2})^{-1}$. Since Equation (53) is associated to the affine Weyl group $E_6^{(1)}$ this is the maximal periodicity one can obtain for the parameters. An analogous result was obtained in Ref. 8 for additive equations belonging to the canonical family (III).

Finally if a square appears in both denominators, there can be no further squares in the numerators before simplification, and thus the only case that does exist, (A1cviii), is $d = h = k$ and $c = 1/g = 1/f$. The form of the equation is again (48) and, after a gauge leading to $ab = 1$, we find $\log d_n = pn + q + \phi_5(n)$, $\log c_n = -p(n - 1/2) - q - \phi_5(n+2)$, $\log f_n = pn + q + \phi_5(n+1) + \phi_5(n-1) - \phi_5(n)$ and $\log h_n = p(n - 1/2) + q + \phi_5(n+1) + \phi_5(n-2) - \phi_5(n+2)$.

Next we consider the second possibility for simplification where we have the general form

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{(1 - y_n/a)(1 - y_n/b)(1 - y_n/c)}{1 - fy_n}, \tag{54a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{(1 - ax_n)(1 - bx_n)(1 - cx_n)}{1 - hx_n} \tag{54b}$$

and the generic case (A1cix) corresponds to a, b, c constant, whereupon a gauge choice let us put $abc = 1$, and where $\log f_n = pn + q + r(-1)^n$ and $\log h_n = pn + s + u(-1)^n$. Here the presence of a square, or higher power, in the numerator before simplification does not change anything. On the other hand squares may exist in the denominators before simplification. When a single one exists, case (A1cx), for instance $h = k = d$, we find, after deautonomisation, $\log f_n = pn + q + r(-1)^n +$

$\phi_3(n)$ and $h_n = f_n f_{n-1}$. When two squares do exist we have the constraint $h = k = d = 1/f = 1/g$. This case can be obtained from the generic (A1cix) one, with $p = 0$ and $q + s = 0$ so as to satisfy $fh = 1$ and thus there is no secular dependence on n.

When there are two simplifications in (41b), $d = k$, $c = h$ the system becomes

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/a)(1 - y_n/b)(1 - y_n/c)(1 - y_n/d)}{(1 - f y_n)(1 - g y_n)}, \tag{55a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - a x_n)(1 - b x_n). \tag{55b}$$

Given the form of (55b), we can solve for x_n in terms of y_n, y_{n-1} and, substituting back into (55a), we obtain, after a rescaling, an equation which is the symmetric form of (42). Upon deautonomisation we find again the full freedom of the (asymmetric) Equation (42). When we have one simplification in the first equation and two in the second one, again, solving the second equation for x_n allows to write the first equation as the symmetric form of (42) and a deautonomisation allows to recover the full freedom of (42). Similarly with two simplifications in each equation,

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = (1 - y_n/a)(1 - y_n/b), \tag{56a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - c x_n)(1 - d x_n), \tag{56b}$$

and when no relation between the a,b,c,d exists, one can solve the second equation for x_n and still recover the symmetric form of (42). A deautonomisation, without any artificial gauge constraint, allows to reconstitute the full freedom of (42). When one equality exists, for instance $c = a$ we go back to the case (A1c) and all its subcases. When $c = a$ and $d = b$ we obtain an equation which should be studied separately in order to avoid a circular reasoning. However the latter, in the gauge $ab = 1$, turns out to be precisely Equation (20) already encountered in Section III.

B. Case B

Case (B) and all its subcases have $\alpha = 0$ and $\gamma\kappa \neq 0$. The case (B1) corresponds to $\beta\delta \neq 0$. The generic case (B1a) is one where no simplification occurs

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/a)(1 - y_n/b)(1 - y_n/c)(1 - y_n/d)}{1 - f_n y_n}, \tag{57a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(1 - a x_n)(1 - b x_n)(1 - c x_n)(1 - d x_n)}{1 - h_n x_n}, \tag{57b}$$

where a,b,c,d are constant, a gauge can be used to put $abcd = 1$, and $\log f_n = pn + q$, $\log h_n = pn + r$. The geometry of the transformations of (57) is described by the affine Weyl group $D_5^{(1)}$ and the same is true for all the equations of the subcase (B1). Case (B1b) corresponds to one simplification in the second equation, $d = h$. When d is different from all a,b,c we have case (B1bi)

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/a)(1 - y_n/b)(1 - y_n/c)(1 - y_n/d_n)}{1 - f_n y_n}, \tag{58a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - a x_n)(1 - b x_n)(1 - c x_n), \tag{58b}$$

where we can again gauge to $abc = 1$. The deautonomisation of this case is $\log d_n = pn + q + r(-1)^n$ and $\log f_n = pn + s - r(-1)^n$. Case (B1bii) has $c = d$ and its deautonomisation leads to

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/a)(1 - y_n/b)(1 - y_n/c_n)(1 - y_n/c_{n+1})}{1 - f_n y_n}, \tag{59a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - a x_n)(1 - b x_n)(1 - c_n x_n), \tag{59b}$$

with $\log c_n = pn + q + \phi_3(n)$, $\log f_n = pn + r + \phi_3(n-1)$, and the gauge freedom allows to take $ab = 1$. Case (B1biii) has $b = c = d$ and when deautonomised leads to

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/a)(1 - y_n/c_{n-1})(1 - y_n/c_n)(1 - y_n/c_{n+1})}{1 - f_n y_n}, \tag{60a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - a x_n)(1 - c_{n-1} x_n)(1 - c_n x_n), \tag{60b}$$

where $a = 1$ by gauge and $\log c_n = pn + q + \phi_4(n)$, $\log f_n = pn + r + \phi_4(n-2)$. Finally in case (B1biv) we have $a = b = c = d$ which is deautonomised to

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/c_{n-1})(1 - y_n/c_n)(1 - y_n/c_{n+1})(1 - y_n/c_{n+2})}{1 - f_n y_n}, \tag{61a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - c_{n-1}x_n)(1 - c_n x_n)(1 - c_{n+1}x_n) \tag{61b}$$

with $\log c_n = pn + q + \phi_5(n)$, $\log f_n = pn + r + \phi_5(n-2)$ and a gauge choice allows to put $r = 0$.

Case (B1c) corresponds to two simplifications, one in each equation, $d = h$ and $c = 1/f$. (The case where $d = h = 1/f$, when d disappears from the equation, leads to a mapping with periodic solution with period 2.) For the generic case, (B1ci) we find

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = (1 - y_n/a)(1 - y_n/b)(1 - y_n/d_n), \tag{62a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - a x_n)(1 - b x_n)(1 - c_n x_n), \tag{62b}$$

where $ab = 1$ by gauge and $\log c_n = pn + q + \phi_3(n)$, $\log(1/d_n) = pn + r + \phi_3(n-1)$, a weakly asymmetric equation. For case (B1cii), i.e., $b = d$, we have

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = (1 - y_n/a)(1 - y_n/b_n)(1 - y_n/b_{n+1}), \tag{63a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - a x_n)(1 - b_n x_n)(1 - c_n x_n) \tag{63b}$$

with $a = 1$ by gauge and $\log c_n = pn + q + \phi_4(n)$, $\log b_n = -pn + r - \phi_4(n-2)$. Case (B1ciii) corresponds to $a = b = d$ and after deautonomisation we find

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = (1 - y_n/b_{n-1})(1 - y_n/b_n)(1 - y_n/b_{n+1}), \tag{64a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - b_{n-1}x_n)(1 - b_n x_n)(1 - c_n x_n) \tag{64b}$$

with $\log b_n = pn + q + \phi_5(n)$, $\log c_n = -pn - r - \phi_5(n-2)$ where a proper choice of gauge allows to take $r = 0$. Finally case (B1civ) corresponds to the choice $b = d$, $a = c$, which, when deautonomised, becomes

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = (1 - y_n/c_n)(1 - y_n/b_n)(1 - y_n/b_{n+1}), \tag{65a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - c_{n-1}x_n)(1 - b_n x_n)(1 - c_n x_n) \tag{65b}$$

with $\log b_n = pn + q + \phi_5(n)$, $\log c_n = -pn - r - \phi_5(n+2)$ where the choice of gauge allows to eliminate one parameter. When moreover we have $a = b = c = d$ we find a symmetric mapping the solutions of which are periodic with period 4.

The case (B2) corresponds to $\delta = 0$ while $\beta \neq 0$. The generic case (B2a) is one where we have no simplifications

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - y_n/a)(1 - y_n/b)(1 - y_n/c)(1 - y_n/d)}{1 - f_n y_n}, \tag{66a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - a x_n)(1 - b x_n)(1 - c x_n)(1 - d x_n), \tag{66b}$$

where a, b, c, d are constant (and $abcd = 1$ by gauge) and $\log f_n = pn + q$. The geometry of the transformations of (66) is described by the affine Weyl group $A_4^{(1)}$ and the same holds for all the equations of the subcase (B2). One simplification occurs when $d = 1/f$, case (B2b). In the absence of squares, or higher powers, (B2bi), we have

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = (1 - y_n/a)(1 - y_n/b)(1 - y_n/c), \tag{67a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - a x_n)(1 - b x_n)(1 - c x_n)(1 - d_n x_n), \tag{67b}$$

where a, b, c are constants gauged to $abc = 1$ and $\log d_n = pn + q + r(-1)^n$. Next we consider the case (B2bii), $c = d$,

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = (1 - y_n/a)(1 - y_n/b)(1 - y_n/c_n), \tag{68a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - a x_n)(1 - b x_n)(1 - c_{n-1}x_n)(1 - c_n x_n), \tag{68b}$$

where $ab = 1$ by gauge and $\log c_n = pn + q + \phi_3(n)$. When $b = c = d$, case (B2biii), we find

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = (1 - y_n/a)(1 - y_n/c_n)(1 - y_n/c_{n+1}), \tag{69a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - ax_n)(1 - c_{n-1}x_n)(1 - c_n x_n)(1 - c_{n+1}x_n), \tag{69b}$$

where $a = 1$ by gauge and $\log c_n = pn + q + \phi_4(n)$. Finally when we take $a = b = c = d$ we obtain a mapping with solutions periodic with period 5.

The case where $\beta = \delta = 0$ is a well-known linearisable mapping

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = (1 - y_n/a)(1 - y_n/b)(1 - y_n/c)(1 - y_n/d), \tag{70a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 - ax_n)(1 - bx_n)(1 - cx_n)(1 - dx_n), \tag{70b}$$

which moreover cannot be extended to a non-autonomous form.

C. Case C

Case (C) as well as its subcases have $\kappa = 0$ and $\alpha\gamma \neq 0$.

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(\delta + \lambda)y_n^3 + (\mu + \epsilon + \alpha)y_n^2 + (\beta + \zeta)y_n + \gamma}{\alpha y_n^2 + \beta y_n + \gamma}, \tag{71a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{\gamma x_n^3 + (\beta + \zeta)x_n^2 + (\mu + \epsilon + \alpha)x_n + \delta + \lambda}{\alpha x_n + \delta}. \tag{71b}$$

In case (C1) neither δ nor $\delta + \lambda$ vanish. In order to simplify the calculations we introduce a more convenient form

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - ay_n)(1 - by_n)(1 - cy_n)}{(1 - fy_n)(1 - gy_n)}, \tag{72a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - b)(x_n - c)}{hx_n - k} \tag{72b}$$

with the autonomous constraint $h = fg$.

The generic case (C1a) is one where no simplification occurs. By choosing the proper gauge we can take $abc = 1$ and deautonomising we obtain

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - ay_n)(1 - by_n)(1 - cy_n)}{(1 - f_n y_n)(1 - g_n y_n)}, \tag{73a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - b)(x_n - c)}{h_n x_n - k_n}, \tag{73b}$$

with $\log f_n = pn + q$, $\log g_n = pn + r$, $\log h_n = p(2n - 1) + q + r$, and $\log k_n = pn + s$. The geometry of the transformations of this equation is described by the affine Weyl group $D_5^{(1)}$, and the same holds true for all equations of the subcase (C1). In this case no other constraint has been imposed. However, the case where $\lambda = 0$ should be specially examined since it leads to the constraint $k = abc$. The deautonomisation in this case, case (C1a′), leads to $k_n = abc = 1$ and $\log f_n = pn + q + s(-1)^n$, $\log g_n = pn + r + s(-1)^n$ and $\log h_n = p(2n - 1) + q + r$.

Case (C1b) corresponds to one simplification in the second equation of the system. We find thus, in the absence of squares in the numerator, case (C1bi)

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - ay_n)(1 - by_n)(1 - c_n y_n)}{(1 - f_n y_n)(1 - g_n y_n)}, \tag{74a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (x_n - a)(x_n - b)/h_n \tag{74b}$$

with $ab = 1$ by gauge and $\log f_n = pn + q + s(-1)^n$, $\log g_n = pn + r + s(-1)^n$, $\log h_n = p(2n - 1) + q + r$, $\log c_n = -pn + u + s(-1)^n$. A case (C1bi′) does also exist, coming from $\lambda = 0$ implying $h_n = ab = 1$, with $\log f_n = q + s(-1)^n$, $\log g_n = -q + s(-1)^n$, and $\log c_n = pn + r + u(-1)^n$. We should point out here that for all "prime" cases in this paragraph one could have solved the second equation for x_n in terms of y_n, y_{n-1} and substituting back into the first equation obtain a symmetric case of the equations analysed under case B. The case (C1bii) corresponds to one square in the numerator, $b = c$. Upon deautonomisation we have

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - a y_n)(1 - c_n y_n)(1 - c_{n+1} y_n)}{(1 - f_n y_n)(1 - g_n y_n)}, \tag{75a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (x_n - a)(x_n - c_n)/h_n \tag{75b}$$

with $a = 1$ by gauge and $\log f_n = pn + q + \phi_3(n)$, $\log g_n = pn + r + \phi_3(n)$, $\log h_n = p(2n - 1) + q + r - \phi_3(n + 1)$, $\log c_n = -pn + s - \phi_3(n + 1)$. The case (C1bii′) does also exist and implies $c_n = h_n$ with the gauge choice $a = 1$. Its deautonomisation gives $\log f_n = pn + q + s(-1)^n + \phi_3(n)$, $\log g_n = pn + r + s(-1)^n + \phi_3(n)$, $\log h_n = p(2n - 1) + q + r - \phi_3(n + 1)$. Finally when $a = b = c$ we have a cube in the numerator and the deautonomisation of this case, (C1biii), is obtained by

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - c_{n-1} y_n)(1 - c_n y_n)(1 - c_{n+1} y_n)}{(1 - f_n y_n)(1 - g_n y_n)}, \tag{76a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (x_n - c_{n-1})(x_n - c_n)/h_n \tag{76b}$$

with $\log f_n = pn + q + \phi_4(n)$, $\log g_n = pn + r + \phi_4(n)$, $\log h_n = p(2n - 1) + q + r + \phi_4(n) + \phi_4(n - 1)$ and $\log c_n = -pn + s - \phi_4(n + 2)$. The case (C1biii′) which would imply $h_n = c_n c_{n-1}$ cannot be extended to a case with secular dependence on n.

Case (C1c) corresponds to one simplification in the first equation of the system with the autonomous constraint $h = cf$. When there are no squares in the numerator we have case (C1ci),

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - a y_n)(1 - b y_n)}{1 - f_n y_n}, \tag{77a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - b)(x_n - c_n)}{h_n x_n - k_n} \tag{77b}$$

with $ab = 1$ and $\log f_n = 2pn + q$, $\log c_n = pn + r + u(-1)^n$, $\log h_n = p(3n - 1) + q + r - u(-1)^n$ and $\log k_n = pn + s$. A case (C1ci′) does also exist when $k_n = abc_n$, or $k_n = c_n$ with the gauge choice $ab = 1$. After deautonomisation we find $\log f_n = 2pn + q + \phi_3(n - 1)$, $\log c_n = pn + r + \phi_3(n)$, $\log h_n = p(3n - 1) + q + r$. The case (C1cii) corresponds to a square in the denominator of the first equation before simplification. We find $\log f_n = p(n + 1/2) + q + \phi_3(n - 1)$, $\log c_n = pn + q + \phi_3(n)$, $\log h_n = 2pn + 2q - \phi_3(n)$ and $\log k_n = 3pn/2 + r$. The case (C1cii′) has $k_n = c_n$ and its deautonomisation gives $\log f_n = p(2n + 1) + q + r(-1)^n$, $\log c_n = 2pn + q + \phi_4(n) + \phi_4(n + 1)$, $\log h_n = 4pn + 2q$. We turn now to the case (C1ciii) where a square exists in the numerator of the first equation before simplification. Upon deautonomisation we have

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - a y_n)(1 - c_n y_n)}{1 - f_n y_n}, \tag{78a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - c_{n-1})(x_n - c_n)}{h_n x_n - k_n} \tag{78b}$$

with $a = 1$ by gauge and $\log f_n = 3pn + q$, $\log c_n = pn + r + \phi_3(n)$, $\log h_n = p(4n - 2) + q + r + \phi_3(n + 1)$, $\log k_n = 3pn + s$. The case (C1ciii′) has $k_n = c_n c_{n-1}$, and when deautonomised leads to $\log f_n = 3pn + q - \phi_4(n - 2)$, $\log c_n = pn + r + \phi_4(n)$, $\log h_n = p(4n - 2) + q$. Finally we examine the case of a cube at the numerator, i.e., $a = b = c$. For case (C1civ) we find

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1 - c_n y_n)(1 - c_{n+1} y_n)}{1 - f_n y_n}, \tag{79a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - c_{n-1})(x_n - c_n)(x_n - c_{n+1})}{h_n x_n - k_n} \tag{79b}$$

with $\log f_n = 4pn + q$, $\log c_n = pn + r + \phi_4(n)$, $\log h_n = p(5n - 2) + q + r + \phi_4(n - 2)$, $\log k_n = 4pn + s$ and a gauge could have been used to eliminate one of the constants. Finally the case (C1civ′) has $k_n = c_{n-1}c_n c_{n+1}$ and its deautonomisation leads to $\log f_n = 4pn + q - \phi_5(n - 2)$, $\log c_n = pn + r + \phi_5(n)$, $\log h_n = p(5n - 2) + q + r$.

Case (C1d) corresponds to one simplification in each of the equations of the system with autonomous constraints either $g = c$, $k = hb$ or $g = c$, $k = hc$. Both cases imply $h = fc$. We start by examining the first constraint. When there are no squares we have case (C1di),

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{(1-ay_n)(1-b_ny_n)}{1-f_ny_n}, \tag{80a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=(x_n-a)(x_n-c_n)/h_n, \tag{80b}$$

with $a=1$ by gauge and $\log f_n=2pn+q+\phi_3(n)$, $\log b_n=-pn+s+\phi_3(n)$, $\log c_n=pn+r-\phi_3(n-1)$, $\log h_n=p(3n-1)+q+r$. Case (C1dii) has a square in the denominator of the first equation before simplification. Upon deautonomisation we find $a=1$ by gauge and $\log f_n=2pn+q-\phi_4(n+1)-\phi_4(n-1)$, $\log b_n=-pn+r+\phi_4(n)$, $\log c_n=p(2n-1)+q+\phi_4(n)+\phi_4(n-1)$, $\log h_n=p(4n-2)+2q$. Case (C1diii) has a square in both numerators before simplification, namely $a=b$. Deautonomising we have

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{(1-a_ny_n)(1-a_{n+1}y_n)}{1-f_ny_n}, \tag{81a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=(x_n-a_n)(x_n-c_n)/h_n \tag{81b}$$

with $\log c_n=pn+q+\phi_4(n+2)$, $\log h_n=3pn+r-\phi_4(n)$, $\log a_n=-pn+s+\phi_4(n)$ and $\log f_n=p(2n+1)+r-q-\phi_4(n)-\phi_4(n+1)$, and a gauge can be used to reduce the number of parameters by one. Case (C1div) has also a square in both numerators before simplification, introduced by $a=c$. Its deautonomisation leads to

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{(1-a_ny_n)(1-b_ny_n)}{1-f_ny_n}, \tag{82a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=(x_n-a_n)(x_n-a_{n-1})/h_n \tag{82b}$$

with $\log b_n=-pn+q-\phi_4(n)$, $\log h_n=p(4n-2)+r$, $\log a_n=pn+s+\phi_4(n+2)$, and $\log f_n=3pn+r-s-\phi_4(n)$ up to a gauge. Case (C1dv) corresponds to the second constraint $g=c$, $k=hc$. In this case a and b are constant and a gauge allows to take $ab=1$. The deautonomisation leads to

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{(1-ay_n)(1-by_n)}{1-f_ny_n}, \tag{83a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=(x_n-a)(x_n-b)/h_n \tag{83b}$$

with $\log f_n=pn+q+s(-1)^n$ and $\log h_n=pn+r+u(-1)^n$. Here the case (C1dv′) also exists where we have, in the gauge where $ab=1$, also $h=1$. Its deautonomisation gives $\log f_n=pn+q+r(-1)^n+\phi_3(n)$. In fact eliminating y we find for x a weakly asymmetric case identical to case (B1ci) above. Finally we have case (C1dvi) where the existence of a square in the denominator before simplification implies $h=f^2$ in the autonomous case. The deautonomisation, with gauge $ab=1$, gives again $\log f_n=pn+q+r(-1)^n+\phi_3(n)$ and $h_n=f_nf_{n-1}$. (A "prime" case with $f=h=1$ does also exist but it leads to a mapping with solutions periodic with period 4.)

Cases (C1e) and (C1f) correspond to two simplifications in the first equation with no simplification and one simplification in the second equation respectively. However, it is not necessary to study them afresh since in both cases one can solve for y from the first equation and obtain for x an equation belonging to case (A1) of Section III, consistent with a $D_5^{(1)}$ geometry.

Case (C2) corresponds to $\delta=0$ with $\lambda\neq 0$ the generic autonomous form of which is

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{(1-ay_n)(1-by_n)(1-cy_n)}{(1-fy_n)(1-gy_n)}, \tag{84a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=\frac{(x_n-a)(x_n-b)(x_n-c)}{hx_n} \tag{84b}$$

with the constraint $h=fg$. We remark that the second equation can never be simplified within case C2. The case (C2a) corresponds to absence of simplification in the first equation. By deautonomising we find

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{(1-ay_n)(1-by_n)(1-cy_n)}{(1-f_ny_n)(1-g_ny_n)}, \tag{85a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - b)(x_n - c)}{h_n x_n}, \tag{85b}$$

where a, b, c are constant and by the proper gauge one can put $abc = 1$ while $\log f_n = pn + q$, $\log g_n = pn + r$, $\log h_n = p(2n - 1) + q + r$. This equation is associated to the affine Weyl group $A_4^{(1)}$, and the same holds true for all equations of case (C2).

Case (C2b) corresponds to one simplification, $c = g$ with the autonomous constraint $h = fc$, and when there is no square in (84) we have case (C2bi). Its deautonomisation gives

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - a y_n)(1 - b y_n)}{1 - f_n y_n}, \tag{86a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - b)(x_n - c_n)}{h_n x_n}, \tag{86b}$$

where a, b are constant, with $ab = 1$ by gauge, and $\log c_n = pn + q + s(-1)^n$, $\log f_n = 2pn + r$, $\log h_n = p(3n - 1) + q + r - s(-1)^n$. Case (C2bii) corresponds to a square in the denominator before simplification, i.e., $f = g = c$, in the gauge $ab = 1$ which leads to $\log f_n = p(n + 1/2) + q + \phi_3(n)$, $\log c_n = pn + q + \phi_3(n + 1)$, and $\log h_n = 2pn + 2q - \phi_3(n + 1)$. When a square, $b = c = g$, is present on the numerator we find, case (C2biii),

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - a y_n)(1 - c_n y_n)}{1 - f_n y_n}, \tag{87a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - c_{n-1})(x_n - c_n)}{h_n x_n} \tag{87b}$$

with $a = 1$, by gauge, and $\log f_n = 3pn + q$, $\log c_n = pn + r + \phi_3(n)$, $\log h_n = p(4n - 2) + q + r + \phi_3(n + 1)$. Finally when a cube is present in the numerator, i.e., $a = b = c = g$, case (C2biv), we find

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - c_n y_n)(1 - c_{n+1} y_n)}{1 - f_n y_n}, \tag{88a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - c_{n-1})(x_n - c_n)(x_n - c_{n+1})}{h_n x_n} \tag{88b}$$

with $\log f_n = 4pn + q$, $\log c_n = pn + r + \phi_4(n)$,and $\log h_n = p(5n - 2) + q + r + \phi_4(n + 2)$. When two simplifications occur in the first equation one can solve for y and obtain for $1/x$ an equation which corresponds to the case (B) of Section III, consistent with an $A_4^{(1)}$ geometry.

Case (C3) corresponds to $\delta + \lambda = 0$ with $\delta \neq 0$, $\alpha + \epsilon + \mu \neq 0$ and its generic autonomous form is

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - a y_n)(1 - b y_n)}{(1 - f y_n)(1 - g y_n)}, \tag{89a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{x_n(x_n - a)(x_n - b)}{h x_n - k} \tag{89b}$$

with the autonomous constraint $h = fg$. All equations belonging to case (C3) are associated to the affine Weyl group $A_4^{(1)}$. In case (C3a) we have no simplifications and a gauge choice can be used to put $ab = 1$. Upon deautonomisation we find

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(1 - a y_n)(1 - b y_n)}{(1 - f_n y_n)(1 - g_n y_n)}, \tag{90a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{x_n(x_n - a)(x_n - b)}{h_n x_n - k_n} \tag{90b}$$

with $\log f_n = pn + q$, $\log g_n = pn + r$, $\log h_n = p(2n - 1) + q + r$, and $\log k_n = pn + s$. Case (C3bi) corresponds to one simplification in the first equation, with $b = g$ without any square (and $h = fb$ at the autonomous limit)

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{1 - a y_n}{1 - f_n y_n}, \tag{91a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=\frac{x_n(x_n-a)(x_n-b_n)}{h_nx_n-k_n}. \tag{91b}$$

We find $a=1$, by gauge, and $\log b_n=pn+q+s(-1)^n$, $\log f_n=2pn+r$, $\log h_n=p(3n-1)+q+r-s(-1)^n$, $\log k_n=2pn+u-s(-1)^n$. Case (C3bii) corresponds to $f=g=b$ and we find, upon deautonomisation, $a=1$, by gauge, and $\log b_n=2pn+q+\phi_3(n)$, $\log f_n=p(2n+1)+q+\phi_3(n-1)$, $\log h_n=4pn+2q-\phi_3(n)$, $\log k_n=3pn+r$. Finally case (C3biii) has a square in the numerator i.e., $a=b=g$ and its deautonomisation gives

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{1-b_ny_n}{1-f_ny_n}, \tag{92a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=\frac{x_n(x_n-b_{n-1})(x_n-b_n)}{h_nx_n-k_n}, \tag{92b}$$

where $\log b_n=pn+q+\phi_3(n)$, $\log f_n=3pn+r$, $\log k_n=3pn+s$, and $\log h_n=p(4n-2)+q+r+\phi_3(n+1)$. Case (C3c) corresponds to one simplification in the second equation. When $a\neq b$ we have case (C3ci)

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{(1-ay_n)(1-b_ny_n)}{(1-f_ny_n)(1-g_ny_n)}, \tag{93a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=\frac{x_n(x_n-a)}{h_n} \tag{93b}$$

with $a=1$ by gauge and $\log f_n=pn+q+r(-1)^n$, $\log g_n=pn+s+r(-1)^n$, $\log h_n=p(2n-1)+q+s$, $\log b_n=-pn+u+s(-1)^n$. When $a=b$ we have case (C3cii)

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{(1-b_{n+1}y_n)(1-b_ny_n)}{(1-f_ny_n)(1-g_ny_n)}, \tag{94a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=\frac{x_n(x_n-b_n)}{h_n}, \tag{94b}$$

where $\log f_n=pn+q+\phi_3(n)$, $\log g_n=pn+r+\phi_3(n)$, $\log h_n=p(2n-1)+q+r-\phi_3(n+1)$, and $\log b_n=-pn+s-\phi_3(n+1)$. Case (C3d) corresponds to one simplification in the first equation $b=g$ and one in the second one. When we simplify by (x_n-a) in the second equation we obtain

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{1-a_ny_n}{1-f_ny_n}, \tag{95a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=\frac{x_n(x_n-b_n)}{h_n} \tag{95b}$$

and we have case (C3di) when no squares exist before simplification. Its deautonomisation gives $\log b_n=pn+q+\phi_3(n)$, $\log a_n=-pn+r-\phi_3(n-1)$, $\log f_n=p(2n+1)+s-\phi_3(n-1)$, and $\log h_n=3pn+q+s$. When a square exists in the denominator of the first equation, i.e., $f=g=b$, we have case (C3dii). Its deautonomisation leads to $\log b_n=2pn+q+\phi_4(n)+\phi_4(n-1)$, $\log a_n=-pn+r+\phi_4(n)$, $\log f_n=p(2n+1)+q-\phi_4(n-1)-\phi_4(n+1)$ and $\log h_n=4pn+2q$. Next we examine the case (C3diii) where we simplify by the factor (x_n-b) in the second equation,

$$(x_{n+1}y_n-1)(y_nx_n-1)=\frac{1-ay_n}{1-f_ny_n}, \tag{96a}$$

$$(y_nx_n-1)(x_ny_{n-1}-1)=\frac{x_n(x_n-a)}{h_n}. \tag{96b}$$

In the absence of squares we find, case (C3div), $a=1$ by gauge and $\log f_n=pn+q+r(-1)^n$, $\log h_n=pn+s+u(-1)^n$. Finally when one square exists in the denominator of the first equation $f=g$ (hence $h=fg$), case (C3dv), we obtain again $a=1$ and $\log f_n=pn+q+r(-1)^n+\phi_3(n)$, $\log h_n=p(2n-1)+2q-\phi_3(n+1)$. When two factorisations exist in the first equation we can solve it for y and obtain an equation belonging to the case (B) of Section III (for the variable $1/x$).

Case (C4) corresponds to $\delta + \lambda = 0$ with $\alpha + \epsilon + \mu = 0$ and where we assume that $\delta \neq 0$ and $\beta + \zeta \neq 0$. The autonomous form of this equation is

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{1 - ay_n}{(1 - fy_n)(1 - gy_n)}, \tag{97a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{x_n^2(x_n - a)}{hx_n - k} \tag{97b}$$

with $h = fg$. The generic case, (C4a), corresponds to the absence of any simplification

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{1 - ay_n}{(1 - f_ny_n)(1 - g_ny_n)}, \tag{98a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{x_n^2(x_n - a)}{h_nx_n - k_n} \tag{98b}$$

with $a = 1$ and $\log g_n = pn + q$, $\log f_n = pn + r$, $\log h_n = p(2n - 1) + q + r$, $\log k_n = pn + s$. It is associated with the affine Weyl group $A_2^{(1)} + A_1^{(1)}$. When we have one simplification in the first equation we must distinguish two cases. When there is no square we have (C4bi), $a = g \neq f$,

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{1}{1 - f_ny_n}, \tag{99a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{x_n^2(x_n - a_n)}{h_nx_n - k_n} \tag{99b}$$

with $\log a_n = pn + q + r(-1)^n$, $\log f_n = 2pn + s$, $\log k_n = 2pn + u$ and $\log h_n = p(3n - 1) + s + q - r(-1)^n$. A gauge choice would allow to put, for instance, $q = 0$ and the same applies to the remaining equations of the (C) case. When a square exists in the denominator of the first equation $a = f = g$ we have, case (C4bii) with $\log a_n = 2pn + q + \phi_3(n)$, $\log f_n = p(2n + 1) + q + \phi_3(n - 1)$, $\log k_n = 3pn + r$ and $\log h_n = 4pn + 2q - \phi_3(n)$. Case (C4c) corresponds to one simplification in the second equation, $k = ah$,

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{1 - ay_n}{(1 - f_ny_n)(1 - g_ny_n)}, \tag{100a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{x_n^2}{h_n}, \tag{100b}$$

where $\log a_n = -pn + q + r(-1)^n$, $\log f_n = pn + s + r(-1)^n$, $\log g_n = pn + u + r(-1)^n$, and $\log h_n = p(2n - 1) + q + u$. Case (C4d) corresponds to a simplification in both equations $a = g$ and $k = ah$ and when there are no squares we have

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{1}{1 - f_ny_n}, \tag{101a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{x_n^2}{h_n} \tag{101b}$$

with $\log f_n = pn + q + r(-1)^n$, $\log h_n = pn + s + u(-1)^n$ and a gauge may be used in order to eliminate one parameter. When a square is present in the denominator of the first equation before deautonomisation the mapping becomes periodic of period 6.

Case (C5) corresponds to $\beta + \zeta = 0$ in addition to the previous constraints. The equation is now

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{1}{(1 - f_ny_n)(1 - g_ny_n)}, \tag{102a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{x_n^3}{h_nx_n - k_n}, \tag{102b}$$

where $\log g_n = pn + q$, $\log f_n = pn + r$, $\log h_n = p(2n - 1) + q + r$, $\log k_n = pn + s$, and is associated with the affine Weyl group $A_1^{(1)} + A_1^{(1)}$.

Finally when $\delta = \lambda = 0$ in addition to $\kappa = 0$, the mapping becomes linearisable, as shown in Ref. 20. Although this paper focuses on discrete Painlevé equations it is interesting to give a few details on this system. The generic case, which is the deautonomisation of the mapping presented in Ref. 20 has the form

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1-ay_n)(1-by_n)}{(1-cz_n y_n)(1-z_n y_n/c)}, \tag{103a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - b)}{z_n z_{n-1}}, \tag{103b}$$

where z_n is now a free function. The limiting cases with vanishing a and/or b do not change anything. However, a simplification is also possible in the first equation leading to a different deautonomisation and resulting in the system

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{1-ay_n}{1-z_n z_{n+1} y_n}, \tag{104a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(bx_n - z_n)}{z_{n-1} z_n z_{n+1}}, \tag{104b}$$

where, again, z_n is a free function, while a and/or b may vanish without any essential change.

The integration of these linearisable cases follows the method presented in Ref. 21. The basic requirement for the implementation of the method is the derivation of the non-autonomous, QRT-type, invariant for the mapping at hand. Without entering into detailed calculations we give the two invariants corresponding to (103) and (104). Since these systems are strongly asymmetric it is mandatory to give explicitly two invariants, K_n and $\tilde{K}_n$ the equations of the system being obtained by the conservation constraints $K_n - \tilde{K}_n = 0$ and $\tilde{K}_n - K_{n+1} = 0$. For (103) we find

$$K_n = z_{n-1}(x_n y_{n-1} - 1) - x_n\left(c + \frac{1}{c}\right) + \frac{(x_n - a)(x_n - b)}{z_{n-1}(x_n y_{n-1} - 1)}, \tag{105a}$$

$$\tilde{K}_n = z_n(x_n y_n - 1) - x_n\left(c + \frac{1}{c}\right) + \frac{(x_n - a)(x_n - b)}{z_n(x_n y_n - 1)}, \tag{105b}$$

while for (104) we have

$$K_n = z_{n-1}(x_n y_{n-1} - 1) - x_n\left(b + \frac{1}{z_n}\right) + \frac{(x_n - a)(bx_n - z_n)}{z_n z_{n-1}(x_n y_{n-1} - 1)}, \tag{106a}$$

$$\tilde{K}_n = z_{n+1}(x_n y_n - 1) - x_n\left(b + \frac{1}{z_n}\right) + \frac{(x_n - a)(bx_n - z_n)}{z_n z_{n+1}(x_n y_n - 1)}. \tag{106b}$$

D. Case D

Case (D) as well as its subcases have $\alpha = \kappa = 0$ while $\gamma \neq 0$,

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(\delta+\lambda)y_n^3 + (\mu+\epsilon)y_n^2 + (\beta+\zeta)y_n + \gamma}{\beta y_n + \gamma}, \tag{107a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{\gamma x_n^3 + (\beta+\zeta)x_n^2 + (\mu+\epsilon)x_n + \delta + \lambda}{\delta}, \tag{107b}$$

which means that δ cannot vanish. In case (D1) neither $\delta + \lambda$ nor β vanish and we can introduce the more convenient form

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{(1-ay_n)(1-by_n)(1-cy_n)}{1-fy_n}, \tag{108a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = h(x_n - a)(x_n - b)(x_n - c). \tag{108b}$$

The generic case (D1a) is one without simplifications. Here a,b,c are constant and we can by gauge take $abc = 1$. Deautonomising we find

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{(1 - ay_n)(1 - by_n)(1 - cy_n)}{1 - f_ny_n}, \tag{109a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = h_n(x_n - a)(x_n - b)(x_n - c) \tag{109b}$$

with $\log f_n = pn + q$, $\log h_n = -pn + r$, an equation associated with the affine Weyl group $A_4^{(1)}$ and the same holds true for all cases under (D1). However, when $\lambda = 0$ we have the autonomous constraint $habc = -1$, case (D1a′) which in the gauge $abc = 1$ leads to $h_n = -1$ and $\log f_n = pn + q + r(-1)^n$. Case (D1b) corresponds to one simplification in the first equation, $c = f$. When no square is present we have case (D1bi),

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = (1 - ay_n)(1 - by_n), \tag{110a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = h_n(x_n - a)(x_n - b)(x_n - c_n) \tag{110b}$$

with $ab = 1$ by gauge and $\log c_n = pn + q + r(-1)^n$, $\log h_n = -2pn + s$. When $\lambda = 0$, case (D1bi′), we have $h_n = -1/c_n$ with $\log c_n = pn + q + \phi_3(n)$. When one square is present in the numerator of the first equation $b = c$ we have case (D1bii),

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = (1 - ay_n)(1 - c_ny_n), \tag{111a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = h_n(x_n - a)(x_n - c_{n-1})(x_n - c_n) \tag{111b}$$

with $a = 1$ by gauge and $\log c_n = pn + q + \phi_3(n)$, $\log h_n = -3pn + r$. The case (D1bii′) has $h_n = -1/(c_nc_{n-1})$ with $\log c_n = pn + q + \phi_4(n)$. Finally when a cube is present in the numerator we have, case (D1biii),

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = (1 - c_ny_n)(1 - c_{n+1}y_n), \tag{112a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = h_n(x_n - c_{n-1})(x_n - c_n)(x_n - c_{n+1}) \tag{112b}$$

with $\log c_n = pn + q + \phi_4(n)$, $\log h_n = -4pn + r$. When moreover $\lambda = 0$, i.e., $hc^3 = -1$, we obtain the mapping $x_{n+1}x_nx_{n-1} - x_{n+1} - x_{n-1} + 1 = 0$, which we have already encountered and the solutions of which are periodic with period 5.

Case (D2) corresponds to $\beta = 0$ and $\delta + \lambda \neq 0$ which leads to the from

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = (1 - ay_n)(1 - by_n)(1 - cy_n), \tag{113a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = h_n(x_n - a)(x_n - b)(x_n - c) \tag{113b}$$

with a,b,c constant (and thus $abc = 1$ by gauge) and $\log h_n = pn + q$, an equation associated to the affine Weyl group $A_2^{(1)} + A_1^{(1)}$. If in addition $\lambda = 0$, which implies $h = 1$, we recover the non-numbered periodic mapping of period 2 encountered in case (B1c) of this section.

Case (D3) corresponds to $\delta + \lambda = 0$ while $\beta \neq 0$ and $\epsilon + \mu \neq 0$. Deautonomising in the case where there is no simplification, case (D3a), we find

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{(1 - ay_n)(1 - by_n)}{1 - f_ny_n}, \tag{114a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = h_nx_n(x_n - a)(x_n - b) \tag{114b}$$

with $ab = 1$ by gauge and $\log f_n = pn + q$, $\log h_n = -pn + r$, again related to $A_2^{(1)} + A_1^{(1)}$. Cases (D3bi) and (D3bii) correspond to one simplification, in the absence or presence of a square. In the former case we have

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = 1 - ay_n, \tag{115a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = h_nx_n(x_n - a)(x_n - b_n) \tag{115b}$$

with $a = 1$ by gauge and $\log b_n = pn + q + r(-1)^n$, $\log h_n = -2pn + s$. In the presence of a square, we have $a = b$ and solving for y we obtain for x the trivial mapping $h_nx_{n+1}x_nx_{n-1} = 1$, with h_n free.

Case (D4) corresponds to $\delta + \lambda = 0$ and $\beta = 0$ while $\epsilon + \mu \neq 0$. We have

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = (1 - ay_n)(1 - by_n), \tag{116a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = h_n x_n (x_n - a)(x_n - b) \tag{116b}$$

with $ab = 1$ by gauge and $\log h_n = pn + q$, an equation associated to $A_1^{(1)} + A_1^{(1)}$.

Case (D5) corresponds to $\delta + \lambda = 0$ and $\epsilon + \mu = 0$ while $\beta \neq 0$ (and also $\beta + \zeta \neq 0$). When no simplification is possible in the first equation, case (D5a), we find

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{1 - a y_n}{1 - f_n y_n}, \tag{117a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = h_n x_n^2 (x_n - a) \tag{117b}$$

with $a = 1$ by gauge and $\log f_n = pn + q$, $\log h_n = -pn + r$. When the first equation is simplified, i.e., $f = a$, which in fact corresponds to $\zeta = 0$, case (D5b), one can eliminate y and obtain for $1/x$ an equation identical to case (E1) of Section III. Similarly for (D6) where we take also $\beta = 0$ but $\zeta \neq 0$, i.e., $f = 0$, we find, for x the q-$\mathrm{P_I}$ equation obtained in case (C4) of Section III. Finally, when we take $\beta \neq 0$ but $\beta + \zeta = 0$ which is tantamount to taking $a = 0$ in (117) we find the same solution for f_n, h_n as in (117) but here a gauge would allow to remove one parameter, resulting to an equation associated to the affine Weyl group $A_1^{(1)}$, case (D7). The case $\beta = \zeta = 0$ reduces to the trivial equation $h_n x_{n+1} x_n x_{n-1} = 1$.

E. Case E

Case (E) corresponds to $\gamma = \kappa = 0$ with $\alpha \neq 0$. The general form of the mapping is

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{(\delta + \lambda) y_n^2 + (\mu + \epsilon + \alpha) y_n + (\beta + \zeta)}{\alpha y_n + \beta}, \tag{118a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{(\beta + \zeta) x_n^2 + (\mu + \epsilon + \alpha) x_n + (\delta + \lambda)}{\alpha x_n + \delta}. \tag{118b}$$

The case (E1) corresponds to $(\beta + \zeta)(\delta + \lambda) \neq 0$ as well as $\beta\delta\lambda\zeta \neq 0$. The generic case (E1a) when there is no simplification can be written most conveniently after a gauge choice which allows it to assume the form

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{c(1 - a y_n)(1 - y_n/a)}{y_n - f}, \tag{119a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{c(1 - a x_n)(1 - x_n/a)}{x_n - h}. \tag{119b}$$

Its deautonomisation leads to

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{c_n(1 - a y_n)(1 - y_n/a)}{y_n - f_n}, \tag{120a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n(1 - a x_n)(1 - x_n/a)}{x_n - h_n}, \tag{120b}$$

where a is a constant, $\log f_n = pn + q$, $\log h_n = pn + r$, $\log c_n = 2pn + s$, and $\log d_n = p(2n - 1) + s$. This equation as well as all equations below belonging to case (E1) are associated to the affine Weyl group $A_4^{(1)}$. When a simplification takes place in the first equation, for instance through $f = 1/a$, we have, in the absence of squares, case (E1bi) the non-autonomous form of which is

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = c_n (y_n - a), \tag{121a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n(1 - b_n x_n)(1 - a x_n)}{x_n - h_n} \tag{121b}$$

with $a = 1$ by gauge and $\log b_n = pn + q + r(-1)^n$, $\log c_n = 2pn + s$, $\log d_n = p(2n - 1) + s - 2r(-1)^n$, $\log h_n = pn + u - r(-1)^n$. When a square is present in the first equation we find, (E1bii),

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = c_n (y_n - a_n) \tag{122a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n(1 - a_n x_n)(1 - a_{n-1} x_n)}{x_n - h_n} \tag{122b}$$

with $\log a_n = pn + q + \phi_3(n)$, $\log h_n = pn + r + \phi_3(n+1)$, $\log c_n = 2pn + s - \phi_3(n)$, and $\log d_n = p(2n-1) + s + 2\phi_3(n+1)$ and here a gauge can be used in order to remove one parameter. When both equations are simplified once, two cases must be distinguished. We can see this easily be referring to Equation (121). Simplifying by the term containing b we find case (E1c)

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = c_n(y_n - a), \tag{123a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = d_n(x_n - 1/a), \tag{123b}$$

where $a = 1$ by gauge and $\log c_n = pn + q + r(-1)^n$, $\log d_n = pn + s + u(-1)^n$. This weakly asymmetric equation is precisely Equation (16) of Section III. Case (E1d) corresponds to a simplification of the term containing a and leads to

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = c_n(y_n - a_n), \tag{124a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = d_n(x_n - b_n) \tag{124b}$$

with $\log a_n = pn + q + \phi_3(n)$, $\log b_n = pn + r + \phi_3(n+1)$, $\log c_n = -3pn + r + s$ and $\log d_n = p(1 - 3n) + q + s$ where one parameter can be removed by the appropriate gauge.

If $(\beta + \zeta)(\delta + \lambda) \neq 0$, $\beta\delta\lambda \neq 0$ but $\zeta = 0$ we have case (E1′), which in the autonomous limit leads to $f = -c$. Without simplification we have, case (E1a′),

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{c_n(1 - a y_n)(1 - y_n/a)}{y_n + c_n}, \tag{125a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n(1 - a x_n)(1 - x_n/a)}{x_n - h_n}, \tag{125b}$$

where a is constant, $\log h_n = pn + q + r(-1)^n$, $\log c_n = p(2n+1) + s$ and $\log d_n = 2pn + s + 2r(-1)^n$. When one simplification is possible in (118b) we have, in the absence of squares, case (E1bi′), and taking $a = 1$ by gauge

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{c_n(1 - b_n y_n)(1 - y_n)}{y_n + c_n}, \tag{126a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = d_n(x_n - 1) \tag{126b}$$

with $\log c_n = 2pn + q + \phi_3(n)$, $\log b_n = pn + r - \phi_3(n)$, $\log d_n = p(2n-1) + q + \phi_3(n+1)$. When a square is present we have case (E1bii′)

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = c_n(y_n - a_n), \tag{127a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n(1 - a_n x_n)(1 - a_{n-1}x_n)}{x_n + d_n} \tag{127b}$$

with $\log a_n = pn + q + \phi_4(n)$, $\log c_n = 2pn + r + \phi_4(n+1) + \phi_4(n-1)$, and $\log d_n = p(2n-1) + r - \phi_4(n) - \phi_4(n-1)$ and where one parameter can be removed by gauge. A simplification in (118a) allows to solve for y in terms of x, in which case the system reduces to case (B1a) of Section III (and when a square is present, i.e., $a^2 = 1$, we find a case (B1c) of Section III). If $(\beta + \zeta)(\delta + \lambda) \neq 0$, $\lambda = 0$ and $\zeta = 0$ we have case (E1″), which in the autonomous limit leads to $f = -c$, $h = -d$,

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{c_n(1 - a y_n)(1 - y_n/a)}{y_n + c_n}, \tag{128a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n(1 - a x_n)(1 - x_n/a)}{x_n + d_n}, \tag{128b}$$

where a is a constant and $\log c_n = 2pn + q + \phi_3(n)$, $\log d_n = p(2n-1) + q + \phi_3(n+1)$. This weakly asymmetric case is nothing but case (B1bii) of Section III. Simplifications in either (128a) or (128b) lead to case (B1c) or (B1a) depending on whether a^2 is equal to 1 or not.

Case (E2) corresponds to $(\beta + \zeta)(\delta + \lambda) \neq 0$ and $\beta \neq 0$ but with $\delta = 0$ which entails that $\lambda \neq 0$. We first examine the case $\zeta \neq 0$. When there is no simplification we have case (E2a) which in the proper gauge can be written as

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{c(1 - ay_n)(1 - y_n/a)}{y_n - f}, \tag{129a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{c(1 - ax_n)(1 - x_n/a)}{x_n}. \tag{129b}$$

Its non-autonomous form is

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{c_n(1 - ay_n)(1 - y_n/a)}{y_n - f_n}, \tag{130a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n(1 - ax_n)(1 - x_n/a)}{x_n}, \tag{130b}$$

where a is a constant, $\log f_n = pn + q$, $\log c_n = 2pn + s$, and $\log d_n = p(2n - 1) + s$. This equation and all equations below belonging to case E2 are associated to the affine Weyl group $A_2^{(1)} + A_1^{(1)}$. When, moreover, $\lambda = 0$ we have case (E2a′) where $f = -c$ which is deautonomised to

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{c_n(1 - ay_n)(1 - y_n/a)}{y_n + c_n}, \tag{131a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n(1 - ax_n)(1 - x_n/a)}{x_n - h_n}, \tag{131b}$$

where a is constant $\log c_n = p(2n + 1) + s$, and $\log d_n = 2pn + s + r(-1)^n$. Simplifying the first equation, i.e., $f = 1/a$, we have, in the absence of squares, a non-autonomous form (E2bi)

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = c_n(y_n - a), \tag{132a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n(1 - b_n x_n)(1 - ax_n)}{x_n} \tag{132b}$$

with $a = 1$ by gauge and $\log b_n = pn + q + r(-1)^n$, $\log c_n = 2pn + s$, $\log d_n = p(2n - 1) + s - 2r(-1)^n$. When a square is present in the first equation we find, (E2bii),

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = c_n(y_n - a_n), \tag{133a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n(1 - a_n x_n)(1 - a_{n-1}x_n)}{x_n} \tag{133b}$$

with $\log a_n = pn + q + \phi_3(n)$, $\log c_n = 2pn + s - \phi_3(n)$, and $\log d_n = p(2n - 1) + s + 2\phi_3(n + 1)$ and where a gauge can be used in order to remove one parameter. When, moreover, $\lambda = 0$ in the last two cases, i.e., $f = -c = 1/a$, it turns out that one can solve the first equation for y and obtain for x an equation of the type (C1a) of Section III but corresponding to the choice $\alpha = \mu = 0$, and a trivial equation $x_{n+1}x_{n-1} = d_n x_n$ respectively.

Case (E3) corresponds to $\delta + \lambda = 0$ with $\alpha + \epsilon + \mu \neq 0$ and $\beta\delta \neq 0$, $\beta + \zeta \neq 0$. We first examine the case $\zeta \neq 0$ with the generic equation being of the form

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{c(1 - ay_n)}{y_n - f}, \tag{134a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{cx_n(x_n - a)}{x_n - h}. \tag{134b}$$

Its deautonomisation leads to, case (E3a),

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = \frac{c_n(1 - ay_n)}{y_n - f_n}, \tag{135a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n x_n(x_n - a)}{x_n - h_n}, \tag{135b}$$

where $a = 1$ by gauge and $\log c_n = 2pn + q$, $\log d_n = p(2n - 1) + q$, $\log f_n = pn + r$, $\log h_n = pn + s$ leading to an equation associated to the group $A_2^{(1)} + A_1^{(1)}$. The case $f = -c$, (E3a′), which is obtained by $\zeta = 0$, has the non-autonomous form

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{c_n(1-ay_n)}{y_n + c_n}, \tag{136a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{d_nx_n(x_n - a)}{x_n - h_n}, \tag{136b}$$

where $a = 1$, $\log c_n = p(2n+1) + s$, $\log d_n = 2pn + s + 2r(-1)^n$, and $\log h_n = pn + q + r(-1)^n$. Case (E3b) corresponds to a simplification in the second equation. We find

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{c_n(1-a_ny_n)}{y_n - f_n}, \tag{137a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = d_nx_n \tag{137b}$$

with $\log a_n = pn + q - r(-1)^n$, $\log c_n = 2pn + s + 2r(-1)^n$, $\log f_n = pn + u + r(-1)^n$, and $\log d_n = p(2n-1) + s$. When $\zeta = 0$, i.e., $f = -c$, we have case (E3b′),

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{c_n(1-a_ny_n)}{y_n + c_n}, \tag{138a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = d_nx_n \tag{138b}$$

with $\log c_n = 2pn + q + \phi_3(n)$, $\log a_n = pn + r - \phi_3(n)$, $\log d_n = p(2n-1) + q + \phi_3(n+1)$, and one parameter can be removed by gauge. Case (E3c) corresponds to a simplification in the first equation, whereupon, after deautonomisation we find

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = c_n, \tag{139a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{d_nx_n(1-a_nx_n)}{x_n - h_n} \tag{139b}$$

$\log a_n = pn + q + r(-1)^n$, $\log c_n = 2pn + s$, $\log d_n = p(2n-1) + s - 2r(-1)^n$ and $\log h_n = pn + u - r(-1)^n$ and, again, one parameter can be removed by gauge. When $\zeta = 0$, i.e., $c = d = 1$, case (E3c′), one can solve the first equation for y in terms of x obtaining an equation of type (C1b) of Section III, again corresponding to $\alpha = \mu = 0$. Finally we can simplify both equations, case (E3d)

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = c_n, \tag{140a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = d_nx_n, \tag{140b}$$

where $\log c_n = pn + q + r(-1)^n$ and $\log d_n = pn + s + u(-1)^n$ up to a gauge which can be used in order to remove one parameter. The case $\zeta = 0$, i.e., $c = 1$ again leads to the trivial equation $x_{n+1}x_{n-1} = d_nx_n$.

Case (E4) corresponds to $\delta + \lambda = 0$ with $\beta + \zeta \neq 0$, $\beta\delta \neq 0$ and $\alpha + \epsilon + \mu = 0$. We first examine the case $\zeta \neq 0$ with the generic equation being of the form

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{c}{y_n - f}, \tag{141a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{cx_n^2}{x_n - h}. \tag{141b}$$

Its deautonomisation leads to

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{c_n}{y_n - f_n}, \tag{142a}$$

$$(y_nx_n - 1)(x_ny_{n-1} - 1) = \frac{d_nx_n^2}{x_n - h_n}, \tag{142b}$$

where $\log c_n = 2pn + q$, $\log d_n = p(2n-1) + q$, $\log f_n = pn + r$, $\log h_n = pn + s$ and where one parameter can be removed by the proper gauge, leading to an equation associated to the group $A_1^{(1)} + A_1^{(1)}$. The case $f = -c$, (E4′), which is obtained by $\zeta = 0$, has the non-autonomous form

$$(x_{n+1}y_n - 1)(y_nx_n - 1) = \frac{c_n}{y_n + c_n}, \tag{143a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n x_n^2}{x_n - h_n} \tag{143b}$$

with $\log c_n = p(2n+1) + s$, $\log d_n = 2pn + s + 2r(-1)^n$ and $\log h_n = pn + q + r(-1)^n$ and a gauge is possible allowing to put one parameter to zero.

Case (E5) corresponds to $\beta + \zeta = \delta + \lambda = 0$ with $\beta\delta \neq 0$. Deautonomising we find

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{c_n y_n}{y_n - f_n}, \tag{144a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n x_n}{x_n - h_n} \tag{144b}$$

with $\log c_n = 2pn + q$, $\log d_n = p(2n-1) + q$, $\log f_n = pn + r$, and $\log h_n = pn + s$. Choosing properly the gauge we can put $s = r - p/2$ in which case we recover Equation (38), case (E1), of Section III. This equation is associated to the affine Weyl group $A_1^{(1)} + A_1^{(1)}$.

Case (E6) corresponds to $\beta = \delta = 0$ with $\zeta\lambda \neq 0$. After deautonomisation we find, with the proper gauge choice,

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{c_n (y_n - a)(y_n - 1/a)}{y_n}, \tag{145a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n (x_n - a)(x_n - 1/a)}{x_n}, \tag{145b}$$

where a is a constant and $\log c_n = 2pn + q$, $\log d_n = p(2n-1) + q$, again an equation associated to $A_1^{(1)} + A_1^{(1)}$, and which, in fact, is nothing but Equation (35) of Section III.

Case (E7) corresponds to $\beta + \zeta = 0$ and $\delta = 0$ with $\beta \neq 0$, $\alpha + \epsilon + \mu \neq 0$ and moreover $\lambda \neq 0$ (the case $\lambda = 0$ being linearisable). Upon deautonomisation and in the absence of simplification we obtain, case (E7a),

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{c_n y_n (1 - a y_n)}{y_n - f_n}, \tag{146a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n (x_n - a)}{x_n}, \tag{146b}$$

where a is a constant which can be gauged to 1 and $\log f_n = pn + q$, $\log c_n = 2pn + s$, $\log d_n = p(2n-1) + s$. With one simplification in the first equation, case (E7b), we find

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = c_n y_n, \tag{147a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n (a_n - x_n)}{x_n} \tag{147b}$$

with $\log a_n = -pn + q + r(-1)^n$, $\log c_n = 2pn + s$, and $\log d_n = p(3n-1) + s - q + r(-1)^n$. Both equations are associated to the affine Weyl group $A_1^{(1)} + A_1^{(1)}$.

Case (E8) corresponds to $\beta + \zeta = 0$, $\delta = 0$ and $\alpha + \epsilon + \mu = 0$, with $\beta\lambda \neq 0$. In non-autonomous form the equation becomes

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{c_n y_n^2}{y_n - f_n}, \tag{148a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{d_n}{x_n} \tag{148b}$$

with $\log c_n = 2pn + q$, $\log d_n = p(2n-1) + q$ and $\log f_n = pn + r$ and one parameter may be removed by gauge, an equation associated to the affine Weyl group $A_1^{(1)}$.

Several linearisable cases do also exist. Since $\gamma = \kappa = 0$ these cases correspond to either $\beta = \zeta = 0$ or, equivalently, $\delta = \lambda = 0$. In fact the general case is:

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{y_n - a}{z_n (z_n y_n - c)}, \tag{149a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{1 - a x_n}{z_n z_{n-1}}, \tag{149b}$$

and the corresponding non-autonomous, QRT-type, invariants are,

$$K_n = z_{n-1}(x_n y_{n-1} - 1) - c x_n + \frac{1 - a x_n}{z_{n-1}(x_n y_{n-1} - 1)}, \tag{150a}$$

$$\tilde{K}_n = z_n(x_n y_n - 1) - c x_n + \frac{(1 - a x_n)}{z_n(x_n y_n - 1)}. \tag{150b}$$

Its limiting cases are obtained either by taking a or c equal to 0, or by taking z infinitely large such that a/z^2 and c/z remain finite (the latter being allowed to go to zero), in which case the right hand sides of (149) become $1/(z_n^2 y_n - c z_n)$ and $x_n/(z_n z_{n-1})$ (with $c = 0$ in the second option just considered).

If the right hand-side of the first equation simplifies, the deautonomisation is different,

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{1}{z_n z_{n+1}}, \tag{151a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{z_n - a x_n}{z_{n-1} z_n z_{n+1}}, \tag{151b}$$

where, again, z_n is a free function, with invariants

$$K_n = z_{n-1}(x_n y_{n-1} - 1) - a x_n + \frac{z_n - a x_n}{z_n z_{n-1}(x_n y_{n-1} - 1)}, \tag{152a}$$

$$\tilde{K}_n = z_{n+1}(x_n y_n - 1) - a x_n + \frac{z_n - a x_n}{z_n z_{n+1}(x_n y_n - 1)}. \tag{152b}$$

When both a and c vanish in (149), or a vanishes in (151) the dependent variables disappear from the right hand side of both equations of each system (149) and (151), and thus both systems become trivial. In fact the two resulting systems are different, with the right hand-sides expressed as different combination of a single function z_n. But these two systems are just special cases of a more general but still trivial system with two free functions, one in the right hand side of each of its equations and is in fact just the weak-asymmetric mapping $(w_{n+1} w_n - 1)(w_n w_{n-1} - 1) = f_n$, with f_n free.

F. Case F

Case (F) as well as its subcases have $\alpha = \gamma = \kappa = 0$, which entails $\beta\delta \neq 0$, in which case we can put $\beta = \delta = 1$. The general form of the mapping is

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = (1 + \lambda) y_n^2 + (\mu + \epsilon) y_n + (1 + \zeta), \tag{153a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = (1 + \zeta) x_n^2 + (\mu + \epsilon) x_n + (1 + \lambda). \tag{153b}$$

When $(1 + \lambda)(1 + \zeta) \neq 0$ we have case (F1) which after a gauge choice can be deautonomised to

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = c_n(1 - a y_n)(1 - y_n/a), \tag{154a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = d_n(1 - a x_n)(1 - x_n/a), \tag{154b}$$

where a is constant and $\log c_n = pn + q$, $\log d_n = pn + r$ an equation which is associated to the affine Weyl group $A_2^{(1)} + A_1^{(1)}$. The case where $\lambda\zeta = 0$ lead back to Equation (20) encountered in Section III and which as noted there cannot have an extension involving secular terms. Case (F2) corresponds to $1 + \zeta = 0$ with $1 + \lambda \neq 0$ and $\mu + \epsilon \neq 0$. Choosing the proper gauge we find

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = c_n y_n(y_n - 1), \tag{155a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = d_n(1 - x_n) \tag{155b}$$

with $\log c_n = pn + q$, $\log d_n = pn + r$, an equation associated to the group $A_1^{(1)} + A_1^{(1)}$. A case (F2′) does also exist with $\lambda = 0$, i.e., $d_n = 1$. Here we can solve for x in terms of y and recover a mapping belonging to the case (C2b) of Section III. Case (F3) corresponds to $1 + \zeta = 0$ and $\mu + \epsilon = 0$ with $1 + \lambda \neq 0$. We have

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = c_n y_n^2, \tag{156a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = d_n \tag{156b}$$

with $\log c_n = pn + q$, $\log d_n = pn + r$ where one parameter can be removed through a gauge. This equation was first derived in Ref. 22. Finally case (F4) has $1 + \zeta = 1 + \lambda = 0$ but here $\mu + \epsilon \neq 0$,

$$(x_{n+1}y_n - 1)(y_n x_n - 1) = c_n y_n, \tag{157a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = d_n x_n \tag{157b}$$

with $\log c_n = 2pn + q$, $\log d_n = p(2n - 1) + q$ which is precisely equation (39) obtained in Section III.

This completes the exploration of limits and degenerate cases of family (IV). Contrary to the case of family (II) and its relative paucity of strongly asymmetric cases here we have more than 70 genuinely strongly asymmetric systems.

V. CONCLUSION

This paper has focused on the derivation of discrete Painlevé equations with particular emphasis on the forms we have dubbed "strongly asymmetric." The symmetric/asymmetric moniker is a direct reference to the QRT terminology. The QRT mappings are presented in either of two forms, the first, symmetric one, expressed as a single mapping while the asymmetric one is presented as a system of two coupled equations. The QRT mappings can be (and have been) classified in a set of families each of which possesses a canonical form which essentially dictates the form of the left hand side of the mapping. The majority of the discrete Painlevé equations to date have been derived by the method of deautonomisation whereupon one starts from an autonomous form (typically a QRT mapping belonging to one of the canonical families), assumes that the parameters are functions of the independent variable, and fixes their precise form by an integrability requirement. Till very recently the deautonomisation method has been applied to mappings of symmetric form, the rationale being that by deautonomising one can obtain coefficients with periodicities ranging from 2 to 8 and thus recover all possible asymmetries. While this argument is in principle valid, its practical application is not always adequate. The reason for this is that there exist systems where the right hand sides of the two equations have different functional forms. These are precisely the systems we call "strongly asymmetric." Given that we almost always work with rational mappings, the strongly asymmetric cases could be accommodated within a symmetric approach provided one allowed for coefficients which are zero or finite depending on the parity of the index. Unfortunately allowing for such a freedom in the deautonomisation procedure would entail considerable difficulties in the application of discrete integrability criteria. This explains why strongly asymmetric forms of discrete Painlevé equations have been largely ignored till recently.

The paper has been devoted to the study of strongly asymmetric forms of equations of q-type. The multiplicative nature of these equations is at the origin of another form of strong asymmetry as can be visualised directly in the case of Equation (15). The latter is the same as Equation (14) with the only difference that one of the prefactors in the right hand side is exactly equal to unity one time out of two. Thus, in the multiplicative case, a coefficient, the value of which is fixed to 1, leads to a strong asymmetry just as in the additive case, such an asymmetry was induced by a coefficient which was put to zero.

The present study has been limited to a thorough examination of the asymmetric forms existing in families II and IV. While the former possesses very few strongly asymmetric cases, the latter is particularly rich: more than 70 discrete Painlevé equations, most of them unknown till now, have been derived. All systems obtained here are related to some affine Weyl group of the Sakai classification. In each case we have indicated this association, which, for the richest equations obtained here, starts at the affine Weyl group $E_6^{(1)}$. More canonical families of QRT mappings do exist, leading upon deautonomisation to discrete Painlevé equations associated to affine Weyl groups lying higher than $E_6^{(1)}$ in the Sakai degeneration cascade, namely $E_7^{(1)}$ and $E_8^{(1)}$. Investigating the strongly asymmetric form of discrete Painlevé equations for these systems is a real challenge, given the bulk of the calculations involved. Still we hope to be able to rise to that challenge one day.

[1] B. Grammaticos and A. Ramani, *Discrete Painlevé Equations* (Springer LNP, 2004), Vol. 644, p. 245.
[2] A. Ramani and B. Grammaticos, Physica A **228**, 160 (1996).
[3] G. R. W. Quispel, J. A. G. Roberts, and C. J. Thompson, Physica D **34**, 183 (1989).
[4] B. Grammaticos, A. Ramani, and V. Papageorgiou, Phys. Rev. Lett. **67**, 1825 (1991).
[5] M. Bellon and C.-M. Viallet, Commun. Math. Phys. **204**, 425 (1999).
[6] J. Satsuma, K. Kajiwara, B. Grammaticos, J. Hietarinta, and A. Ramani, J. Phys. A: Math. Gen. **28**, 3541 (1995).
[7] A. Ramani, B. Grammaticos, T. Tamizhmani, and K. M. Tamizhmani, J. Phys. A: Math. Gen. **33**, 579 (2000).
[8] B. Grammaticos, A. Ramani, K. M. Tamizhmani, T. Tamizhmani, and J. Satsuma, J. Math. Phys. **55**, 053503 (2014).
[9] A. Ramani, S. Carstea, B. Grammaticos, and Y. Ohta, Physica A **305**, 437 (2002).
[10] A. Ramani, B. Grammaticos, and J. Hietarinta, Phys. Rev. Lett. **67**, 1829 (1991).
[11] M. Jimbo and H. Sakai, Lett. Math. Phys. **38**, 145 (1996).
[12] H. Sakai, Commun. Math. Phys. **220**, 165 (2001).
[13] M. D. Kruskal, K. M. Tamizhmani, B. Grammaticos, and A. Ramani, Regul. Chaotic Dyn. **5**, 273 (2000).
[14] K. M. Tamizhmani, B. Grammaticos, A. S. Carstea, and A. Ramani, Regul. Chaotic Dyn. **9**, 13 (2004).
[15] T. Tsuda, K. Okamoto, and H. Sakai, Math. Ann. **331**, 713 (2005).
[16] A. Ramani, B. Grammaticos, and T. Tamizhmani, J. Phys. A: Math. Gen. **33**, 3033 (2000).
[17] K. Kajiwara, M. Noumi, and Y. Yamada, Lett. Math. Phys. **62**, 259 (2002).
[18] A. Ramani, R. Willox, B. Grammaticos, A. S. Carstea, and J. Satsuma, Physica A **347**, 1 (2005).
[19] B. Grammaticos, A. Ramani, and Y. Ohta, J. Nonlinear Math. Phys. **10**, 215 (2003).
[20] A. Ramani, B. Grammaticos, J. Satsuma, and N. Mimura, J. Phys. A: Math. Gen. **44**, 425201 (2011).
[21] A. Ramani, B. Grammaticos, K. M. Tamizhmani, and T. Tamizhmani, J. Phys. A: Math. Gen. **46**, 065201 (2013).
[22] B. Grammaticos, T. Tamizhmani, A. Ramani, A. S. Carstea, and K. M. Tamizhmani, J. Phys. Soc. Jpn. **71**, 443 (2002).