Discrete Applied Mathematics 159 (2011) 1053–1057

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Total domination dot-stable graphs

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ARTICLE INFO

Article history:
Received 28 June 2010
Received in revised form 28 February 2011
Accepted 3 March 2011
Available online 31 March 2011

Keywords:
Domination dot-critical
Total domination
Total domination dot-critical graph
Total domination dot-stable graph

ABSTRACT

A set S of vertices in a graph G is a total dominating set if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a total dominating set of G is the total domination number of G. Two vertices of G are said to be dotted (identified) if they are combined to form one vertex whose open neighborhood is the union of their neighborhoods minus themselves. We note that dotting any pair of vertices cannot increase the total domination number. Further we show it can decrease the total domination number by at most 2. A graph is total domination dot-stable if dotting any pair of adjacent vertices leaves the total domination number unchanged. We characterize the total domination dot-stable graphs and give a sharp upper bound on their total domination number. We also characterize the graphs attaining this bound.

1. Introduction

For notation and graph theory terminology not defined here, we refer the reader to [6]. Let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from the context, we simply write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively. The degree of a vertex v in G is $deg_G(v) = |N_G(v)|$. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v\in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. An *S-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to v but to no other vertex of S. The set of all S-external private neighbors of $v \in S$ is called the *S-external private neighbor set* of v and is denoted as $\text{epn}(v, S)$. A set S is a *dominating set* if $N[S] = V$, and a *total dominating set* if $N(S) = V$. A total dominating set, abbreviated as TDS, was first defined by Cockayne et al. [5]. Every graph without isolated vertices has a TDS, since $V(G)$ is such a set. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a TDS of G, and a TDS of G of cardinality $\gamma_t(G)$ is called a *$\gamma_t(G)$-set*. For more details on total domination in graphs, see the two books [6,7] and a recent survey [9].

Graphs for which a given graph parameter changes (respectively, remains the same) upon a graph modification are called critical (respectively, stable). Both criticality and stability of the domination number of a graph have been studied for various graph modifications including adding an edge, removing an edge, and removing a vertex. Chapter 5 of [6] is a survey of such results. Burton and Sumner [1] studied the effects of a different graph modification, which they called dotting, on the domination number of a graph. In particular they investigated the criticality question, that is, they considered when dotting changed the domination number. *Dotting* (also referred to as *identifying* in the literature) can be described as combining two vertices to form one vertex whose open neighborhood is the union of their neighborhoods minus themselves. More formally, *dotting* (respectively, *identifying*) two vertices u and v forms a new vertex, denoted as (uv), whose open neighborhood is $(N(u) \cup N(v)) \setminus \{u, v\}$. The graph formed from G by dotting u and v is denoted as $G.uv$. Henning and Rad [10] investigated

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2011.03.006

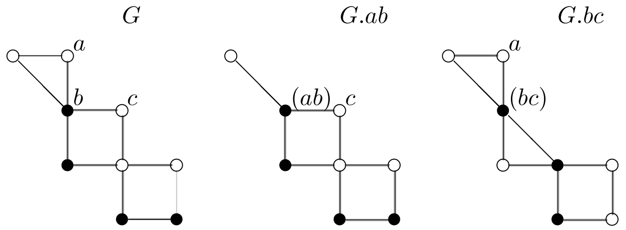

Fig. 1. Dotting example (the darkened vertices represent γ_t-sets of the respective graphs).

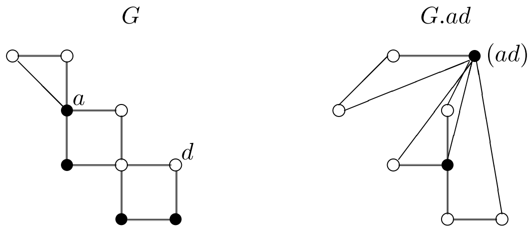

Fig. 2. Example where $\gamma_t(G.ad) = \gamma_t(G) - 2$.

criticality of the total domination number when the graph is modified by dotting adjacent vertices. In this paper, we consider stability of the total domination number with the dotting modification.

Since a graph with a TDS can have no isolated vertices, for the remainder of this paper, we require that dotting two vertices in a graph G does not produce an isolate in $G.uv$. We note that dotting two vertices cannot increase the total domination number, but as shown in Fig. 1, the total domination number can decrease or remain the same. For the graphs in Fig. 1, $\gamma_t(G) = 4$, $\gamma_t(G.ab) = 4$, and $\gamma_t(G.bc) = 3$. In Fig. 2, dotting vertices a and d decreases the total domination number by 2. Our first result shows that in fact, dotting any pair of vertices can decrease the total domination number by at most 2.

Proposition 1. *Let G be a connected graph of order $n \geq 3$. For any two vertices a and b, $\gamma_t(G) - 2 \leq \gamma_t(G.ab) \leq \gamma_t(G)$.*

Proof. Clearly dotting two vertices does not increase the total domination number, so the upper bound holds.

For the lower bound, let M be a $\gamma_t(G.ab)$-set. We consider two cases.

Case 1: $(ab) \in M$. Let $S = (M \setminus \{(ab)\}) \cup \{a, b\}$. If $N_G(a) \cap S \neq \emptyset$ and $N_G(b) \cap S \neq \emptyset$, then S is a TDS of G, implying that $\gamma_t(G) \leq |S| = |M| + 1 = \gamma_t(G.ab) + 1$. Without loss of generality, assume $N_G(a) \cap S = \emptyset$. Since M is a TDS of $G.ab$, we know that (ab) has at least one neighbor in M. It follows that $N_G(b) \cap S \neq \emptyset$. Since G has no isolates, a has a neighbor, say y, in $V \setminus S$. Thus $S \cup \{y\}$ is a TDS of G, and $\gamma_t(G) \leq |S| + 1 = |M| + 2 = \gamma_t(G.ab) + 2$.

Case 2: $(ab) \notin M$. Consider M in G. If M total dominates G, then $\gamma_t(G) \leq \gamma_t(G.ab)$. Assume that M does not total dominate G. Since M is a TDS of $G.ab$, renaming the vertices if necessary, we may assume that M total dominates $G - \{a\}$ but does not dominate a. Let $x \in N_G(a)$. Then $x \in V \setminus M$ and x has a neighbor in M. Therefore $S = M \cup \{x\}$ is a TDS of G, and hence $\gamma_t(G) \leq |M| + 1 = \gamma_t(G.ab) + 1$. □

Considering the cases in the proof, we deduce the following corollary.

Corollary 2 (*[10]*). *Let G be a connected graph with order $n \geq 3$. If a and b are adjacent vertices of G, then $\gamma_t(G) - 1 \leq \gamma_t(G.ab) \leq \gamma_t(G)$.*

As defined in [1], a graph is *domination dot-critical* if dotting any pair of adjacent vertices decreases the domination number. This concept was further investigated in [2,4,3,11]. *Total domination dot-critical graphs* were defined similarly and studied in [10]. Here we also restrict our attention to dotting only adjacent vertices, and we study stability for total domination. A graph G is *total domination dot-stable*, abbreviated as γ_t*-dot-stable*, if for any pair of *adjacent* vertices $a, b \in V(G)$, $\gamma_t(G.ab) = \gamma_t(G)$.

In Section 2, we characterize the total domination dot-stable graphs. Then, in Section 3, we give an upper bound on the total domination number of a total domination dot-stable graph and characterize the graphs achieving this bound.

2. Total domination dot-stable graphs

We first show that no graph with an odd total domination number is stable.

Proposition 3. *If G is γ_t-dot-stable, then $\gamma_t(G)$ is even.*

Proof. We prove the contrapositive. Assume that $\gamma_t(G)$ is odd, and let S be a $\gamma_t(G)$-set. Since $\gamma_t(G)$ is odd, $G[S]$ has an odd component. The new set formed by dotting any two adjacent vertices, say x and y, in the odd component of S is a TDS for $G.xy$, so $\gamma_t(G.xy) < |S| = \gamma_t(G)$. Hence G is not γ_t-dot-stable. □

Theorem 4. *A graph G is γ_t-dot-stable if and only if for every $\gamma_t(G)$-set S the induced subgraph $G[S]$ is a set of independent edges.*

Proof. The result is straightforward for graphs with total domination number 2, so assume that $\gamma_t(G) \geq 4$. Let G be a γ_t-dot-stable graph, and let S be a $\gamma_t(G)$-set. Let x_1 and x_2 be adjacent vertices in S. Notice that because G is γ_t-dot-stable, the set $S' = (S \setminus \{x_1, x_2\}) \cup \{(x_1x_2)\}$ is not a TDS of $G.x_1x_2$. Since S' is a dominating set of $G.x_1x_2$, it must be the case that (x_1x_2) is not total dominated. Thus in G, x_1 and x_2 have no neighbors in $S \setminus \{x_1, x_2\}$. Since S, x_1, and x_2 are arbitrary, the necessity follows.

For the sufficiency, assume that every $\gamma_t(G)$-set induces a set of independent edges. Assume, to the contrary, that G is not γ_t-dot-stable. Thus there exists a pair of adjacent vertices a and b such that $\gamma_t(G.ab) = \gamma_t(G) - 1$. Let M be a $\gamma_t(G.ab)$-set. We consider two cases.

Case 1: $(ab) \in M$.
Since M is a TDS of $G.ab$, (ab) is adjacent to a vertex, say x, in M. Notice that, in G, x is adjacent to a or b. Thus $M' = (M \setminus \{(ab)\}) \cup \{a, b\}$ is a TDS of G with cardinality $\gamma_t(G.ab) + 1 = \gamma_t(G)$. But then M' is a $\gamma_t(G)$-set where $\{x, a, b\}$ induces a P_3 or a K_3 in $G[M']$, contradicting that every $\gamma_t(G)$-set induces an independent set of edges.

Case 2: $(ab) \notin M$.
Since M is a TDS of $G.ab$; (ab) is adjacent to a vertex, say x, in M. Also x is adjacent to a vertex, say y, in M. Thus, renaming the vertices if necessary, we may assume that x is adjacent to a in G. It follows that $M' = M \cup \{a\}$ is a $\gamma_t(G)$-set for which $\{a, x, y\}$ induces a P_3 or a K_3 in $G[M']$, again a contradiction. $\square$

We conclude this section with a characterization of the γ_t-dot-stable paths and cycles. The total domination number of paths and cycles is well-known.

Observation 5 ([8])**.** *For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.*

Since dotting two vertices of a P_n (respectively, C_n) yields a P_{n-1} (respectively, C_{n-1}), Observation 5 yields the following corollary.

Corollary 6. *Paths P_n and cycles C_n with $n \geq 3$ are γ_t-dot-stable if and only if $n \equiv 0, 3 \bmod 4$.*

3. An upper bound

In this section, we establish a sharp upper bound on the total domination number of γ_t-dot-stable graphs that offers a slight improvement over the following known upper bound for arbitrary graphs.

Theorem 7 ([8])**.** *For any graph G with no isolates, $\gamma_t(G) \leq \frac{2n}{3}$.*

To help establish our upper bound, we first state a couple of observations and a lemma. We note that a star is the only connected graph for which every TDS contains at least one leaf. In particular we observe:

Observation 8. *If G is a connected graph with $\gamma_t(G) \geq 4$, then G has a $\gamma_t(G)$-set containing no leaves.*

Theorem 4 and Observation 8 imply the following result.

Observation 9. *If G is a connected, γ_t-dot-stable graph with $\gamma_t(G) \geq 4$, then there exists a $\gamma_t(G)$-set S such that each vertex in S has a neighbor in $V \setminus S$.*

For a set S, we say that a vertex $x \in V \setminus S$ is *component common* if x is adjacent to vertices in different components of $G[S]$.

Lemma 10. *Let G be a γ_t-dot-stable graph with $\gamma_t(G) \geq 4$ and $\gamma_t(G)$-set S. If u and v are adjacent vertices in S and u is adjacent to a component common vertex, then $\mathrm{epn}(v, S) \neq \emptyset$.*

Proof. Let u and v be adjacent vertices in S. Without loss of generality, assume that u is adjacent to a component common vertex, say x. Notice that $S' = (S \setminus \{v\}) \cup \{x\}$ is a $\gamma_t(G)$-set if $\mathrm{epn}(v, S) = \emptyset$. Since by definition x is adjacent to a vertex in some other component of $G[S]$, $G[S']$ does not induce a set of independent edges, contradicting Theorem 4. Thus $\mathrm{epn}(v, S) \neq \emptyset$. $\square$

For γ_t-dot-stable graphs, we can now make a slight improvement on the upper bound of Theorem 7.

Theorem 11. *If G is a γ_t-dot-stable graph of order n and $\gamma_t(G) \geq 4$, then $\gamma_t(G) \leq \frac{2(n-1)}{3}$.*

Proof. Let G be a γ_t-dot-stable graph. By Observation 9, we choose S to be a $\gamma_t(G)$-set such that every vertex in S has a neighbor in $V \setminus S$. By Proposition 3, $\gamma_t(G) = 2k$ for some integer $k \geq 2$. By Theorem 4, $G[S] = kK_2$. Label the vertices of S as u_i and v_i, where u_i is adjacent to v_i in the ith component of $G[S]$, $1 \leq i \leq k$. To establish our bound, we count the vertices of $V \setminus S$. If $u_i \in S$ and $\mathrm{epn}(u_i, S) \neq \emptyset$, then we associate a unique vertex from $\mathrm{epn}(u_i)$ with u_i. We do the same for $v_i \in S$. Let P be the set of S-external private neighbors.

Suppose that $\mathrm{epn}(u_i, S) = \emptyset$ and $\mathrm{epn}(v_i, S) = \emptyset$ for some pair u_iv_i. Then Lemma 10 implies that neither u_i nor v_i is adjacent to a component common vertex in $V \setminus S$. Hence, $N(u_i) \cap (V \setminus S) = N(v_i) \cap (V \setminus S)$. Let A_i denote $N(u_i) \cap (V \setminus S)$. We claim that $|A_i| \geq 2$. To see this, assume that $A_i = \{x\}$. Since G is connected with $\gamma_t(G) \geq 4$, x has a neighbor, say y, in $V \setminus S$. Moreover, since S is a TDS, y has a neighbor in $S \setminus \{u_i, v_i\}$. Hence, $(S \setminus \{u_i, v_i\}) \cup \{x, y\}$ is a TDS of G which does not induce a set of independent edges, contradicting Theorem 4. Therefore $|A_i| \geq 2$. Let $A = \bigcup_{i=1}^{k} A_i$. Note that A does not contain a component common vertex, $A_i \cap A_j = \emptyset$ for all i and j, and $A \cap P = \emptyset$. Thus again we can count two unique vertices in A for each such u_iv_i component.

Hence the only vertices in S that we have not associated with a unique vertex in $V \setminus S$ are the ones with no S-external private neighbor while their neighbor in S has an S-external private neighbor. Renaming the vertices if necessary, we may assume that $\mathrm{epn}(u_i, S) \neq \emptyset$ and $\mathrm{epn}(v_i, S) = \emptyset$. Recall that for u_i, we have associated a unique vertex in P.

First assume that v_i is not adjacent to a component common vertex. Then $N(v_i) \cap (V \setminus S) \subseteq N(u_i) \cap (V \setminus S)$. Let $B_i = |N(v_i) \cap N(u_i) \cap (V \setminus S)|$. Note that for this case, we can count a unique vertex of B_i associated with v_i. Let $B = \bigcup_{i=1}^{k} B_i$. Note that B does not contain a component common vertex, $B_i \cap B_j = \emptyset$ for all i and j, $B \cap P = \emptyset$, and $B \cap A = \emptyset$.

The only remaining case is where v_i is adjacent to a component common vertex. Let j be the number of such vertices in S, and let c be the number of component common vertices in $V \setminus S$. Now we may assume that $c \geq 1$; for otherwise, $|V \setminus S| = |A| + |B| + |P| \geq |S|$ implying that $\gamma_t(G) \leq \frac{n}{2}$, and we are finished. Moreover, Lemma 10 implies that $j \leq \frac{|S|}{2} = k$. Hence $n = |S| + |V \setminus S| = 2k + |A| + |B| + |P| + c \geq 2k + (2k - j) + c = 4k - j + c$. To minimize $4k - j + c$, we must maximize j and minimize c. Thus $n \geq 4k - k + 1 = 3k + 1$. Hence $\gamma_t(G) = 2k \leq \frac{2(n-1)}{3}$. □

We conclude by characterizing the graphs achieving the bound of Theorem 11. Let $\mathcal{G}_4$ be the family of connected graphs of order 4. Let $\mathcal{T}$ be the family of trees that can be obtained by subdividing each edge of a non-trivial star exactly twice.

Theorem 12. *A connected γ_t-dot-stable graph G of order n has $\gamma_t(G) = 2(n-1)/3$ if and only if G is the cycle C_7 or $G \in \mathcal{G}_4 \cup \mathcal{T}$.*

Proof. If $G \in \mathcal{T} \cup \{C_7\}$, then it is straightforward to see that $\gamma_t(G) = 2(n-1)/3$ and every $\gamma_t(G)$-set induces a set of independent edges. By Theorem 4, G is γ_t-dot-stable. If $G \in \mathcal{G}_4$, then $\gamma_t(G) = 2 = 2(n-1)/3$ and G is γ_t-dot-stable.

For the sufficiency, assume that G is a connected γ_t-dot-stable graph with $\gamma_t(G) = 2(n-1)/3$. By Proposition 3, $\gamma_t(G)$ is even. If $\gamma_t(G) = 2$, then $n = 4$. Thus $G \in \mathcal{G}_4$, and the result holds.

Assume that $\gamma_t(G) = 2k \geq 4$. By Observation 9, G has a $\gamma_t(G)$-set where every vertex in the set has a neighbor outside the set. Among all $\gamma_t(G)$-sets with this property, let S be one containing the maximum number of vertices adjacent to component common vertices. Label the vertices of S as we did in the proof of Theorem 11. We prove a series of claims.

Claim 1. *At least one vertex of each component of $G[S]$ has an S-external private neighbor.*

Proof of claim. Suppose to the contrary that for some i, $\mathrm{epn}(u_i, S) = \mathrm{epn}(v_i, S) = \emptyset$. Lemma 10 implies that neither u_i nor v_i is adjacent to a component common vertex in $V \setminus S$. Since every vertex in S has a neighbor in $V \setminus S$, $N(u_i) \cap (V \setminus S) = N(v_i) \cap (V \setminus S)$. Let $X = N(u_i) \cap (V \setminus S)$. Moreover since G is connected with $\gamma_t(G) \geq 4$, at least one vertex, say x, in X has a neighbor, say y, in $(V \setminus S) \setminus X$. Note that x is not component common in $V \setminus S$ and y is dominated by $S \setminus \{u_i, v_i\}$. Then $S' = (S \setminus \{u_i\}) \cup \{x\}$ is a $\gamma_t(G)$-set where every vertex in S' has a neighbor in $V \setminus S'$ and $\{x, v_i\}$ is a component in $G[S']$. Since y is adjacent to x and at least one vertex in $S' \setminus \{x, v_i\}$, y is a component common vertex for the set S'. It follows that S' has at least one more vertex, namely x, with a component common neighbor than S has, contradicting our choice of S. Hence at least one vertex of each component of $G[S]$ has an external private neighbor with respect to S. □

Renaming the vertices if necessary, we may assume that $\mathrm{epn}(u_i, S) \neq \emptyset$ for $1 \leq i \leq k$. Let $\mathrm{epn}(u_i, S) = \{x_i\}$. Since $\gamma_t(G) = 2k = 2(n-1)/3$, we have $n = 3k + 1$ and $|V \setminus S| = k + 1$. Thus there is only one vertex, say x, in $V \setminus S$ that can be adjacent to v_i for $1 \leq i \leq k$. Since $k \geq 2$, it follows that $v_ix \in E(G)$ for $1 \leq i \leq k$ and x is a component common vertex. Note that $N(v_i) \cap (V \setminus S) = \{x\}$ and $\mathrm{epn}(v_i, S) = \emptyset$ for $1 \leq i \leq k$.

Claim 2. *No u_i is adjacent to x.*

Proof of claim. Suppose to the contrary that $u_ix \in E(G)$ for some i. Then $(S \setminus \{v_i\}) \cup \{x\}$ is a $\gamma_t(G)$-set that does not induce a set of independent edges, a contradiction. □

Claim 3. *If $\gamma_t(G) \geq 6$, then $V \setminus S$ is an independent set.*

Proof of claim. Suppose to the contrary that S is not an independent set. If $x_ix_j \in E(G)$ for some i and j, then $(S \setminus \{u_i, v_i, u_j, v_j\}) \cup \{x, x_i, x_j\}$ is a $\gamma_t(G)$-set with cardinality less than $\gamma_t(G)$, a contradiction. If $x_ix \in E(G)$ for some i, then $(S \setminus \{u_i\}) \cup \{x\}$ is a $\gamma_t(G)$-set that does not induce a set of independent edges, contradicting Theorem 4. □

Returning to the proof of the theorem, note that if $\gamma_t(G) \geq 6$, then $G \in \mathcal{T}$ and we are finished. Assume that $\gamma_t(G) = 4$. If $x_ix \in E(G)$ for some i, then as before G has a $\gamma_t(G)$-set that does not induce a set of independent edges, contradicting Theorem 4. Hence $x_ix \notin E(G)$. If $x_1x_2 \in E(G)$, then $G = C_7$. If $x_1x_2 \notin E(G)$, then $G \in \mathcal{T}$. In both cases, the result holds. □

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