Vibrations of Circular Plates Resting on Elastic Foundation with Elastically Restrained Edge Against Translation

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Abstract: The present paper deals with exact solutions for the free vibration characteristics of thin circular plates resting on Winkler-type elastic foundation based on the classical plate theory elastically restrained against translation. Parametric investigations are carried out for estimating the influence of edge restraint against translation and stiffness of the elastic foundation on the natural frequencies of circular plates. The elastic edge restraint against translation and the presence of elastic foundation has been found to have a profound influence on vibration characteristics of the circular plate undergoing free transverse vibrations. Computations are carried out for natural frequencies of vibrations for varying values of translational stiffness ratio and stiffness parameter of Winkler-type foundation. Results are presented for twelve modes of vibration both in tabular and graphical form for use in the design. Extensive data is tabulated so that pertinent conclusions can be arrived at on the influence of translational edge restraint and the foundation stiffness ratio of the Winkler foundation on the natural frequencies of uniform isotropic circular plates.

Keywords: Plate; Frequency; Elastic edge; Translational stiffness; Elastic foundation.

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اهتزازات الالواح الدائرية المثبتة على قاعدة مرنة مع حافة مقيدة المرونة اثناء الحركة

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الملخص : تتناول الدراسة البحثية الحلول الصحيحة لخصائص الاهنزازات الحرة للالواح الدائرية الرقيقة المثبتة على قاعدة مرنة من نوع وينكلر ذات حافة مقيدة المرونة اثناء الحركة بموجب نظرية الالواح التقليدية. تم إجراء ابحاث معاملية (بارامترية) لتقديرمدى تأثير تقييد الحافة اثناء الحركة وصلابة القاعدة المرنة على الترددات الطبيعية للألواح الدائرية. و توصل البحث الى أن الحافة مقيدة المرونة أثناء عملية الحركة مع وجود القاعدة المرنة لها تأثير كبير على خصائص اهتزاز اللوح الدائري التي يتعرض لاهتزازات عرضية حرة. وتم إجراء الحسابات للترددات الطبيعية للاهتزازات لقيم متفاوتة من معدل الصلابة المتعدية ومعامل صلابة القاعدة من نوع وينكلر. و قدم البحث النتائج لاثني عشر نمطا من أنماط الاهتزازات في شكل جداول و رسومات بيانية لاستخدامها في التصميم. كما تم جدولة بيانات شاملة بحيث يمكن التوصل منها إلى استنتاجات ذات صلة بتأثير تقييد الحافة المتنقلة ونسبة صلابة القاعدة من نوع وينكلر على الترددات الطبيعية للوحات الدائرية المتماثلة الموحدة.

الكلمات المفتاحية : لوح، التردد ،حافة مرنة، صلابة متنقلة ، قاعدة مرنة

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Nomenclature

h	Thickness of a plate, mm
a	Radius of a plate, mm
ν	Poisson's ratio
E	Young's modulus, N/mm²
ρ	Density of a material, kg/mm³
$W(r,\theta)$	Transverse deflection of the plate, mm
D	Flexural rigidity of a plate, N.mm²
K_T	Translational spring stiffness, N/mm
K_w	Stiffness of Winkler foundation, N/mm²/mm
T	Translational spring stiffness ratio
ξ	Foundation stiffness ratio
ω_{mn}	Natural frequency, rad/sec
λ_{mn}	Eigenvalue without foundation, cycles/sec
λ_{mn}^{*}	Eigenvalue with Winkler foundation, cycles/sec m, n positive integers corresponding to the number of concentric circles and nodal diameters in each flexural mode.

1. Introduction

Circular plates resting on elastic foundation have a wide range of applications in the static and dynamic design of linear/nonlinear vibration absorbers, dynamic exciters, telephone receiver diaphragms, computer discs, printed circuit boards etc. (Leissa 1969). Due to the essential use of vibration data in the computation of stresses in such structures, reliable prediction of vibration data is of great importance. In view of its importance in engineering design, the problem of vibration of circular plates on elastic foundation has attracted the focus and attention of many researchers.

Some of the recent studies have reinstated the classical approach efficiency in analyzing the vibrations of variety of structures. Circular plate problems allow for significant simplification in view of their symmetry, but still many difficulties arise when the plate boundary conditions become complex involving linear and rotational restraints. A recent review of literature shows that very few studies exist on the study of circular plates resting on elastic foundations. Wang and Wang (2003), who observed the switching between axisymmetric and asymmetric vibration modes, have recently investigated the effect of internal elastic translational supports.

The vibration characteristics of plates resting on an elastic medium are different from those of the plates supported only on the boundary. Leissa (1993) discussed the vibration of a plate supported laterally by an elastic foundation. Leissa deduced that the effect of Winkler foundation merely increases the square of the plate natural frequency by a constant. Salari *et al.* (1987) speculated the same conclusion. Ascione and Grimaldi (1984) studied unilateral frictionless contact between a circular plate and a Winkler foundation using a variational formulation. Leissa (1969), who tabulated a frequency parameter for four vibration modes of a simply supported circular plate with varying rotational stiffness, presented one of the earliest formulations of this problem. Kang and Kim (1996) presented an extensive review of the modal properties of the elastically restrained beams and plates.

Zheng and Zhou (1988) studied the large deflection of a circular plate resting on Winkler foundation. Ghosh (1997) studied the free and forced vibration of circular plates on Winkler foundation by an exact analytical method. (Chang and Wickert (2001); Kim *et al.* (2001) and Tseng and Wickert (1994) studied the dynamic characteristics of bolted flange connections involving circular plates displaying beating type of repeat frequencies and typical mode shapes of vibration. Bolted flange connections are practically the best examples for the elastically restrained boundary conditions of circular plates on partial or continuous Winkler type elastic foundation.

The most general soil model used in practical applications is the Winkler (1867) model in which the elastic medium below a structure is represented by a system of identical but mutually independent elastic linear springs. Recent investigations have reiterated the efficiency of the classical approach (Soedel 1993) in analyzing the behavior of structures under vibrations. There are other works (Weisman 1970; Dempsey *et al.* 1984; Celep *et al.* 1988) dealing with the study of plates on a Winkler foundation. In general, those dealing with vibrating plates, shells and beams are concerned with the determination of eigenvalues and mode shapes (Leissa 1969).

A good number of studies was conduced (Wang and Lin 1996; Kim and Kim 2001; Yayli *et al.* 2014) using the method of Fourier series for estimating the frequencies of beams with generally restrained end conditions including the effect of elastic soil foundation. The method includes the use of Stoke's transformation in suitably modifying the complex boundary conditions. Very much similar to the dynamic stiffness matrix approach, the elements of the matrix involving infinite Fourier series are explicitly obtained in these studies. The determinant of this matrix for each case considered leads to the frequency equation and the same can be solved using well known numerical methods. The results obtained for various elastically restrained beam cases in these studies tallied well with those available in the literature establishing the efficiency of this method.

In view of the necessity of using complex combinations of rotational and translational springs at the circular plate boundary to suitably simulate the practical non-classical boundary connections being adopted in a wide range of industrial applications (Bhaskara Rao *et al.* 2009; Bhaskara Rao *et al.* 2010; Lokavarapu and Chellapilla 2013; Bhaskara Rao *et al.* 2015; Lokavarapu *et al.* 2015; Rao *et al.* 2016), the use of exact method of solution becomes imperative

and hence the same is adopted in this paper. Even though the method adopted here is classical, the particular case of vibration of elastically restrained circular plate resting on elastic foundation considered here is not addressed within the available literature.

Utilizing the classical plate theory, this paper deals with an exact method of solution for the analysis of thin circular plate free transverse vibrations that is elastically restrained against translation and resting on Winkler-type elastic foundation. For estimating the influence of edge restraint against the elastic foundation translation and stiffness on circular plates natural frequencies, parametric investigations are carried out by varying the values of elastic edge restraint stiffness against the elastic foundation translation and stiffness. The results obtained on natural frequencies of vibration clearly show that the vibration characteristics of the circular plate undergoing free transverse vibrations are found to be profoundly influenced by these variations. Computations are carried out for natural frequencies of vibrations for varying values of translational stiffness ratio and stiffness parameter of Winkler-type foundation. The results that are presented for twelve modes of vibration both in tabular and graphical forms are believed to be quite useful for designers in this area.

2. Mathematical Formulation of the System

The considered elastic thin circular plate is supported on a Winkler foundation as shown in Fig. 1. In the classical plate theory (Leissa 1969), the following fourth order differential equation describes free flexural vibrations of a thin circular uniform plate:

$$D.\nabla^4 w(r,\theta,t) + \rho h \partial^2 w(r,\theta,t)/\partial t^2 = 0 \qquad (1)$$

where $D = Eh^3/12(1-v^2)$ is the flexural rigidity of a plate and a,h,ρ,E,v are the plate's radius, thickness, density, Young's modulus and Poisson's ratio, respectively.

The homogeneous equation for Kirchhoff's plate on one parameter elastic foundation is given by the following equation:

$$D.\nabla^4 w(r,\theta,t) + K_w w(r,\theta,t) + \qquad (2)$$
$$\rho h \partial^2 w(r,\theta,t)/\partial t^2 = 0$$

Displacement in (2) can be presented as a combination of spatial and time dependent components as follows:

Let $w(r,\theta,t) = W(r,\theta)e^{i\omega t}$ (3)

Now substitute (3) in (2)

$$D.\nabla^4 W(r,\theta) + (K_w - \rho h\omega^2).W(r,\theta) = 0 \qquad (4)$$

The solution of the equation takes the following form

$$W_{mn}(r,\theta) = A_{mn}\left[\frac{J_n\left(\frac{\lambda_{mn}r}{a}\right) + C_{mn}I_n}{\left(\frac{\lambda_{mn}r}{a}\right)}\right].cos\, n\theta\,, n>0$$

$$(5)$$

where A_{mn} and C_{mn} are constants, J_n is Bessel function of the first kind of first order and I_n is modified Bessel function of the first kind of first order, indexes m and n are positive integers and correspond to the number of concentric circles and nodal diameters in each flexural mode. Considering an elastically supported plate as shown in Fig. 1, boundary conditions can be formulated at $r = a$, in terms of translational stiffness (K_T) as follows:

$$M_r(a,\theta) = 0 \qquad (6)$$

$$V_r(a,\theta) = K_T W(a,\theta) \qquad (7)$$

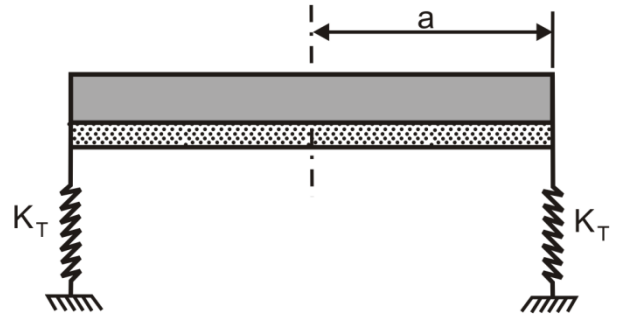where the Kelvin-Kirchhoff and bending moment are defined as follows:

Figure 1. A thin circular plate with translational elastic edge restraint and supported on elastic foundation.

$$M_r(a,\theta) = -D.\left[\frac{\partial^2 W(a,\theta)}{\partial r^2} + v\left(\frac{1}{r}\frac{\partial W(a,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W(a,\theta)}{\partial \theta^2}\right)\right] \quad (8)$$

$$V_r(a,\theta) = -D.\left[\frac{\partial}{\partial r}\nabla^2 W(a,\theta) + (1-v)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial^2 W(a,\theta)}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial W(a,\theta)}{\partial \theta}\right)\right] \quad (9)$$

By applying Eqs. (6) and (8), we obtain the following equation

$$\left[\frac{\partial^2 W(a,\theta)}{\partial r^2} + v\left(\frac{1}{r}\frac{\partial W(a,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W(a,\theta)}{\partial \theta^2}\right)\right] = 0 \quad (10)$$

From Eqs. (5) and (10), we derive the following equation

$$C_{mn} = \frac{-Q_{mn} + \dfrac{2v}{\lambda_{mn}}P_{mn} + \left[2 + \dfrac{4vn^2}{\lambda_{mn}^2}\right]J_n(\lambda_{mn})}{T_{mn} + \dfrac{2v}{\lambda_{mn}}S_{mn} + \left[2 - \dfrac{4vn^2}{\lambda_{mn}^2}\right]I_n(\lambda_{mn})} \quad (11)$$

where $P_{mn} = J_{n+1}(\lambda_{mn}) - J_{n-1}(\lambda_{mn})$;

$Q_{mn} = J_{n+2}(\lambda_{mn}) + J_{n-2}(\lambda_{mn})$;

$S_{mn} = I_{n+1}(\lambda_{mn}) + I_{n-1}(\lambda_{mn})$;

$T_{mn} = I_{n+2}(\lambda_{mn}) + I_{n-2}(\lambda_{mn})$;

From Eqs. (7) and (9), we get the following

$$-D.\left[\frac{\partial}{\partial r}\nabla^2 W(a,\theta) + (1-v)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial^2 W(a,\theta)}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial W(a,\theta)}{\partial \theta}\right)\right] = -K_T.W(a,\theta) \quad (12)$$

From Eqs. (5) and (12), we derived the following equation

$$C_{mn} = \frac{R_{mn} - \dfrac{2Q_{mn}}{\lambda_{mn}} - \left[3 + \dfrac{4+4(2-v)n^2}{\lambda_{mn}^2}\right]P_{mn} - \left[\dfrac{8(3-v)n^2}{\lambda^3} - \dfrac{4}{\lambda_{mn}} - \dfrac{8}{\lambda_{mn}^3}T\right]J_n(\lambda_{mn})}{U_{mn} + \dfrac{2T_{mn}}{\lambda_{mn}} + \left[3 - \dfrac{4-4(2-v)n^2}{\lambda_{mn}^2}\right]S_{mn} + \left[\dfrac{8(3-v)n^2}{\lambda_{mn}^3} + \dfrac{4}{\lambda_{mn}} - \dfrac{8}{\lambda_{mn}^3}T\right]I_n(\lambda_{mn})} \quad (13)$$

where, $T = \dfrac{a^3 K_T}{D}$

$P_{mn} = J_{n+1}(\lambda_{mn}) - J_{n-1}(\lambda_{mn})$; $Q_{mn} = J_{n+2}(\lambda_{mn}) + J_{n-2}(\lambda_{mn})$; $R_{mn} = J_{n+3}(\lambda_{mn}) - J_{n-3}(\lambda_{mn})$;

$S_{mn} = I_{n+1}(\lambda_{mn}) + I_{n-1}(\lambda_{mn})$; $T_{mn} = I_{n+2}(\lambda_{mn}) + I_{n-2}(\lambda_{mn})$; $U_{mn} = I_{n+3}(\lambda_{mn}) + I_{n-3}(\lambda_{mn})$;

If $K_T \to \infty$ then this case becomes a simply supported boundary condition as shown in Fig. 2. The frequency equation can be calculated From Eqs. (11) and (13), which allows determining eigenvalues λ_{mn}. The mode shape parameters C_{mn} can be determined corresponding to these eigenvalues by using either Eq. (11) or Eq. (13). The amplitude of each vibration mode in Eq. (5) is set by the normalization constant A_{mn} determined from the following condition.

$$\int_0^{2\pi}\int_0^a W_{mn}(r,\theta).W_{pq}(r,\theta)rdrd\theta = M_{mn}\delta_{mp}\delta_{nq} \quad (14)$$

where, M_{mn} is a mass of the plate,

$\delta_{mp} = \delta_{nq} = 1$ if $m = p, n = q$ and $\delta_{mp}\delta_{nq} = 0$ if $m \neq p$ or $n \neq q$.
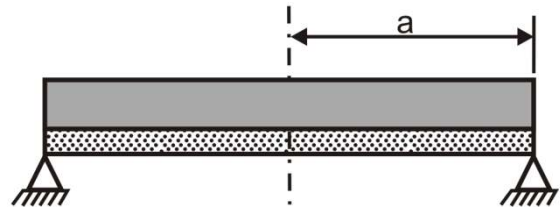

Figure 2. A simply supported thin circular plate resting on elastic foundation.

The normalization constant A_{mn} can be derived using Eqs. (5) and (14) as given below:

$$A_{mn} = \left[\frac{1}{\pi a^2} \cdot \int_0^{2\pi} \int_0^a \left(\begin{array}{c} J_n\left(\frac{\lambda_{mn}r}{a}\right) + \\ C_{mn} . I_n\left(\frac{\lambda_{mn}r}{a}\right) \end{array} \right)^2 . \cos n\theta \right)^2 rdrd\theta \right]^{-1}$$

(15)

In Eq (4), the natural frequency is defined as

$$\omega_{mn} = \left(\frac{\lambda_{mn}^2}{a^2} \right) \sqrt{\left(\frac{D}{\rho h} \right)}$$

(16)

It is clear from Eq. (16) that the natural frequency parameter ω_{mn} is dependent on the plate radius 'a'.

From Eq. (16) we can express

$$\lambda_{mn}^4 = \frac{\rho h a^4 \omega_{mn}^2}{D}$$

(17)

$$\lambda_{mn}^{*4} = \lambda_{mn}^4 + \xi^2$$

(18)

where $\xi^2 = \frac{K_w a^4}{D}$

(19)

$$\lambda_{mn}^* = \left[\lambda_{mn}^4 + \xi^2 \right]^{\frac{1}{4}}$$

(20)

3. Solution

Using Matlab programming, computer software with symbolic capabilities, solves the above set of equations. The program determines eigenvalues (λ_{mn}^*), for a given range of boundary conditions. The boundary's linear translational non-dimensional restraint parameter can be defined as follows:

$$T = \frac{K_T a^3}{D}$$

(21)

$$\xi^2 = \frac{K_w a^4}{D}$$

(22)

The following represent the input parameters to the program; (i) Translational stiffness ratio (T); (ii) Foundation ratio (ξ); (iii) Poisson's ratio (v); (iv) Upper bound for eigenvalues (N); (v) Suggested accuracy for eigenvalues (d); (vi) Number of mode shape parameters (n);. The program finds eigenvalues λ_{mn}^* by using Matlab root finding function.

4. Results and Discussion

The code developed is used to determine eigenvalues of any set or range of translational and foundation constraints. This code is also implemented for various plate materials by adjusting Poisson's ratio. Such a wide range of results is not available in the literature. The eigenvalues for the plate edge, which is elastically restrained against translation and fully resting on the elastic foundation, at various values of the translational stiffness ratios, are computed and the results are given in Table 1. The effects of the translational stiffness ratios are plotted in Fig. 3. As seen from Fig. 3, eigenvalues increase with an increment in the translational stiffness ratio, and the plates become unstable in the region when the translational stiffness ratio exceeds a certain value. Twelve vibration modes are presented in Fig. 3. The smoothened stepped variation is observed in Fig. 3. The stepped region increases with increase in translational stiffness ratio and vibration modes. The location of the stepped region with respect to T changed gradually from the range of 0.01526 [†] [9.9997] [‡] – 5587.5316 [10] to 16.62296 [14.6739] – 611824.96917 [16.75055]. Here [†] represents translational stiffness ratio and [‡] represents Eigen values throughout the text. Here the value in the bracket represents eigenvalue. The simply supported boundary conditions (Fig. 2) could be accounted for by setting ($K_T \rightarrow \infty$) shown in Fig. 1. The frequency in this case is 2.23175 and this is in good agreement with the results published by Wang (2005). Another result, considered for comparison, is from Rao and Rao (2009) on a study of the case of vibrations of elastically restrained circular plates supported on partial Winkler foundation. When the

Table 1. Eigenvalues for different Translational stiffness ratio for ξ =100 & ν =0.33.

$\log_{10} T$	λ_{00}	λ_{10}	λ_{20}	λ_{01}	λ_{11}	λ_{21}	λ_{02}	λ_{12}	λ_{22}	λ_{03}	λ_{13}	λ_{23}
-3	10	10.0205	10.3517	10.10358	10.79568	12.46774	10.29701	11.43857	13.52984	10.63672	12.2505	14.67332
-2	10	10.02051	10.35171	10.10359	10.79568	12.46775	10.29702	11.43857	13.52985	10.63673	12.25051	14.67332
-1	10.00005	10.02061	10.3518	10.10368	10.79576	12.46779	10.29711	11.43864	13.52989	10.63682	12.25056	14.67335
0	10.00047	10.02162	10.35263	10.10462	10.79649	12.46827	10.29803	11.43927	13.53026	10.63769	12.25109	14.67365
1	10.003	10.03287	10.36125	10.11468	10.804	12.47302	10.30745	11.44568	13.53403	10.64649	12.25645	14.67667
2	10.00568	10.1199	10.46887	10.2319	10.89095	12.52351	10.42003	11.5759	13.57333	10.74585	12.31326	14.70772
3	10.00614	10.203	11.00421	10.49407	11.61576	13.1793	10.91134	12.27823	14.08459	11.41839	12.99719	15.0966
4	10.00619	10.21322	11.14468	10.53794	11.94459	14.15323	11.03916	12.88272	15.33948	11.70347	13.90786	16.53545
5	10.00619	10.21423	11.15725	10.54212	11.97273	14.2438	11.05093	12.93527	15.47979	11.72934	13.99516	16.74092
6	10.00619	10.21433	11.15849	10.54254	11.97546	14.25215	11.05209	12.94027	15.49231	11.73186	14.00329	16.75866
12	10.00619	10.21434	11.15862	10.54258	11.97576	14.25307	11.05222	12.94082	15.49369	11.73214	14.00418	16.7606

Table 2. Eigenvalues for different Foundation stiffness ratio for T =100 & V =0.33.

$\text{Log}_{10}\ \zeta$	λ_{00}	λ_{10}	λ_{20}	λ_{01}	λ_{11}	λ_{21}	λ_{02}	λ_{12}	λ_{22}	λ_{03}	λ_{13}	λ_{23}
-3	2.18341	4.70075	6.69703	5.56683	7.98678	10.99197	6.50356	9.33292	12.43922	7.59879	10.67534	13.84974
-2	2.18341	4.70075	6.69703	5.56683	7.98678	10.99197	6.50356	9.33292	12.43922	7.59879	10.67534	13.84973
-1	2.18365	4.70077	6.69704	5.56684	7.98679	10.99197	6.50357	9.33292	12.43922	7.59879	10.67535	13.84974
0	2.20704	4.70315	6.69786	5.56827	7.98727	10.99216	6.50447	9.33323	12.43935	7.59936	10.67555	13.84983
1	3.3284	4.92488	6.77875	5.7064	8.03541	11.01074	6.5926	9.36352	12.45218	7.65514	10.69583	13.84914
2	10.00568	10.1199	10.46887	10.2319	10.89095	12.52351	10.42003	11.5159	13.57333	10.74585	12.31326	14.70772
3	31.62296	31.62664	31.63867	31.63037	31.6549	31.73756	31.63691	31.68259	31.81038	31.6491	31.72496	31.90972

Table 3. Eigenvalues for different Translation and Foundation stiffness ratios for V =0.33.

$\log_{10}\xi$ & $\log_{10}T$	λ_{00}	λ_{10}	λ_{20}	λ_{01}	λ_{11}	λ_{21}	λ_{02}	λ_{12}	λ_{22}	λ_{03}	λ_{13}	λ_{23}
-3	0.2115	3.0115	6.2054	4.52915	7.73687	10.9091	5.93655	9.18564	12.3826	7.27469	10.57846	13.80849
-2	0.37648	3.01187	6.20544	4.52925	7.73689	10.90911	5.93659	9.18566	12.38261	7.27471	10.57846	13.8085
-1	0.67602	3.0156	6.20584	4.53032	7.73709	10.90918	5.93708	9.18578	12.38266	7.27499	10.57855	13.80854
0	1.30374	3.067	6.21074	4.54336	7.73963	10.91008	5.94302	9.18733	12.38328	7.27834	10.57959	13.80899
1	3.25317	3.90325	6.34932	4.87899	7.81237	10.93614	6.10341	9.23131	12.40119	7.36838	10.60869	13.82199
2	10.00568	10.1199	10.46887	10.2319	10.89095	12.52351	10.42003	11.5152	13.57333	10.74585	12.31326	14.70772
3	31.62297	31.62939	31.65958	31.63958	31.68744	31.78104	31.65573	31.72292	31.85232	31.67796	31.76831	31.94926

support position is in full span, which means that when $b = 1$, the case becomes a circular plate having full foundation support with elastically restrained edge against translation. For this case, the frequency is 2.1834 and that is in good agreement with the frequency of 2.18341 obtained from the present study.

The eigenvalues at various values of the foundation stiffness ratios for $T = 100 \, \& \, \nu = 0.33$ are computed and the results are given in Table 2. The effects of the foundation stiffness ratio on eigenvalues are plotted in Fig. 4. As seen from Fig. 4, the eigenvalue increases with increase in the foundation stiffness ratio, and the plate becomes stiffer and stronger as the value of foundation stiffness becomes greater than 10^2. As seen from the Tables 1 and 2, the influence of foundation stiffness ratio on eigenvalue is relatively greater than that of the translation stiffness ratio in increasing the overall natural frequencies of the plate support system. As seen from Fig. 4, for all the modes considered here, up to a value of 10 the eigenvalues stay constant and beyond this value all the curves tend to converge to a constant eigenvalue as the foundation stiffness

ratio increases up to 10^3. The convergence starts from 1.07897 [2.0325779] and continues up to a constant value of 9.63274 [13.84796].

The eigenvalues at various values of the translational stiffness ratios and foundation stiffness ratios are computed and the results are given in Table 3. The effects of the translation and foundation stiffness ratios on eigenvalues are clearly observed in Fig. 5, eigenvalues increases with an increment in both the translational and foundation stiffness ratios. As observed from the Table 1 and 3, the influence of foundation stiffness ratio on eigenvalue is more predominant than that of translation stiffness ratio alone. From the results presented in Tables 1 and 3, we can see that the influence of foundation stiffness ratio on eigenvalues is more predominant than that of translation stiffness ratio. From the results given in Tables 1 to 3, one can easily find that the eigenvalues become lower for lower values of foundation and translation stiffness ratios. As seen from Fig. 5, all the curves are stable up to a certain region beyond which the curves tend to converge for increasing values of translation and foundation stiffness ratios.
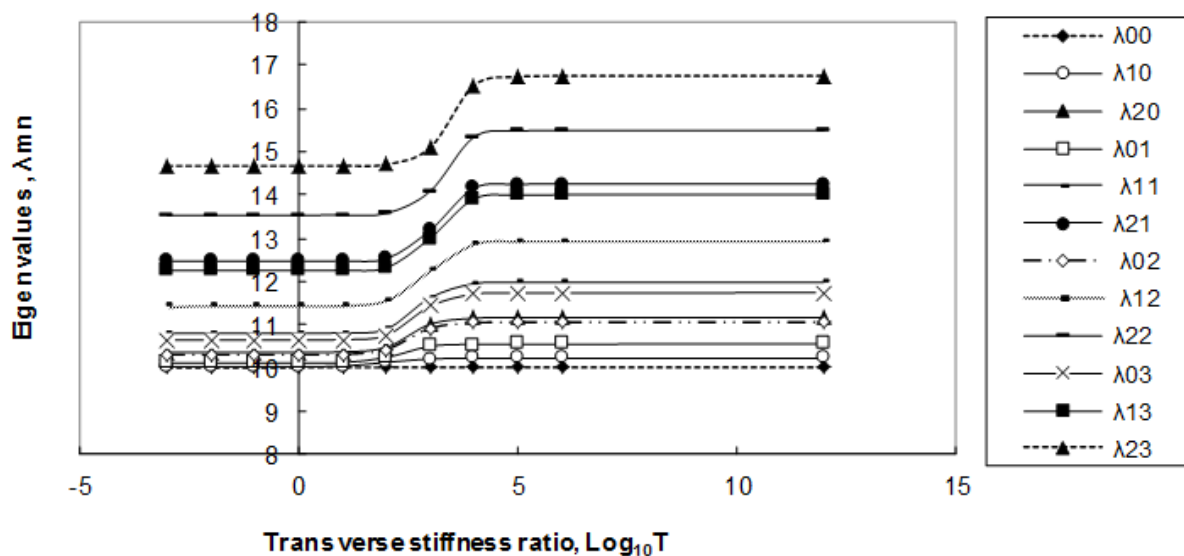

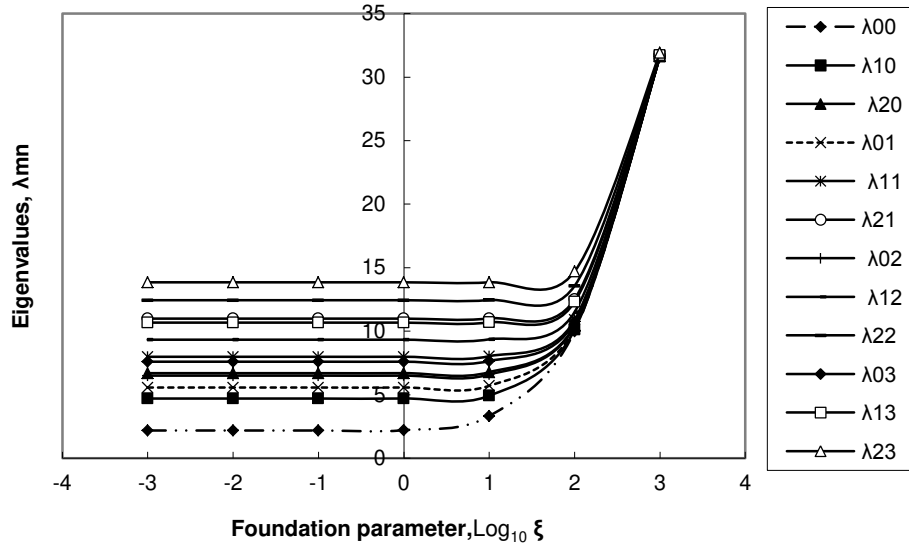**Figure 3**. Effect of translational stiffness ratio T on eigenvalues, λ_{mn} .

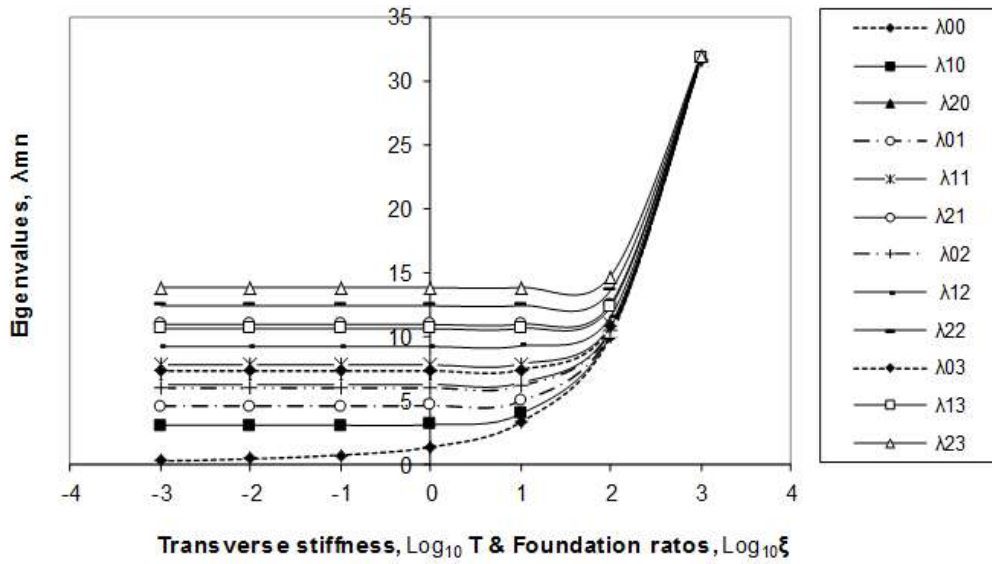**Figure 4**. Effect of foundation stiffness ratio, ξ on eigenvalues, λ_{mn}.

Figure 5. Effect of translational, T and foundation, ξ stiffness ratios on eigenvalues, λ_{mn}.

Table 4. Eigenvalues for different Poisson ratios.

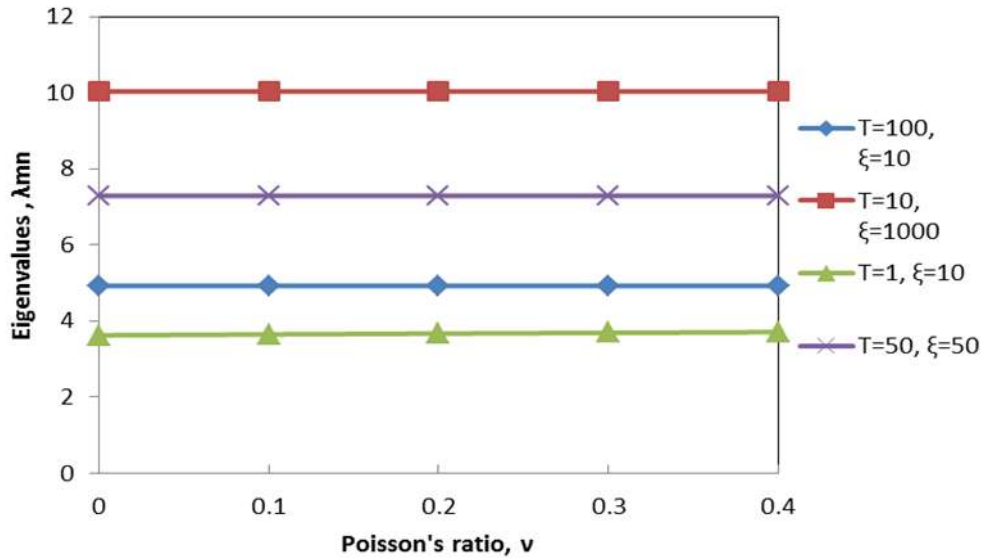| v	$T = \xi = 1000$	$T = 100, \xi = 10$	$T = 10, \xi = 1000$	$T = 1, \xi = 10$	$T = 50, \xi = 50$
0	31.62925	4.92456	10.03027	3.62553	7.28924
0.1	31.62929	4.92466	10.03111	3.64904	7.2896
0.2	31.62934	4.92476	10.0319	3.6708	7.28993
0.3	31.62938	4.92485	10.03265	3.69097	7.29025
0.4	31.62942	4.92494	10.03336	3.70974	7.29056

Figure 6. Effect of Poisson ratio, ν on eigenvalues, λ_{mn}.

5. Conclusion

This work deals with a method of computation of eigenvalues of flexural vibrations of a circular plate with translational edge supported and resting on Winkler foundation using a specifically written MATLAB code. The computed numerical results are presented in a tabular format to enable estimating the accuracy of approximate methods being used by other researchers for solving such problems. Two-dimensional plots of eigenvalues are drawn for a wide range of translational and foundation stiffness ratios facilitating their use in design. From the numerical and graphical results presented in this paper, it can be easily observed that the eigenvalues remain constant only for a limited range of constraints specific to each vibration mode and then steeply increase with the increasing values of foundation stiffness ultimately converging towards a constant value. It is also observed that the influence of foundation stiffness ratio on eigenvalues is more predominant than that of translational stiffness ratio.

Conflict of Interest

The authors declare no conflicts of interest.

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