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Weaker cyclic (φ, ϕ)-contractive mappings with an application to integro-differential equations

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Abstract. We introduce a new variant of cyclic contractive mapping in a metric space and originate existence and uniqueness results of fixed points for such mappings. Examples are given to support the usability of our results. After these results, an application to integro-differential equations is given.

Keywords: fixed point, cyclic contraction, integro-differential equation.

1 Introduction and preliminaries

The Banach Contraction Principle (BCP) is a very popular tool which is used to solve existence problems in many branches of Mathematical Analysis and its applications. There is a great number of generalizations of this fundamental principle. In particular, obtaining the existence and uniqueness of fixed points for self-maps on a metric space by altering distances between the points with the use of a certain control function is an interesting aspect. In this direction, Khan et al. [1] addressed a new category of fixed point problems for a single self-map with the help of a control function which they called an altering distance function.

Definition 1. (See [1].) A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(a) φ is continuous and non-decreasing, and
(b) $\varphi(t) = 0 \Leftrightarrow t = 0$.

Rhoades [2] extended BCP by introducing weakly contractive mappings in complete metric spaces.

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Definition 2. (See [2].) Let $(\mathcal{X}, d)$ be a metric space. A mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ is called weakly contractive if

$$d\big(\mathcal{T}x,\mathcal{T}y\big)\leqslant d(x,y)-\varphi\big(d(x,y)\big)$$

for all $x,y\in\mathcal{X}$, where φ is an altering distance function.

Theorem 1. (*See* [2, *Thm.* 2].) *Let $(\mathcal{X}, d)$ be a complete metric space. If $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ is a weakly contractive mapping, then $\mathcal{T}$ has a unique fixed point.*

Dutta and Choudhury in [3] obtained the following generalization of Theorem 1.

Theorem 2. (*See* [3, *Thm.* 2.1].) *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ satisfy that*

$$\psi\big(d(\mathcal{T}x,\mathcal{T}y)\big)\leqslant\psi\big(d(x,y)\big)-\varphi\big(d(x,y)\big)$$

for all $x,y\in\mathcal{X}$, where ψ and φ are altering distance functions. Then $\mathcal{T}$ has a unique fixed point.

Weak inequalities of the above type have been used to establish fixed point results in a number of subsequent works, some of which are noted in [4] and references cited therein.

On the other hand, cyclic representations and cyclic contractions were introduced by Kirk et al. [5].

Definition 3. (See [5, 11].) Let $(\mathcal{X}, d)$ be a complete metric space. Let p be a positive integer, $\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_p$ be nonempty subsets of $\mathcal{X}$, $\mathcal{Y}=\bigcup_{i=1}^{p}\mathcal{A}_i$ and $\mathcal{T}:\mathcal{Y}\to\mathcal{Y}$. Then $\mathcal{Y}$ is said to be a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$ if

(a) $\mathcal{A}_i$, $i=1,2,\dots,p$, are nonempty closed sets, and
(b) $\mathcal{T}(\mathcal{A}_1)\subseteq\mathcal{A}_2,\dots,\ \mathcal{T}(\mathcal{A}_{p-1})\subseteq\mathcal{A}_p,\ \mathcal{T}(\mathcal{A}_p)\subseteq\mathcal{A}_1$.

$\mathcal{T}$ is called a cyclic contraction if, moreover, there exists $k\in(0,1)$ such that $d(\mathcal{T}x,\mathcal{T}y)\leqslant kd(x,y)$ for all $x\in\mathcal{A}_i$ and $y\in\mathcal{A}_{i+1}$, $i=1,\dots,p$.

Notice that although a contraction is continuous, cyclic contractions need not be. This is one of the important gains of this approach.

Following [5], a number of fixed point theorems on cyclic contractions have appeared (see, e.g., [6–14]).

In this paper, we introduce a new variant of cyclic contractive mappings, named as weaker cyclic (φ,ϕ)-contractive mappings, modifying the conditions used in [9]. Then we derive the existence and uniqueness of fixed points for such mappings. Our main result generalizes and improves many existing theorems in the literature. Some examples are provided to demonstrate the validity of our results. Finally as an application of the presented theorems, we obtain existence and uniqueness of solutions of an integro-differential equation.

2 Main results

All the way through this paper, by $\mathbb{R}^+$, we designate the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers.

We state the notion of weaker cyclic (φ, ϕ)-contraction mapping as follows:

Definition 4. Let $(\mathcal{X}, d)$ be a metric space. Let p be a positive integer, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p$ be nonempty subsets of $\mathcal{X}$ and $\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$. An operator $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is called weaker cyclic (φ, ϕ)-contractive (in short WCC), if

(a) $\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$,

(b) for any $(x, y) \in \mathcal{A}_i \times \mathcal{A}_{i+1}$, $i = 1, 2, \dots, p$ (with $\mathcal{A}_{p+1} = \mathcal{A}_1$),

$$\varphi\bigl(d(\mathcal{T}x, \mathcal{T}y)\bigr) \leqslant \varphi\bigl(\Theta(x, y)\bigr) - \phi\bigl(\Theta(x, y)\bigr), \tag{1}$$

where

$$\Theta(x, y) = \max\biggl\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{1}{2}\bigl[d(y, \mathcal{T}x) + d(x, \mathcal{T}y)\bigr]\biggr\}, \tag{2}$$

(c) $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function with $\varphi(t) = 0$ if and only if $t = 0$,

(d) $\phi : [0, +\infty) \to [0, +\infty)$ is a function with $\phi(t) = 0$ if and only if $t = 0$, and $\liminf_{n\to\infty} \phi(\alpha_n) > 0$ if $\lim_{n\to\infty} \alpha_n = \alpha > 0$,

(e) $\phi(\alpha) > \varphi(\alpha) - \varphi(\alpha-)$ for any $\alpha > 0$, where $\varphi(\alpha-)$ is the left limit of φ at α.

Note that there exist examples of WCC type mappings which do not satisfy conditions given in [9, Thm. 2.1] (see further Example 2).

Our main result is the following.

Theorem 3. *Let $(\mathcal{X}, d)$ be a complete metric space, $p \in \mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p$ be nonempty closed subsets of $\mathcal{X}$ and $\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$. Suppose $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is a WCC mapping. Then $\mathcal{T}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{T}$ belongs to $\bigcap_{i=1}^{p} \mathcal{A}_i$.*

Proof. It should be noted that there exists the left limit of φ at a by the monotonicity of φ.

Let $x_0 \in \mathcal{A}_1$ (such a point exists since $\mathcal{A}_1 \neq \emptyset$). Define the sequence $\{x_n\}$ in $\mathcal{X}$ by

$$x_{n+1} = \mathcal{T}x_n, \quad n = 0, 1, 2, \dots.$$

First, we will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3}$$

If for some k, we have $x_{k+1} = x_k$, then (3) follows immediately. So, we can suppose that

$$\Theta(x_n, x_{n-1}) > 0 \tag{4}$$

for all $n \geqslant 1$. From condition (a), we observe that for all n, there exists $i = i(n) \in \{1, 2, \dots, p\}$ such that $(x_n, x_{n+1}) \in \mathcal{A}_i \times \mathcal{A}_{i+1}$. Then, from condition (b), we have

$$\varphi\bigl(d(x_n, x_{n+1})\bigr) \leqslant \varphi\bigl(\Theta(x_{n-1}, x_n)\bigr) - \phi\bigl(\Theta(x_{n-1}, x_n)\bigr), \quad n = 1, 2, \dots. \tag{5}$$

On the other hand, we have

$$\begin{aligned}\Theta(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_{n+1}, x_n), \frac{1}{2}d(x_{n-1}, x_{n+1})\right\}\\ &\leqslant \max\bigl\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\bigr\}.\end{aligned}$$

Now we claim that

$$d(x_{n+1}, x_n) \leqslant d(x_n, x_{n-1}) \tag{6}$$

for all $n \geqslant 1$. Suppose that $\max\{d(x_{k-1}, x_k), d(x_k, x_{k+1})\} = d(x_k, x_{k+1})$ for some $k \in \mathbb{N}$. Then $\Theta(x_{k-1}, x_k) = d(x_k, x_{k+1})$, hence

$$\varphi\bigl(d(x_k, x_{k+1})\bigr) \leqslant \varphi\bigl(d(x_k, x_{k+1})\bigr) - \phi\bigl(\Theta(x_{k-1}, x_k)\bigr).$$

This implies $\phi(\Theta(x_{k-1}, x_k)) = 0$. By a property of ϕ, we have $\Theta(x_{k-1}, x_k) = 0$, which contradicts to (4). Therefore, (6) is true and so the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing and bounded. Thus there exists $\rho \geqslant 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = \rho$. Therefore, by (2)

$$\begin{aligned}&\lim_{n\to\infty} d(x_{n+1}, x_{n-1})\\ &\quad= \lim_{n\to\infty} d\bigl(\mathcal{T}(x_n), \mathcal{T}(x_{n-1})\bigr) \leqslant \lim_{n\to\infty} \Theta(x_n, x_{n-1})\\ &\quad= \lim_{n\to\infty} \max\biggl\{d(x_n, x_{n-1}), d(x_n, \mathcal{T}x_n), d(x_{n-1}, \mathcal{T}x_{n-1}),\\ &\qquad\qquad \frac{1}{2}\bigl[d(x_{n-1}, \mathcal{T}x_n) + d(x_n, \mathcal{T}x_{n-1})\bigr]\biggr\}\\ &\quad= \lim_{n\to\infty} \max\left\{d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-2}), d(\mathcal{T}x_{n-1}, \mathcal{T}x_n), \frac{1}{2}d(\mathcal{T}x_{n-2}, \mathcal{T}x_n)\right\}.\end{aligned}$$

This implies $\rho \leqslant \lim_{n\to\infty} \Theta(x_n, x_{n-1}) \leqslant \rho$ and so $\lim_{n\to\infty} \Theta(x_n, x_{n-1}) = \rho$. Now we claim that $\rho = 0$. By (5), we have

$$\varphi\bigl(d(\mathcal{T}x_n, \mathcal{T}x_{n-1})\bigr) \leqslant \varphi\bigl(\Theta(x_n, x_{n-1})\bigr) - \phi\bigl(\Theta(x_n, x_{n-1})\bigr)$$

and taking limit as $n \to \infty$, we have

$$\varphi(\rho+) \leqslant \varphi(\rho+) - \liminf_{n\to\infty} \phi\bigl(\Theta(x_n, x_{n+1})\bigr)$$

which is contradictory unless $\rho = 0$. Hence

$$\rho = 0 = \lim_{n\to\infty} d(x_{n+1}, x_n).$$

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$. Suppose to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \qquad d(x_{m(k)}, x_{n(k)}) \geqslant \varepsilon, \qquad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{7}$$

Using (7) and the triangle inequality, we get

$$\begin{aligned}\varepsilon \leqslant d(x_{n(k)}, x_{m(k)}) &\leqslant d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})\\ &< \varepsilon + d(x_{n(k)}, x_{n(k)-1}).\end{aligned}$$

Thus we have

$$\varepsilon \leqslant d(x_{n(k)}, x_{m(k)}) < \varepsilon + d(x_{n(k)}, x_{n(k)-1}).$$

Passing to the limit as $k \to \infty$ in the above inequality and using (3), we obtain

$$\lim_{k\to\infty} d\big(x_{n(k)}, x_{m(k)}\big) = \varepsilon^+. \tag{8}$$

On the other hand, for all k, there exists $j(k) \in \{1, \ldots, p\}$ such that $n(k)-m(k)+j(k) \equiv 1[p]$. Then $x_{m(k)-j(k)}$ (for k large enough, $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labelled sets $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ for certain $i \in \{1, \ldots, p\}$. Using (1), we obtain

$$d(x_{m(k)-j(k)+1}, x_{n(k)+1}) \leqslant \Theta(x_{m(k)-j(k)}, x_{n(k)}) \tag{9}$$

for all k, where

$$\begin{aligned}&\Theta(x_{m(k)-j(k)}, x_{n(k)})\\ &\quad= \max\biggl\{d(x_{m(k)-j(k)}, x_{n(k)}), d(x_{m(k)-j(k)+1}, x_{m(k)-j(k)}), d(x_{n(k)+1}, x_{n(k)}),\\ &\qquad\frac{1}{2}\bigl[d(x_{m(k)-j(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)-j(k)+1})\bigr]\biggr\},\end{aligned}$$

for all k. Using the triangle inequality, we get

$$\begin{aligned}&\bigl|d(x_{m(k)-j(k)}, x_{n(k)}) - d(x_{n(k)}, x_{m(k)})\bigr|\\ &\quad\leqslant d(x_{m(k)-j(k)}, x_{m(k)}) \leqslant \sum_{l=0}^{j(k)-1} d(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1})\\ &\quad\leqslant \sum_{l=0}^{p-1} d(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}) \to 0 \quad \text{as } k \to \infty \text{ (from (3))},\end{aligned}$$

which, by (8), implies that

$$\lim_{k\to\infty} d(x_{m(k)-j(k)}, x_{n(k)}) = \varepsilon. \tag{10}$$

Using (3), we have

$$\lim_{k\to\infty} d(x_{m(k)-j(k)+1}, x_{m(k)-j(k)}) = 0$$

and

$$\lim_{k\to\infty} d(x_{n(k)+1}, x_{n(k)}) = 0. \tag{11}$$

Again, using the triangle inequality, we get

$$\left|d(x_{m(k)-j(k)}, x_{n(k)+1}) - d(x_{m(k)-j(k)}, x_{n(k)})\right| \leqslant d(x_{n(k)}, x_{n(k)+1}).$$

Passing to the limit as $k \to \infty$ in the above inequality, and using (11) and (10), we get

$$\lim_{k\to\infty} d(x_{m(k)-j(k)}, x_{n(k)+1}) = \varepsilon.$$

Similarly, we have

$$\left|d(x_{n(k)}, x_{m(k)-j(k)+1}) - d(x_{m(k)-j(k)}, x_{n(k)})\right| \leqslant d(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}).$$

Passing to the limit as $k \to \infty$, and using (3) and (10), we obtain

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)-j(k)+1}) = \varepsilon. \tag{12}$$

Similarly, we have

$$\lim_{k\to\infty} d(x_{m(k)-j(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{13}$$

Passing to the limit as $k \to \infty$ in (9), and using (12), (13), we obtain

$$\varepsilon \leqslant \lim_{k\to\infty} \Theta(x_{m(k)-j(k)}, x_{n(k)}) \leqslant \varepsilon,$$

and so

$$\lim_{k\to\infty} \Theta(x_{m(k)-j(k)}, x_{n(k)}) = \varepsilon.$$

If there exists a subsequence $\{k(p)\}$ of $\{k\}$ such that $\varepsilon < d(x_{n(k(p)+1)}, x_{m(k(p)+1)})$ for any p, then by (b) we get

$$\begin{aligned}
\varphi(\varepsilon+) &= \limsup_{k\to\infty} \varphi\big(d(x_{n(k)+1}, x_{m(k)+1})\big) \\
&= \limsup_{k\to\infty} \varphi\big(d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)})\big) \\
&\leqslant \limsup_{k\to\infty} \varphi\big(d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)+1}) + d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1})\big) \\
&= \limsup_{k\to\infty} \varphi\big(d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1})\big) \\
&\leqslant \limsup_{k\to\infty} \big[\varphi\big(\Theta(x_{n(k)}, x_{m(k)-1})\big) - \phi\big(\Theta(x_{n(k)}, x_{m(k)-1})\big)\big] \\
&= \varphi(\varepsilon+) - \liminf_{k\to\infty} \phi\big(\Theta(x_{n(k)}, x_{m(k)-1})\big),
\end{aligned}$$

which is a contradiction. We repeat the procedure if there exists a subsequence $\{k(p)\}$ of $\{k\}$ such that $\varepsilon < d(x_{n(k(p)+1)}, x_{m(k(p)+2)})$ for any p or $\varepsilon < d(x_{n(k(p)+2)}, x_{m(k(p)+1)})$ for any p. Therefore, we can suppose that

$$d(x_{n(k(p)+1)}, x_{m(k(p)+1)}) = \varepsilon, \qquad d(x_{n(k(p)+2)}, x_{m(k(p)+1)}) \leqslant \varepsilon$$
$$d(x_{n(k(p)+1)}, x_{m(k(p)+2)}) \leqslant \varepsilon$$

for any $k \geqslant k_1$. Then $\Theta(x_{n(k)}, x_{m(k)}) = \varepsilon$ for $k \geqslant k_3 = \max\{k_1, k_2\}$, where k_2 is such that $d(x_{k+1}, x_{k+2}) < \varepsilon$ for all $k \geqslant k_2$. Substituting $x = x_{n(k)}$, $x = x_{m(k)}$ in (b), we have

$$\varphi\big(d(x_{n(k)+2}, x_{m(k)+2})\big) \leqslant \varphi(\varepsilon) - \phi(\varepsilon)$$

for any $k \geqslant k_2$. Obviously $d(x_{n(k)+2}, x_{m(k)+2}) < \varepsilon$, otherwise we have $\phi(\varepsilon) = 0$. Letting $k \to \infty$ we obtain

$$\varphi(\varepsilon-) \leqslant \varphi(\varepsilon) - \phi(\varepsilon),$$

which contradicts hypothesis (d). Thus $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$.

Since $(\mathcal{X}, d)$ is complete, there exists $x^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{14}$$

We shall prove that

$$x^* \in \bigcap_{i=1}^{p} \mathcal{A}_i. \tag{15}$$

From condition (a), and since $x_0 \in \mathcal{A}_1$, we have $\{x_{np}\}_{n\geqslant 0} \subseteq \mathcal{A}_1$. Since $\mathcal{A}_1$ is closed, from (14), we get that $x^* \in \mathcal{A}_1$. Again, from the condition $(*)$, we have $\{x_{np+1}\}_{n\geqslant 0} \subseteq \mathcal{A}_2$. Since $\mathcal{A}_2$ is closed, from (14), we get that $x^* \in \mathcal{A}_2$. Continuing this process, we obtain (15).

Now, we shall prove that x^* is a fixed point of $\mathcal{T}$. Indeed, from (15), since for all n, there exists $i(n) \in \{1, 2, \dots, p\}$ such that $x_n \in \mathcal{A}_{i(n)}$, applying (b) with $x = x^*$ and $y = x_n$, we obtain

$$\varphi\big(d(\mathcal{T}x^*, x_{n+1})\big) = \varphi\big(d(\mathcal{T}x^*, \mathcal{T}x_n)\big) \leqslant \varphi\big(\Theta(x^*, x_n)\big) - \phi\big(\Theta(x^*, x_n)\big), \tag{16}$$

for all n. On the other hand, we have

$$\Theta(x^*, x_n) = \max\left\{d(x^*, x_n), d(x^*, Tx^*), d(x_n, x_{n+1}), \frac{d(x^*, x_{n+1}) + d(x_n, \mathcal{T}x^*)}{2}\right\}.$$

Passing to the limit as $n \to \infty$ in the above inequality and using (14), we obtain that

$$\lim_{n\to\infty} \Theta(x^*, x_n) = \max\left\{d(x^*, \mathcal{T}x^*), \frac{1}{2}d(x^*, \mathcal{T}x^*)\right\}. \tag{17}$$

Passing to the limit as $n \to \infty$ in (16), and using (17) and (14), we get

$$\begin{aligned}&\varphi\big(d(x^*, \mathcal{T}x^*)-\big)\\ &\quad\leqslant \varphi\left(\max\left\{d(x^*, \mathcal{T}x^*), \frac{1}{2}d(x^*, \mathcal{T}x^*)\right\}\right) - \phi\left(\max\left\{d(x^*, \mathcal{T}x^*), \frac{1}{2}d(x^*, \mathcal{T}x^*)\right\}\right).\end{aligned}$$

Suppose that $d(x^*, \mathcal{T}x^*) > 0$. In this case, we have

$$\max\left\{d(x^*, \mathcal{T}x^*), \frac{1}{2}d(x^*, \mathcal{T}x^*)\right\} = d(x^*, \mathcal{T}x^*),$$

which implies that

$$\varphi\big(d(x^*,\mathcal{T}x^*)-\big) \leqslant \varphi\big(d(x^*,\mathcal{T}x^*)\big) - \phi\big(d(x^*,\mathcal{T}x^*)\big),$$

which contradicts hypothesis (d). Thus we have $d(x^*,\mathcal{T}x^*) = 0$, that is, x^* is a fixed point of $\mathcal{T}$.

Finally, we prove that x^* is the unique fixed point of $\mathcal{T}$. Assume that y^* is another fixed point of $\mathcal{T}$, that is, $\mathcal{T}y^* = y^*$. By the condition (a), this implies that $y^* \in \bigcap_{i=1}^p \mathcal{A}_i$. Then we can apply (b) for $x = x^*$ and $y = y^*$. We obtain

$$\varphi\big(d(x^*,y^*)\big) = \varphi\big(d(\mathcal{T}x^*,\mathcal{T}y^*)\big) \leqslant \varphi\big(\Theta(x^*,y^*)\big) - \phi\big(\Theta(x^*,y^*)\big).$$

Since x^* and y^* are fixed points of $\mathcal{T}$, we can show easily that $\Theta(x^*,y^*) = d(x^*,y^*)$. If $d(x^*,y^*) > 0$, we get

$$\begin{aligned}\varphi\big(d(x^*,y^*)\big) &= \varphi\big(d(\mathcal{T}x^*,\mathcal{T}y^*)\big) \leqslant \varphi\big(\Theta(x^*,y^*)\big) - \phi\big(\Theta(x^*,y^*)\big)\\ &= \varphi\big(d(x^*,y^*)\big) - \phi\big(d(x^*,y^*)\big)\end{aligned}$$

a contradiction unless $d(x^*,y^*) = 0$, that is, $x^* = y^*$. Thus we have proved the uniqueness of the fixed point. □

This theorem generalizes, e.g., results from [6–16].

We present here a corollary concerning mappings satisfying a general contractive condition of integral type in a complete metric space [17].

Corollary 1. *Let $\mathcal{T}$ as well as φ, ϕ, $\Theta(x,y)$ satisfy the conditions of Theorem* 3*, except that condition* (b) *is replaced by the following*: *there exists a nonnegative Lebesgue integrable function u on $\mathbb{R}_+$ such that $\int_0^\varepsilon u(t)\,\mathrm{d}t > 0$ for each $\varepsilon > 0$ and that*

$$\int\limits_0^{\varphi(d(\mathcal{T}x,\mathcal{T}y))} u(t)\,\mathrm{d}t \leqslant \int\limits_0^{\varphi(\Theta(x,y))} u(t)\,\mathrm{d}t - \int\limits_0^{\phi(\Theta(x,y))} u(t)\,\mathrm{d}t. \tag{18}$$

Then $\mathcal{T}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{T}$ belongs to $\bigcap_{i=1}^p A_i$.

Proof. Define $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ by $\Lambda(x) = \int_0^x u(t)\,\mathrm{d}t$. Then Λ is continuous and nondecreasing with $\Lambda(0) = 0$. Condition (18) becomes

$$\Lambda\big(\varphi\big(d(\mathcal{T}x,\mathcal{T}y)\big)\big) \leqslant \Lambda\big(\varphi\big(\Theta(x,y)\big)\big) - \Lambda\big(\phi\big(\Theta(x,y)\big)\big),$$

which can be further written as

$$\varphi_1\big(d(\mathcal{T}x,\mathcal{T}y)\big) \leqslant \varphi_1\big(\Theta(x,y)\big) - \phi_1\big(\Theta(x,y)\big),$$

where $\phi_1 = \Lambda \circ \phi$ and $\varphi_1 = \Lambda \circ \varphi$. Clearly, ϕ_1, φ_1 are control functions with $\phi_1(0) = 0 = \varphi_1(0)$. Hence by Theorem 3, $\mathcal{T}$ has a fixed point. □

3 Examples

The following example (which is inspired by [18]) demonstrates the validity of Theorem 3.

Example 1. Let $\mathcal{X} = \ell^1$ be endowed with the standard metric

$$d\big((x_n),(y_n)\big) = \sum_{n=1}^{\infty} |x_n - y_n|.$$

Let $\alpha \in (0,1)$ be fixed, denote $\mathbf{0} = (0)_{n=1}^{\infty}$ and consider the subsets $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{X}$ defined by $\mathcal{A}_1 = \mathcal{A}' \cup \{\mathbf{0}\}$, $\mathcal{A}_2 = \mathcal{A}'' \cup \{\mathbf{0}\}$, where

$$\mathcal{A}' \ni x^l = (x_n^l)_{n=1}^{\infty} \quad \text{iff } x_n^l = \begin{cases} 0, & n < 2l \vee n = 2k-1,\ k \in \mathbb{N}, \\ \alpha^n, & n = 2k \geqslant 2l, \end{cases} \qquad l = 1,2,\dots,$$

and

$$\mathcal{A}'' \ni x^l = (x_n^l)_{n=1}^{\infty} \quad \text{iff } x_n^l = \begin{cases} 0, & n < 2l-1 \vee n = 2k,\ k \in \mathbb{N}, \\ \alpha^n, & n = 2k-1 \geqslant 2l-1, \end{cases} \qquad l = 1,2,\dots.$$

Denote $\mathcal{Y} = \mathcal{A}_1 \cup \mathcal{A}_2$ (obviously $\mathcal{A}_1 \cap \mathcal{A}_2 = \{\mathbf{0}\}$).

Consider the mapping $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ given by:

$$\begin{aligned}
&\mathcal{T}(\mathbf{0}) = \mathbf{0},\\
&\mathcal{T}\big((\underbrace{0,\dots,0}_{2l-1}, \alpha^{2l}, 0, \alpha^{2l+2}, 0, \dots)\big) = \big(\underbrace{0,\dots,0}_{2l}, \alpha^{2l+1}, 0, \alpha^{2l+3}, 0, \dots\big),\\
&\mathcal{T}\big((\underbrace{0,\dots,0}_{2l}, \alpha^{2l+1}, 0, \alpha^{2l+3}, 0, \dots)\big) = \big(\underbrace{0,\dots,0}_{2l+1}, \alpha^{2l+2}, 0, \alpha^{2l+4}, 0, \dots\big).
\end{aligned}$$

Obviously, $\mathcal{T}(\mathcal{A}_1) \subset \mathcal{A}_2$ and $\mathcal{T}(\mathcal{A}_2) \subset \mathcal{A}_1$, hence $\mathcal{Y} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$.

Take $\varphi(t) = kt$, $k > 0$ and $\phi(t) = ht$ for some $h > 0$, $h \leqslant k(1-\alpha)$. Let us check the contractive condition (b) of Theorem 3. Take

$$\begin{aligned}
x &= \big(\underbrace{0,\dots,0}_{2l-1}, \alpha^{2l}, 0, \alpha^{2l+2}, 0, \dots\big) \in \mathcal{A}_1,\\
y &= \big(\underbrace{0,\dots,0}_{2m}, \alpha^{2m+1}, 0, \alpha^{2m+3}, 0, \dots\big) \in \mathcal{A}_2
\end{aligned}$$

and assume, e.g., that $l < m$ (the case $l \geqslant m$ is treated similarly, as well as the case when x or y is equal to $\mathbf{0}$). Then

$$\begin{aligned}
d(x,y) &= \alpha^{2l} + \dots + \alpha^{2m-2} + \frac{\alpha^{2m}}{1-\alpha},\\
d(\mathcal{T}x,\mathcal{T}y) &= \alpha^{2l+1} + \dots + \alpha^{2m-1} + \frac{\alpha^{2m+1}}{1-\alpha} \leqslant \frac{\alpha^{2l+1}}{1-\alpha},
\end{aligned}$$

$$\begin{aligned} &d(x,\mathcal{T}x)=\frac{\alpha^{2l}}{1-\alpha}, \qquad d(y,\mathcal{T}y)=\frac{\alpha^{2m+1}}{1-\alpha},\\ &d(x,\mathcal{T}y)=\alpha^{2l}+\alpha^{2l+2}+\cdots+\alpha^{2m},\\ &d(y,\mathcal{T}x)=\alpha^{2l+1}+\alpha^{2l+3}+\cdots+\alpha^{2m-1},\\ &\Theta(x,y)=\frac{\alpha^{2l}}{1-\alpha}. \end{aligned}$$

Hence,

$$\begin{aligned} \varphi\bigl(d(\mathcal{T}x,\mathcal{T}y)\bigr) &\leqslant k\frac{\alpha^{2l+1}}{1-\alpha}=k\alpha\frac{\alpha^{2l}}{1-\alpha}=k\alpha\Theta(x,y)\\ &\leqslant (k-h)\Theta(x,y)=\varphi\bigl(\Theta(x,y)\bigr)-\phi\bigl(\Theta(x,y)\bigr). \end{aligned}$$

Thus, all the conditions of Theorem 3 are satisfied. Obviously, $\mathcal{T}$ has a unique fixed point $\mathbf{0}$.

The following example (inspired from [4]) demonstrates the validity of Theorem 3 when φ is nonlinear and discontinuous.

Example 2. Let $\mathcal{X}=[0,1]$ be equipped with the standard metric and consider the following mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ and functions $\varphi,\phi:[0,+\infty)\to[0,+\infty)$:

$$\mathcal{T}x=\begin{cases}\frac{1}{2}, & 0\leqslant x<1,\\ 0, & x=1,\end{cases} \qquad \varphi(t)=\begin{cases}\frac{7}{5}t, & 0\leqslant t<\frac{1}{2},\\ \frac{\sqrt{2}}{2}, & t=\frac{1}{2},\\ \frac{2t+3}{5}, & \frac{1}{2}<t<+\infty,\end{cases} \qquad \phi(t)=\frac{1}{10}\,t^2.$$

Taking $\mathcal{A}_1=[0,1/2]$ and $\mathcal{A}_2=[1/2,1]$ we obtain a cyclic representation $\mathcal{X}=\mathcal{A}_1\cup\mathcal{A}_2$ with respect to $\mathcal{T}$. Conditions (c) and (d) of Definition 4 are obvious. The only point of discontinuity of φ is $1/2$ and it is $\phi(1/2)=0.025>\sqrt{2}/2-0.7=\varphi(1/2)-\varphi(1/2-)$, hence condition (e) is also satisfied.

Since $\phi(t)\leqslant\varphi(t)$ for all $t\in[0,1]$, the only nontrivial case when the contractive condition (b) has to be checked is when $x\in[0,1/2)$, $y=1$ (or vice versa). Since $\mathcal{T}x=1/2$, $\mathcal{T}y=0$ and $\Theta(x,y)=1$, it becomes

$$\varphi\bigl(d(\mathcal{T}x,\mathcal{T}y)\bigr)=\varphi\left(\frac{1}{2}\right)=\frac{\sqrt{2}}{2}<\frac{9}{10}=\varphi(1)-\phi(1)=\varphi\bigl(\Theta(x,y)\bigr)-\phi\bigl(\Theta(x,y)\bigr).$$

Thus, the conditions of Theorem 3 are fulfilled and the mapping $\mathcal{T}$ has a unique fixed point (which is $1/2$).

Note that this example is not covered by Theorem 2.1 of [9], since the function φ is not right-continuous at the point $1/2$.

4 An application to integro-differential equations

In this section we present examples of certain Volterra and Fredholm type integro-differential equations. The examples are inspired by [19].

Consider the nonlinear Volterra and Fredholm type integro-differential equations of the forms

$$x(t) = g(t) + \int_a^t f\bigl(t, s, x(s), x'(s)\bigr)\,\mathrm{d}s, \tag{19}$$

and

$$x(t) = g(t) + \int_a^b f\bigl(t, s, x(s), x'(s)\bigr)\,\mathrm{d}s, \tag{20}$$

for $-\infty < a \leqslant t \leqslant b < \infty$, where x, g, f are real functions. We shall denote $J = [a, b]$. The functions $g(t)$ $(t \in J)$ and $f(t, s, u, v)$ $(a \leqslant s \leqslant t \leqslant b,\ u, v \in \mathbb{R})$ are supposed to be continuous and continuously differentiable with respect to t.

For a real-valued function $x(t)$, $t \in J$, continuous together with its first derivative $x'(t)$ for $t \in J$, we denote $|x(t)|_1 = |x(t)| + |x'(t)|$. Denote by $\mathcal{E}$ the space of functions which fulfill the condition

$$\bigl|x(t)\bigr|_1 = O\bigl(\exp(\lambda t)\bigr), \quad t \in J, \tag{21}$$

where λ is a positive constant. Define the norm in the space $\mathcal{E}$ as

$$|x|_{\mathcal{E}} = \max_{t \in J}\bigl\{\bigl|x(t)\bigr|_1 \exp(-\lambda t)\bigr\}. \tag{22}$$

It is easy to see that $\mathcal{E}$ with the norm defined in (22) is a Banach space. We note that the condition (21) implies that there is a constant $N \geqslant 0$ such that $|x(t)|_1 \leqslant N \exp(\lambda t)$, $t \in J$. Using this fact in (22) we observe that

$$|x|_{\mathcal{E}} \leqslant N. \tag{23}$$

Define a mapping $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ by

$$(\mathcal{T}x)(t) = g(t) + \int_a^t f\bigl(t, s, x(s), x'(s)\bigr)\,\mathrm{d}s \tag{24}$$

for $x \in \mathcal{E}$. Note that, if $u^* \in \mathcal{E}$ is a fixed point of $\mathcal{T}$, then u^* is a solution of the problem (19). We shall prove the existence of a fixed point of $\mathcal{T}$ under the following conditions.

(I) There exist $(\alpha, \beta) \in \mathcal{E}^2$, $(\alpha_0, \beta_0) \in \mathbb{R}^2$ such that

$$\alpha_0 \leqslant \alpha(t) \leqslant \beta(t) \leqslant \beta_0, \quad \alpha_0 \leqslant \alpha'(t) \leqslant \beta'(t) \leqslant \beta_0, \quad t \in J,$$

and for all $t \in J$, we have

$$\alpha(t) \leqslant g(t) + \int_a^t f\bigl(t, s, \beta(s), \beta'(s)\bigr)\,\mathrm{d}s,$$

$$\alpha'(t) \leqslant g'(t) + f\bigl(t, t, \beta(t), \beta'(t)\bigr) + \int_a^t \frac{\partial}{\partial t} f\bigl(t, s, \beta(s), \beta'(s)\bigr)\,\mathrm{d}s, \quad t \in J,$$

and

$$\beta(t) \geqslant g(t) + \int_a^t f\big(t, s, \alpha(s), \alpha'(s)\big)\,\mathrm{d}s,$$
$$\beta'(t) \geqslant g'(t) + f(t, t, \alpha(t), \alpha'(t)) + \int_a^t \frac{\partial}{\partial t} f\big(t, s, \alpha(s), \alpha'(s)\big)\,\mathrm{d}s, \quad t \in J.$$

(II) $f : J \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and nonincreasing with respect to the third and fourth variables, that is, for $u, v \in \mathcal{E}$,

$$\begin{aligned} &u(t) \geqslant v(t) \quad \text{and} \quad u'(t) \geqslant v'(t) \quad \text{for } t \in J \\ &\quad \Longrightarrow \quad f\big(t, s, u(s), u'(s)\big) \leqslant f\big(t, s, v(s), v'(s)\big) \quad \text{and} \\ &\qquad \frac{\partial}{\partial t} f\big(t, s, u(s), u'(s)\big) \leqslant \frac{\partial}{\partial t} f\big(t, s, v(s), v'(s)\big), \quad a \leqslant s \leqslant t \leqslant b. \end{aligned}$$

(III) The function f and its derivative satisfy the conditions

$$\big|f(t, s, u, v) - f(t, s, \overline{u}, \overline{v})\big| \leqslant h_1(t, s)\big[|u - \overline{u}| + |v - \overline{v}|\big],$$
$$\left|\frac{\partial}{\partial t} f(t, s, u, v) - \frac{\partial}{\partial t} f(t, s, \overline{u}, \overline{v})\right| \leqslant h_2(t, s)\big[|u - \overline{u}| + |v - \overline{v}|\big],$$

for $a \leqslant s \leqslant t \leqslant b$, $u, v, \overline{u}, \overline{v} \in \mathcal{E}$, where $h_i \in C(J^2, \mathbb{R}^+)$ for $i = 1, 2$.

(IV) There exist nonnegative constants γ_1, γ_2 such that $\gamma_1 + \gamma_2 < 1$ and

$$\int_a^t h_1(t, s) \exp(\lambda s)\,\mathrm{d}s \leqslant \gamma_1 \exp(\lambda t),$$
$$h_1(t, t) \exp(\lambda t) + \int_a^t h_2(t, s) \exp(\lambda s)\,\mathrm{d}s \leqslant \gamma_2 \exp(\lambda t),$$

for $t \in J$, where λ is given in (21).

(V) There exist nonnegative constants δ_1, δ_2 such that

$$\big|g(t)\big| + \int_a^t \big|f(t, s, 0, 0)\big|\,\mathrm{d}s \leqslant \delta_1 \exp(\lambda t),$$
$$\big|g'(t)\big| + \big|f(t, s, 0, 0)\big| + \int_a^t \left|\frac{\partial}{\partial t} f(t, s, 0, 0)\right|\mathrm{d}s \leqslant \delta_2 \exp(\lambda t),$$

for $a \leqslant s \leqslant t \leqslant b$, where λ is given in (21).

We have the following result for the set

$$\mathcal{P} = \big\{u \in \mathcal{E}\colon\ \alpha(t) \leqslant u(t) \leqslant \beta(t),\ \alpha'(t) \leqslant u'(t) \leqslant \beta'(t),\ t \in J\big\}.$$

Theorem 4. *Under the assumptions* (I)–(V)*, the integro-differential problem* (19) *has a unique solution in the set* $\mathcal{P}$.

Proof. The proof of the theorem is divided into three parts.

(A) First we show that $\mathcal{T}$ maps $\mathcal{E}$ into itself.

Differentiating both sides of (24) with respect to t we get

$$(\mathcal{T}x)'(t) = g'(t) + f\big(t, t, x(t), x'(t)\big) + \int_a^t \frac{\partial}{\partial t} f\big(t, s, x(s), x'(s)\big)\,\mathrm{d}s. \tag{25}$$

Evidently, $\mathcal{T}x, (\mathcal{T}x)'$ are continuous on J. We verify that (21) is fulfilled. From (22), (25) and using conditions (IV), (V) and (23) we have

$$\begin{aligned}
\big|(\mathcal{T}x)(t)\big| &\leqslant \big|g(t)\big| + \int_a^t \big|f(t, s, x(s), x'(s)\big) - f(t, s, 0, 0) + f(t, s, 0, 0)\big|\,\mathrm{d}s \\
&\leqslant \big|g(t)\big| + \int_a^t \big|f(t, s, 0, 0)\big|\,\mathrm{d}s + \int_a^t h_1(t, s)\big|x(s)\big|_1\,\mathrm{d}s \\
&\leqslant \delta_1 \exp(\lambda t) + |x|_{\mathcal{E}} \int_a^t h_1(t, s) \exp(\lambda s)\,\mathrm{d}s \\
&\leqslant [\delta_1 + N\gamma_1] \exp(\lambda t),
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\big|(\mathcal{T}x)'(t)\big| &\leqslant \big|g'(t)\big| + \big|f(t, t, x(t), x'(t)\big) - f(t, t, 0, 0) + f(t, t, 0, 0)\big| \\
&\quad + \int_a^t \left|\frac{\partial}{\partial t} f\big(t, s, x(s), x'(s)\big) - \frac{\partial}{\partial t} f(t, s, 0, 0) + \frac{\partial}{\partial t} f(t, s, 0, 0)\right|\mathrm{d}s \\
&\leqslant \big|g'(t)\big| + \big|f(t, t, 0, 0)\big| + \int_a^t \left|\frac{\partial}{\partial t} f(t, s, 0, 0)\right|\mathrm{d}s + h_1(t, t)\big|x(t)\big|_1 \\
&\quad + \int_a^t h_2(t, s)\big|x(s)\big|_1\,\mathrm{d}s \\
&\leqslant \delta_2 \exp(\lambda t) + |x|_{\mathcal{E}} h_1(t, t) \exp(\lambda t) + |x|_{\mathcal{E}} \int_a^t h_2(t, s) \exp(\lambda s)\,\mathrm{d}s \\
&\leqslant [\delta_2 + N\gamma_2] \exp(\lambda t).
\end{aligned} \tag{27}$$

Combining (26) and (27) we get

$$\big|(\mathcal{T}x)(t)\big|_1 \leqslant \big[\delta_1 + \delta_2 + N(\gamma_1 + \gamma_2)\big] \exp(\lambda t). \tag{28}$$

It follows from (28) that $\mathcal{T}x \in \mathcal{E}$. This proves that $\mathcal{T}$ maps $\mathcal{E}$ into itself.

(B) Define closed subsets of $\mathcal{E}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ by

$$\mathcal{A}_1 = \bigl\{u \in \mathcal{E}\colon\ u(t) \leqslant \beta(t),\ u'(t) \leqslant \beta'(t) \text{ for } t \in J\bigr\}$$

and

$$\mathcal{A}_2 = \bigl\{u \in \mathcal{E}\colon\ u(t) \geqslant \alpha(t),\ u'(t) \geqslant \alpha'(t) \text{ for } t \in J\bigr\}.$$

We shall prove that

$$\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{A}_2 \quad \text{and} \quad \mathcal{T}(\mathcal{A}_2) \subseteq \mathcal{A}_1. \tag{29}$$

Let $u \in \mathcal{A}_1$, that is,

$$u(t) \leqslant \beta(t) \quad \text{and} \quad u'(t) \leqslant \beta'(t) \quad \text{for all } t \in J.$$

Using condition (II), we obtain that

$$f\bigl(t, s, u(s), u'(s)\bigr) \geqslant f\bigl(t, s, \beta(s), \beta'(s)\bigr) \tag{30}$$

and

$$\frac{\partial}{\partial t} f\bigl(t, s, u(s), u'(s)\bigr) \leqslant \frac{\partial}{\partial t} f\bigl(t, s, \beta(s), \beta'(s)\bigr) \tag{31}$$

for $a \leqslant s \leqslant t \leqslant b$. The inequality (30) with condition (I) imply that

$$(\mathcal{T}u)(t) = g(t) + \int_a^t f\bigl(t, s, u(s), u'(s)\bigr)\,\mathrm{d}s \geqslant g(t) + \int_a^t f\bigl(t, s, \beta(s), \beta'(s)\bigr)\,\mathrm{d}s \geqslant \alpha(t)$$

for all $t \in J$. The inequality (31) with condition (I) imply that

$$\begin{aligned}
(\mathcal{T}u)'(t) &= g'(t) + f\bigl(t, t, u(t), u'(t)\bigr) + \int_a^t \frac{\partial}{\partial t} f\bigl(t, s, u(s), u'(s)\bigr)\,\mathrm{d}s \\
&\geqslant g'(t) + f\bigl(t, t, \beta(t), \beta'(t)\bigr) + \int_a^t \frac{\partial}{\partial t} f\bigl(t, s, \beta(s), \beta'(s)\bigr)\,\mathrm{d}s \geqslant \alpha'(t)
\end{aligned}$$

for all $t \in J$. Hence, we have $\mathcal{T}u \in \mathcal{A}_2$.

Similarly, if $u \in \mathcal{A}_2$, it can be proved that $\mathcal{T}u \in \mathcal{A}_1$ holds. Thus, (29) is fulfilled.

(C) We verify that the operator $\mathcal{T}$ is a WCC map.

Let $(u, v) \in \mathcal{A}_1 \times \mathcal{A}_2$, that is, for all $t \in J$,

$$\begin{aligned}
u(t) &\leqslant \beta(t) \leqslant \beta_0, & u'(t) &\leqslant \beta'(t) \leqslant \beta_0, \\
v(t) &\geqslant \alpha(t) \geqslant \alpha_0, & v'(t) &\geqslant \alpha'(t) \geqslant \alpha_0.
\end{aligned}$$

Using the properties (24) and (25) of $\mathcal{T}$ and conditions (III), (IV) and (V), we conclude that

$$\begin{aligned}\big|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\big| &\leqslant \int_a^t \big|f\big(t,s,u(s),u'(s)\big) - f\big(t,s,v(s),v'(s)\big)\big|\mathrm{d}s \\ &\leqslant \int_a^t h_1(t,s)\big|u(s)-v(s)\big|_1 \mathrm{d}s \\ &\leqslant |u-v|_{\mathcal{E}} \int_a^t h_1(t,s)\exp(\lambda s)\,\mathrm{d}s \leqslant |u-v|_{\mathcal{E}}\gamma_1 \exp(\lambda t), \qquad (32)\end{aligned}$$

and

$$\begin{aligned}\big|(\mathcal{T}u)'(t) - (\mathcal{T}v)'(t)\big| &\leqslant \big|f\big(t,t,u(t),u'(t)\big) - f\big(t,t,v(t),v'(t)\big)\big| \\ &\quad + \int_a^t \left|\frac{\partial}{\partial t} f\big(t,s,u(s),u'(s)\big) - \frac{\partial}{\partial t} f\big(t,s,v(s),v'(s)\big)\right|\mathrm{d}s \\ &\leqslant h_1(t,t)\big|u(t)-v(t)\big|_1 + \int_a^t h_2(t,s)\big|u(s)-v(s)\big|_1\,\mathrm{d}s \\ &\leqslant |u-v|_{\mathcal{E}} h_1(t,t)\exp(\lambda t) + |u-v|_{\mathcal{E}} \int_a^t h_2(t,s)\exp(\lambda s)\,\mathrm{d}s \\ &\leqslant |u-v|_{\mathcal{E}}\gamma_2\exp(\lambda t), \qquad (33)\end{aligned}$$

for $t \in J$. Combining (32) and (33) we get

$$\big|(\mathcal{T}u)(t) - \big(\mathcal{T}v)(t)\big|_1 \leqslant |u-v|_{\mathcal{E}}(\gamma_1+\gamma_2)\exp(\lambda t). \qquad (34)$$

From (34) we obtain (with $k = \gamma_1 + \gamma_2 < 1$)

$$\begin{aligned}\big|\mathcal{T}u - \mathcal{T}v\big|_{\mathcal{E}} &\leqslant k|u-v|_{\mathcal{E}} \\ &\leqslant k\max\left\{|u-v|_{\mathcal{E}}, |u-\mathcal{T}u|_{\mathcal{E}}, |v-\mathcal{T}v|_{\mathcal{E}}, \frac{1}{2}\big[|u-\mathcal{T}v|_{\mathcal{E}} + |v-\mathcal{T}u|_{\mathcal{E}}\big]\right\}.\end{aligned}$$

Consider the functions $\varphi, \phi : [0,+\infty) \to [0,+\infty)$ defined by:

$$\varphi(t) = t \quad \text{and} \quad \phi(t) = (1-k)t.$$

Then the contractive condition takes the form

$$\varphi\big(|\mathcal{T}u - \mathcal{T}v|_{\mathcal{E}}\big) \leqslant \varphi\big(\Theta(u,v)\big) - \phi\big(\Theta(u,v)\big).$$

Using the same technique, we can show that the above inequality also holds if we take $(u,v) \in \mathcal{A}_2 \times \mathcal{A}_1$. All other conditions of Theorem 3 are fulfilled for the complete metric space $(\mathcal{A}_1 \cup \mathcal{A}_2, |\cdot|_{\mathcal{E}})$ and $\mathcal{T}$ restricted to $\mathcal{A}_1 \cup \mathcal{A}_2$ (with $p = 2$).

We conclude that the operator $\mathcal{T}$ has a unique fixed point u^* and, hence, the integro-differential equation (19) has a unique solution in the set $\mathcal{P}$. □

Remark 1. A similar result can be shown for equation (20).

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